**Inlämningsuppgifter**

Differential Calculus

5/9 2002

due 10/9 2002

*starred exercises are meant for the curious*

**1** Let $g(x)$ be a function on the unit circle ($|x| = 1$) satisfying
i) $g(0,1) = g(1,0) = 0$
ii) $g(-x) = -g(x)$
a) Give an example of such a function
Define a function $f(x)$ on the whole plane as follows

$$f(x) = |x|g(\frac{x}{|x|}) \quad x \neq 0$$
$$f(0,0) = 0$$

b) Show that $f$ is continous at $(0,0)$ iff $g$ is bounded
c) Show that $f$ is continous everywhere iff $g$ is continous.
d) Show that $f$ has directional derivatives for every direction, regardless of $g$. (Thus in particular even if $f$ is not even continous!)
e) Show that $f$ is not differerentiable at $(0,0)$ unless $g$ is identically equal to zero!

**2** Find the derivatives of the following functions
a) $f(x,y,z) = x^y$
b) $f(x,y,z) = x^{y^z}$
c) $f(x,y) = (\sin(xy), sin(x\sin(y)), x^y)$
d) $f(x,y) = \int_a^{xy} g(t)dt$

**3** Find the partial derivatives of $f$ in terms of the derivatives of $g$ and $h$ of the following
a) $f(x,y) = g(x + h(y))$
b) $f(x,y) = g(x)h(y)$

**4** Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the co-ordinate axis and use that to compare the two mixed partials $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$

**5** A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be homogenous of degree $m$ if $f(tx) = t^m f(x)$. Show that a differentiable homogenous funtion of degree $m$ satisfies the Euler identity

$$\sum_{i+1}^{n} x_i \frac{\partial f}{\partial x_i} = mf(x)$$

**6** Let $F(x, y) = (x + y, xy)$ Compute its derivative at each point $(x, y)$ and express it as a map $DF : \mathbb{R}^2 \to \mathbb{R}^4$ then compute $D^2F$

**7** Let $A, B$ be $n \times n$ matrices. Show that the map $(A, B) \mapsto AB$ is a differentiable map from $R^{2n^2}$ to $R^{n^2}$ and compute its matrix!

**8** Let $A, B$ be skew-symmetric $3 \times 3$ matrices, (i.e. $A^T = -A$) show that $(A, B) \mapsto AB - BA$ maps pairs of skew-symmetric matrics to skewsymmetric matrices and can be considered as a map from $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ and is differentiable. Compute the derivative!

**9** * If you are familiar with the quaternions $\mathbb{H} \cong \mathbb{R}^4$ note that the left multiplications $x \mapsto ax$ form a 4-dimensional subspace $L \cong \mathbb{H}$ of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{H})$ (this is simply the space of all $4 \times 4$ matrices). Say that a differentiable map $f : \mathbb{H} \to \mathbb{H}$ is left-quaternionic iff $Df \in L$.

a) Show that the sum of two left-quaternionic maps is left- quaternionic.

b) What about the product?