Problems October 22

Points on Conics

1 The conic $2x^2 = y^2 + z^2$ has an obvious solution. Use this one to get a parametrization of all its rational solutions in terms of binary quadrics. Discuss this solution with the problem of finding integral solutions to Pells equation $2n^2 - m^2 = 1$

2 A pencil of conics $t_0C_0 + t_1C_1$ can be thought of as a single conic C with koefficients in the field $k(t)$. What is meant by a solution $A(t), B(t), C(t)$ to the conic, and how can we interpret it in terms of k (i.e. interpreting the conic as a pencil of conics).

3 Show that every conic in $P^2(\mathbb{F}_q)$ has a solution over $\mathbb{F}_q$ and hence exactly $q + 1$ solutions.

Hint :If C is a conic with no solutions over $\mathbb{F}_q$ chose two distinct lines L_1L_2. None of them intersect C in a rational point, because C has no rational point! Consider the pencil spanned by C and L_1L_2 it can only have one singular fiber consisting of two rational defined lines, while the other two must consist of single points. Use this to give an upper bound on the number of points on the projective plane in terms of adding points on fibers and get a contradiction.

Remark 1 : This is a somewhat special proof of a special case of a theorem due to Chevalley, saying that any form of degree n in $n+1$ homogenous variables has a non-trivial zero.

4 Show that every irreducible singular plane cubic can be parametrized by binary cubics. In particular show that a nodal cubic over a finite field with q elements has either q or $q + 2$ elements, and explain the difference.

euler numbers

5 There if a $P^5(\mathbb{C})$ of conics. Inside it we have a hypersurface of singular conics compute its euler-number as well as that of the complement

Hint : A singular conic consists of two lines. Show that the euler number is hence given by that of the quotient of $P^2 \times P^2$ given by interchanging the two lines.

6 The same set-up as in the previous, but now with $\mathbb{R}, \mathbb{F}_q$ instead of the complex numbers. Note that in the case of a finite field, the euler-number is simply the number of points.

7 Compute the euler-number of the 8-dimensional hypersurface in P^9 of singular cubics.

Hint :: Show that this variety is the union of 6-dimensional planes parametrized by P^2. (Show that the linear space of cubics singular at a given point has co-dimension three.) The complication arises because those planes are not disjoint. A cubic consisting of a non-singular conic and a line not tangent to it, lies in two planes, the intersection of three distinct lines not going through a point, lies in three planes, and finally cubics containing a double line lies in a whole P^1 of planes. By 'blowing up' in those strata you obtain a space which is indeed a disjoint union of 6-planes.

Singularities of plane curves

8 Find the singular points (if any) of the cubic

$$xy(x+y) + z(x-y)^2 = 0$$

and determine their types.

9 Show that the quartic curve

$$y^2z^2 + x^2y^2 + z^2x^2 + \lambda xyz(x+y+z) = 0$$

has three double points. Give a condition on λ to ensure that those double points are cusps.

Hint :Note that the quartic is invariant under permutation of the variables which constitutes a group of six elements. As a quartic cannot have six singularities unless it contains a multiple component (exclude that case) the singular points need to be of a special form visavi the action.

10 Give a list of all irreducible plane curves of degree n and given by two monomials. Show in particular that if $n > 2$ they are necessarily singular and describe their singularities.

11 Give conditions on $f_3(x,y)$ and $f_4(x,y)$ such that the curve given by

$$x^2 + f_3 + f_4 + \dots$$

has a tacnode (two tangent branches -a_3) or an a_4 singularity. In particular write down irreducible quartics with such singularities.

12 Find an irreducible quartic with a triple point, and show that it is necessarily rational. By that is meant that one can parametrize it by binary quartics. For your quartic give such a parametrization.

13 Consider a circle with radius 1 and let a circle with radius 1/3 roll inside it. Give an equation for the curve traced out by a point on the smaller circle and find its singularities. Such a curve is called a hypocycloid.

Canonical divisors and genera of curves

14 Find the canonical divisor of $P^1 \times P^1$ in terms of the generators of the two rulings. And in particular find the genus of a smooth curve of bidegree (m, n).

15 Find the genus of the smooth curve given by a quadric and a cubic intersecting transversally.

Hint : Find its bi-degree as a curve on a quadric ($P^1 \times P^1$). Or consider the hint to a problem below

16 Given a generic pencil of quadrics in P^3 find the genus of the base curve!

17 Find the genus of a smooth curve given as the intersection of a hypersurfaces of degree n and m respectively in P^3

Hint :Use the adjunction formula twice after having found the canonical divisor on P^3. First on one of the hypersurfaces and then on the intersection curve.

18 Find the canonical divisor of $P^1 \times P^1 \times P^1$ and compute the genus of the intersection of two hypersurfaces of tri-degree (n_1, n_2, n_3) and (m_1, m_2, m_3).

19 Let Q be a smooth quartic in P^3 containing a line L

a) compute L^2

b) Compute the degree, the genus and the self-intersection of $H - L$ and interpret it geometrically.

c) Compute the intersections of $(H - 2L)H$ and $(H - 2L)L$ and show that both are positive, show nevertheless that the linear system $H - 2L$ does not have any effective curves.

Hint :Compute the canonical divisor of a smooth quartic. As to c) find some effective curve D such that $(H - 2L)D < 0$ or simply compute the degree of $H - 2L$.

20 Two quartics intersect transversally, except at three points, where they have common tangent-planes giving nodes. What is the genus of the smooth resolution of the intersection?

21 Consider two cubics that contain the line L and intersect in $C \cup L$ where C is smooth. Compute the genus of C!

Hint :To compute the genus and self-intersection of $3H = L + C$ is easy (see exercise 17). Furthermore by adjunction KL is computable as we know the genus of L and this gives us KC. If we knew LC we would be done. But we know HL and L^2 and thus we get C^2.