Inlämningsuppgifter

Differential Calculus

19/9 2003

due 26/9 2003

starred exercises are meant for the curious

1 Consider the maps $P(r,\theta) = (r\cos\theta, r\sin\theta)$ and

$$S(r,\theta,\psi) = (r\cos\theta\cos\psi, r\sin\theta\cos\psi, r\sin\psi)$$

and compute their Jacobians. Those are refered to as polar and spherical coordinates respectively. Can you generalize this to four dimensions and compute the associated Jacobian?

2 Use the previous exercise, or by other means, determine for what n respectively the mutiple integrals

$$\int_1^\infty \int_1^\infty (x^2+y^2)^n dxdy$$

and

$$\int_1^\infty \int_1^\infty \int_1^\infty (x^2+y^2+z^2)^n dxdydz$$

exist

Can you generalize to higher dimensions?

3 Define the function $F(x) = \int_x^1 \sin t^2 dt$ and compute the integral $\int_0^1 F(x)dx$. *Hint: Use Fubini!*

4 The series $\sum_{n=1}^\infty (-1)^n \frac{1}{n}$ is conditionally convergent. Write down an explicit rearrangement of the ordering of the sum so that the series diverges!

5 Try to give an example of a doubly indexed series $a_{n.m}$ such that the double sum $\sum_n^\infty(\sum_m^\infty a_{n,m})$ converges (i.e. each series $\sum_m^\infty a_{n.m}$ converges to b_n and $\sum_n^\infty b_n$ converges as well.) but that the double sum $\sum_m^\infty(\sum_n^\infty a_{n,m})$ does not. If you are more ambitious, try to find an example where both order of summations yield convergent sums, but the sums differ!

Can you use those examples to give examples where

$$\int_0^\infty (\int_0^\infty f(x,y)dx)dy \neq \int_0^\infty (\int_0^\infty f(x,y)dy)dx$$

where we define $\int_0^\infty f(t)dt = \lim_{N\to\infty} \int_{\frac{1}{N}}^N f(t)dt$ conditionally (i.e. we do not require that $\lim_{N\to\infty} \int_{\frac{1}{N}}^N |f(t)|dt$ exists.)

6 * Show that the set of all numbers which do not have a seven in their decimal expansions form a set of measure zero.

7 * Consider rational numbers $r_{q,n}$ of form $\frac{q}{2^n}$ with q odd in the unit interval $[0,1]$. To each such number we associate the open interval

$$I_{q,n} = (r_{q,n} - 2^{-3n}, r_{q,n} + 2^{-3n})$$

Let $U = \bigcup_{q,n} I_{q.n}$.

a) Show that U is a dense open subset of the unit interval $[0,1]$

b) Try to describe the connected component, i.e. the largest open interval containing $\frac{1}{2}$ and contained in U. Can you describe its endpoints and show that those are not rational?

c) More generally let $J_{q,n}$ denote the largest interval containing $R_{q.n}$. Give an estimate of its length.

d) Can you determine for reasonably small N the numbers q such that $r_{q,N} \notin J_{q,n}$ for $n < N$? ($N = 3,4,5$ can be considered reasonably small, if you are ambitious try larger N!)

e) Does $1/\sqrt{2}$ belong to U?

f) Can you convince yourself that the boundary of U has positive measure?