Inlämningsuppgifter

Differential Calculus

6/10 2003

due 13/10 2003

starred exercises are meant for the curious

1 Decide which of the following forms are closed and which are exact. In order to claim exactness you have to find an explicit form whose differential is the given.
a) dx_1 b) $dx \wedge dy$ c) xdx d) ydx

2 A cube consists of six squares, twelve edges and eight vertices. I.e. I^3 would split up in $6I^2, 12I^1, 8I^0$. Can you generalize this to four-dimensions? In fact can you write a formula for any I^k?

It might be helpful to note that the vertices of a $k-cube$ consist of all arrays of k numbers $(a_1, a_2 \dots a_k)$ each being either zero or one. When are two such vertices joined by an edge? Which vertices make up a square and so on.

Finally compute the volume of a k-cube, whatever that is meant, and the length of the longest diagonal.

* In order to challenge your powers of visualization. What figure do you get when you cut a cube with a plane through its center and orthogonal to one of the long diagonals through it. What will happen when you move the plane in the direction of its orthogonal diagonal. Try to do the same for a 4-dimensional cube.

3 Let U be an open set with the property that each closed form is exact and let f be a diffeomorphism. Show that $f(U)$ has the same property. Give examples of such open sets which are not starshaped.

4 Show that being closed is a local property. In other words if the restriction of ω to an open set is closed, this property remains, whatever the values of ω outside the open set. Thus a form is closed if and only if its restriction to each open set is closed. Show that this does not hold for the property of being exact.

5 The tangent space $T(S^1)$ of the circle S^1 can be described as the subset of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ consisting of vectors (v, w) such that

$$\text{i)} < v, v > = 1$$
$$\text{ii)} < v, w > = 0$$

a) Show that $T(S^1)$ is two-dimensional and that there is a natural projection $\pi : T(S^1) \to S^1$ with fibers equal to $\mathbb{R}$ (the tangents space at the corresponding point).

b) Try to show that in effect $T(S^1) \equiv S^1 \times \mathbb{R}$

c) Define the 1-form ω on S^1 by $\omega_v(w) = (v, w)$ and show that ω is closed.

d) Given the map $E : \mathbb{R} \to S^1$ defined by $E(t) = (\cos t, \sin t)$. Compute $E^*(\omega)$ in terms of dt and conclude that the pullback of a form can be exact without the form being exact.

e) The form ω is usually refered to as $d\theta$. Why is that?

f) Try to extend the form $d\theta$ to the whole punctured plane so that it becomes closed. Why can it not be exact? Conclude that you cannot extend it as a closed form to the entire plane.

6 Show that if a differentiable form is defined on the entire plane and closed except possibly at an isolated point (say the origin) then it is closed everywhere.

7 * Consider the group $SO(3)$ of orthogonal 3×3 matrices of determinant one. The tangent plane at the identity can be naturally identified with the 3×3 skew-symmetric matrices. Let L be a linear form on those (e.g. associating the entry a_{13} (Why is the trace not an interesting linear form in this particular case?)). Define a 1-form on the whole space by defining $L(g) = g^*L$ for every element $g \in SO(3)$. Try to compute the pull-back T^*L on the vector space of skew-symmetric matrices (isomorphic to $\mathbb{R}^3$) under the map $T(A) = (I + A)(I - A)^{-1}$