Inlämningsuppgifter

Differential Calculus

10/9 2003

due 17/9 2003

starred exercises are meant for the curious

1 Use the implicit function thereom to show that if the gradient of $F(x_1, \dots x_n)$ is non-zero for all points on the level set $F = 0$ the latter can be described as a manifold. Try and generalize this to the intersections of several functions.

2 Let K be a closed subset of $\mathbb{R}^n$ and define the distance $d_K(x)$ of $x \in \mathbb{R}^n$ to K as

$$d_K(x) = \inf_{y \in K} |x - y|$$

a) Show that d_K is a continous function and that the levelset $\{x : d_K(x) = 0\} = K$

b) Let K be a circle or a square respectively, what can you say about the differentiability of d_K respectively?

3 * Assume that for each open disc $D_{a,r} = \{x : |x - a| < r\}$ we can find a differentiable function $\phi_{a,r}$ such that

$$\phi_{a,r}(x) \begin{cases} > 0 & |x - a| < r \\ 0 & |x - r| \geq r \end{cases}$$

Show that for each closed set K we can find a differentiable function Φ whose zeroset is exactly K.

4 Let Λ be the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ (i.e. all points with integral co-ordinates), and let Λ_N denote $\frac{1}{N}\Lambda$ (i.e. all points with rational co-ordinates with denominator N).

a) Show that for an integrable function f we have

$$\int f = \lim_{N \to \infty} \frac{1}{N^n} \sum_{x \in \Lambda_N} f(x)$$

and give examples of when the right-hand side has a limit, although the left-hand side is not defined!

b) Define $f_N(x) := f(x/N)$ and reformulate the expression in a) using Λ instead of Λ_N. If f is a characteristic equation for a set S show that f_N is the characteristic equation of a set S_N. Describe that set in terms of S.

c)* Indicate why the number of lattice points inside a circle gives a good relative approximation of the area of that circle provided the radius is big. In particular show that

$$\lim_{R \to \infty} \frac{1}{R^2} \#\{(m, n) \in \mathbb{Z}^2 : m^2 + n^2 < R\} = \pi$$

and try to get an estimate of the error in terms of R !

5 Define $F(x) = \int_{-1}^{x} \phi(t)dt$ and try to compute $F'(0)$ in the following cases.
a)

$$\phi(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

b)

$$\phi(t) = \begin{cases} 1 & t \leq 0 \\ \sin \frac{1}{t} & t > 0 \end{cases}$$

6 Show that the volume $V(S^n, r)$ of an n-dimensional sphere S^n of radius r will be given by a formula $V(S^n, r) = \pi_n r^n$. By using Fubini set up a recursive formula for the coefficient π_n, and in particular compute the volume of the 3-dimensional unit sphere S^3.

7 Let $f \geq 0$ be an integrable function such that $\int f = 0$. Show that the set $A = \{x : f(x) > 0\}$ has measure zero.

8 Let $\Gamma(f)$ be the graph of an integrable function (i.e. the set $\{(x, f(x))\}$. Show that $\Gamma(f)$ has measure zero. On the other hand give an example of a differentiable function on $[0, 1]$ whose graph has infinity length.

9 Let $\phi_1(x) = \frac{1}{3}x$ and $\phi_2(x) = \frac{1}{3}x + \frac{2}{3}$ and define inductively

$$K_{n+1} = \phi_1(K_n) \cup \phi_2(K_n)$$

starting with $K_0 = [0, 1]$.

a) Show that each K_n is compact, $K_{n+1} \subset K_n$ and that the length of K_{n+1} is two-thirds of the length of K_n.

b) Show that $C = \cap_n K_n$ (the famous Cantorset) is a compact, non-empty (in fact uncountable) compact subset of $[0, 1]$ of measure zero.

Let $\phi_1, \phi_2 : \mathbb{R}^2 \to \mathbb{R}^2$ as follows

$$\phi_1(x, y) = (\frac{1}{3}x, \frac{1}{2}y)$$
$$\phi_2(x, y) = (-\frac{1}{3}x + 1, 1 - \frac{1}{2}y)$$

Set

$$I_0 = \{(x, y) : \frac{1}{3} \leq x \leq \frac{2}{3}, y = \frac{1}{2}\}$$

and recursively

$$I_{n+1} = \phi_1(I_n) \cup \phi_2(I_n)$$

c) Show that $\overline{\cup_n I_n}$ defines the graph of a non-decreasing function $F : [0, 1] \to [0, 1]$ ((*Try to draw a picture!*)

d) Show that F is continous and that it is furthermore differentiable outside the Cantorset with derivative identically equal to zero!

e) Show that if $y \neq p/2^n$ for some integers p, n then $F^{-1}(y)$ consists of just one point, otherwise it consists of a closed interval. Define $G(y)$ to be the length of $F^{-1}(y)$, and determine where it is continous.

f) Compute the integrals $\int_{[0,1]} F$ and $\int_{[0,1]} G$!

g) * Consider the restriction of F to the Cantorset. What is its image? Is it 1-1?

Note: The function F is known as the Devils staircase.