Problems November 12

Projective linear transformations

1 Let V be a vector space and Φ a linear automorphism. This induces a map $P\Phi$ on the projective space $P(V)$ which will be referred to as a projective linear automorphism

a) Show that $P\Phi = P\Phi'$ iff $\Phi = \lambda\Phi'$ for some $\lambda \neq 0$.

b) Show that the projective linear automorhisms form a group $PGL(V)$ isomorphic to $GL(V)/D$ where D is the subgroup of scalar matrices λI where I is the identity. (Show in fact that D is the center of $GL(V)$)

c) Show that the automorphisms of the Riemann sphere $\mathbb{C}P^1$ form a 3-dimensional group isomorphic with the well-known group of Möbius transformations.

d) If $k = \mathbb{F}_2$ show that the Möbius transformations in this case become S_3 acting triply transitively on the space $\mathbb{F}_2P^1$.

e) Show that if $q = p^n$ is odd that $PSL(2, \mathbb{F}_q)$ is a non-trivial normal subgroup of $PGL(2, \mathbb{F}_q)$ and act as permutations on $\mathbb{F}_qP^1$. What kind of permutations?

2 Let C be a conic in P^2.

a) Show that $\Phi(C) := C(\phi)$ defines an action of the 8-dimensional group of projective transformations of P^2 and that it has three orbits (in the complex case) and identify them.

b) Describe the stabilizers of each type of orbits, and in particular show that the stabilizer to a non-singular conic is isomorphic to the Möbius transformations.

c) The action of $PGL(3, \mathbb{C})$ can be extended to pairs of conics. Show that there are bound to be infinitely many orbits.

d) Given a generic pencil spanned by the conics C, D. If we normalize C to $x^2 + y^2 + z^2 = 0$ and three base points to $(-1, 0, i), (0, 1, i), (1, 0, i)$ (the images of $), 1, \infty$ under the parametrization $(s, t) \mapsto (t^2 - s^2, 2st, i(t^2 + s^2))$ what are the possible choices for D?

e) Are any two generic pencils projectively equivalent? What is the stabilizer of a pencil?

Hessians

3 The cubic $x^3 + y^3 + z^3 - 3xyz$ coincides with its Hessian. What is going on?

4 Recall that two plane curves $F = 0, G = 0$ are said to be projectively equivalent iff there is a projective linear map Φ such that $F = \Phi^*(G) = G(\Phi)$. If two curves are projectively equivalent, is the same true for their Hessians?

5 Is the Hessian of the Hessian of a Cubic the original cubic?

Nets of Conics

6 Let N be a net of conics with a base-point p.

a) Show that there is a singular conic in the net with its singular point at p and that this conic corresponds to the singular point of the discriminant cubic.

b) Show that if this singular conic consists of two distinct lines then the singularity of the discriminant cubic is a node (wit two tangent directions).

c) Is it possible to find a net whose discriminant cubic is a cuspidal cubic?

Dual Curves

7 Consider the cuspidal cubic (C) given by $xy^2 - z^3$ parametrized by P^1 via $(s,t) \mapsto (s^3, t^3, st^2$.

a) For each point $p = (\lambda_0, \lambda_1,]ambda_2)$ find the intersection of the polar with the curve and show that of the six intersection points three are absorbed by the cusp.

b) How should p be positioned visavi C for the polar to have four intersection points with the cusp?

c) Find the region in the real case, through which we can find three tangents go through each of its points.

d) Give the equation for the dual curve C^*.