Inlämningsuppgifter

Differential Calculus

26/9 2003

due 3/10 2003

starred exercises are meant for the curious

1 Let V be a 3-dimensional vectorspace with a volume form Ω. Show that each $\omega \in \bigwedge^2$ defines a vector $v_\omega \in V$ by

$$\omega \wedge \psi = \psi(v_\omega)\Omega$$

for every element $\psi \in \bigwedge^2(V) \equiv V^*$

2 Let V be as before and let $\psi \in \bigwedge^2(V)$ be defined as

$$\psi(v_1, v_2) = \det \begin{vmatrix} v_1 \\ v_2 \\ a \end{vmatrix}$$

and $\theta \in \bigwedge^1(V)$ be defined by $\theta(v) =< b, v >$ compute $\psi \wedge \theta$!

3 Define the crossproduct by

$$< v \times u, z >= \det \begin{vmatrix} v \\ u \\ z \end{vmatrix}$$

Show the standard identities involving the cross-product and the inner-product. In particular derive

a) $|v \times u|^2 =< v, v >< u, v > - < v, u >^2$

b) $v \times (u \times w) =< v, w > u- < v, u > w$

4 Let $Av = a \times v, Bv = b \times v$ be linear maps from $\mathbb{R}^3$ to $\mathbb{R}^3$.

a) Show that A, B are represented by skew-symmetric matrices, i.e. matrices such that $A^T = -A$

b) Show that the skew-symmetric matrices are in fact parametrised by a 3-dimensional vectorspace and set up an isomorphism with $\mathbb{R}^3$

c) Show that the commutator $[A, B] = AB - BA$ is a skew-symmetric matrix and describe it in terms of the vectors a, b.

d)* Show that in addition to $a \times b = -b \times a$ we have the following strange identity

$$(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0$$

refered to as the Jacobi identity. Thus the cross-product (of which you may familiar with from high-school), in facts turns $\mathbb{R}^3$ into an interesting Lie-algebra, (something you were never told in school).

5 Let V be a 4-dimensional vectorspace with a volume form Ω. Show that for $\omega\eta \in \bigwedge^2(V)$ we can define a bi-linear form B on that vectorspace by

$$\omega \wedge \eta = B(\omega, \eta)\Omega$$

Furthermore show a) B is symmetric and non-degenerate (i.e. if $B(\omega, \eta) = 0 \quad \forall \eta$ then $\wedge = 0$)

b) If ω is decomposable, i.e. it can be written as $\psi_1 \wedge \psi_2$ then $\omega \wedge \omega = 0$. Is the converse true? In particular show that B is not positive definite.

c) Try to determine the kernel of the linear map $x \mapsto \omega \wedge x$ at least for ω decomposable. Can the kernel ever consist of only zero?

d) Show that every element ω can be written as a sum of three decomposable elements. Can it always be written as the sum of two?

6 Let $F(x, y)$ be a function, and consider the vector field $V(F) = (-\frac{\partial F}{\partial y}, \frac{\partial F}{\partial x})$.

a) Determine the integral curves of that field. Can they ever intersect.

b) Determine whether $V(F)$ can ever be a gradient vector-field of some function G, and if so show that V(G) has the same property. Do you recognise those particular functions? Whatever, show that they can never attain maxima and minima.

c) Give explicit examples of such pairs F, G.

Hint: Show that you can find examples where F, G are quadratic polynomials

7 Show that the integral curves of two orthogonal vector-fields are orthogonal to each other. Give non-trivial examples of orthogonal vector-fields.

8 Show that the tangentspace to $\mathbb{R}^2$ can naturally be identified with the space of all linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ or equivalently with all 2×2 matrices. Describe a vector-field on the tangentspace by pairs (A, A). Can you describe the integral curves.