Summing inverse squares by euclidean geometry

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We give a simple proof of a generalization of Euler's famous identity[1]

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}. \tag{1}$$

First notice that an equivalent form of (1) is

$$1 + \frac{1}{9} + \frac{1}{25} + \cdots = \frac{\pi^2}{8}. \tag{2}$$

The reason for this is that the terms occurring in (1) but not in (2) are $1/4 + 1/16 + \ldots$, which are $1/4$ times the terms in (1). Therefore the left hand-side of (2) is $3/4$ times the left hand-side of (1), and $(3/4)\cdot\pi^2/6 = \pi^2/8$. Equation (2) in turn is equivalent to

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{4}. \tag{3}$$

[1] The essence of the proof comes from [2] and the simplified version [1]. What is new here is the presentation that allows us to replace the trigonometry of [1] and [2] by euclidean geometry. From a strictly mathematical point of view it is not clear whether this is an improvement over the very brief proof in [1], but it may help in visualizing what is going on. In any case the fact that the famous identity (1) is not *that* hard to prove is something that deserves to be better known.

In (3) we sum the inverse squares of all odd integers including the negative ones. Since the inverse square of a negative number is equal to the inverse square of the corresponding positive number, (3) is twice (2). Finally we can simplify (3) by multiplying each term by 4, obtaining

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-1/2)^2} = \pi^2. \tag{4}$$

Equation (4) is thus equivalent to Euler's identity (1). We are going to prove a generalization of (4) consisting in replacing the number 1/2 by an arbitrary real number x, with the only restriction that x cannot be an integer since that would lead to a division by zero in one of the terms.

Theorem 1. *If x is a real number which is not an integer, then*

$$\sum_{n=-\infty}^{+\infty} \frac{1}{(n-x)^2} = \left(\frac{\pi}{\sin \pi x}\right)^2. \tag{5}$$

Since $\sin(\pi/2) = 1$, (5) specializes to (4) if $x = 1/2$. Let us first briefly discuss how one might prove this identity.

Clearly there is no way of starting from the left hand-side of (5), or any of equations (1)–(4), and arriving through purely algebraic operations at the right hand-side. This is what makes these identities amazing. Proofs involve computing something in two different ways, finding it to be equal to the left hand-side through one calculation, and to the right hand-side through another. Finding what to compute may require some creativity and imagination, but the equations give some clues. Apart from inverse squares they involve integers and the number π. To get all ingredients into the mix we should compute something that involves inverse squares in a model where some objects are counted and where there is a circle.

Inverse squares occur in physics, and we are going to exploit this by describing a problem in terms of a physical system. The apparent brightness of a star is proportional to the inverse square of its distance. Consider a system of N stars of equal luminosity uniformly spaced in a common circular orbit around their center of gravity as in Figure 1. We investigate the total amount of light received at an arbitrary point P in the orbit. First we choose units conveniently.

As unit of distance we choose the distance between neighboring stars measured along the circle. Thus the perimeter of the circle is N and its diameter N/π. The amount of light received at any point is the sum of the inverse squares of the distances to the stars (along straight lines). We let x define the displacement of the point P relative to the stars in such a way that P is at distance x to one of the stars, again measured along the circle. We may assume without loss of generality that we measure the displacement relative to the closest star so that $|x| \leq 1/2$. We define $f_N(x)$ to be the amount of light received at a point of displacement x in a system of N stars, and we would like to determine $f_N(x)$ for any N and x.

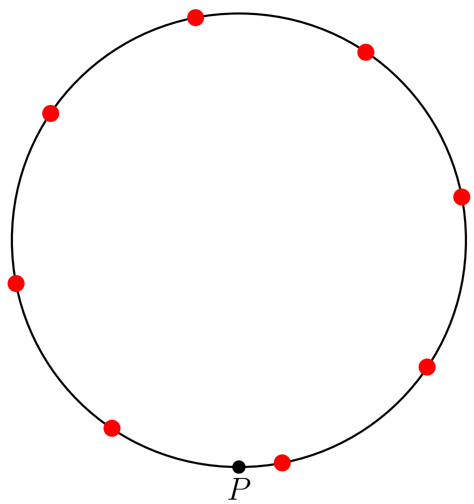

Figure 1: A circular system of N stars. Here $N = 8$. The point P is an arbitrary point on the circle.

Exact solution

Our first approach is to use euclidean geometry in order to calculate $f_N(x)$ explicitly. As it turns out, the method works only when N is a power of 2, but this will suffice. First consider the case $N = 1$. The unit of distance is the distance of the star to itself along the circle, in other words the circumference of the circle.

Lemma 2.

$$f_1(x) = \left(\frac{\pi}{\sin \pi x}\right)^2.$$

Proof. The circumference of the circle is 1 and consequently the radius is $1/(2\pi)$. The displacement x is the distance from P to the star along the circle, and it follows that the *angular* displacement α is $x/r = 2\pi x$, see Figure 2 (left). We find as in Figure 2 (right) that the distance from P to the star along a straight line is

$$2r \sin\frac{\alpha}{2} = \frac{\sin \pi x}{\pi},$$

and consequently

$$f_1(x) = \left(\frac{\pi}{\sin \pi x}\right)^2.$$

□

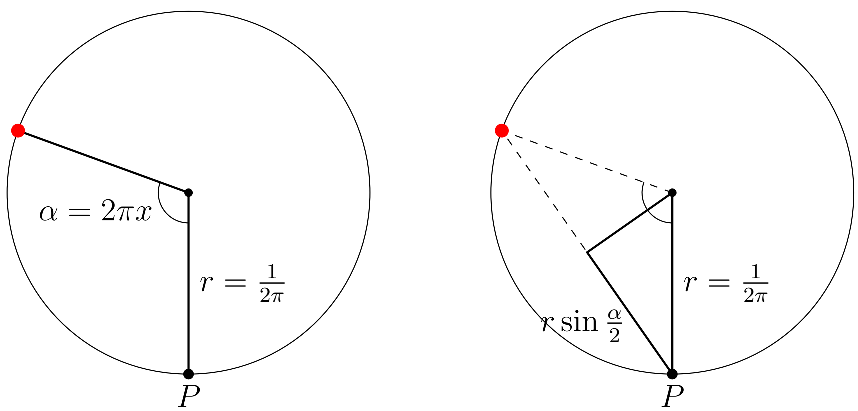

Figure 2: The case $N = 1$.

In the following we repeatedly apply a theorem which has been called the *Inverse Pythagorean Theorem*[2] since it relates three inverse squares in a right triangle, rather than three squares. We phrase the theorem in terms of light received from stars.

Proposition 3. *If two stars are located at points A and B, and the point C is such that the angle at C in the triangle ABC is right, then at the point C, the light received from the stars at A and B together is equal to the light that*

[2]Not to be confused with the *converse* of the Pythagorean theorem, which states that whenever the three sides of a triangle satisfy $a^2 + b^2 = c^2$, the angle opposite c is right.

would be received from a star at the point X of projection of C onto the line through A and B.

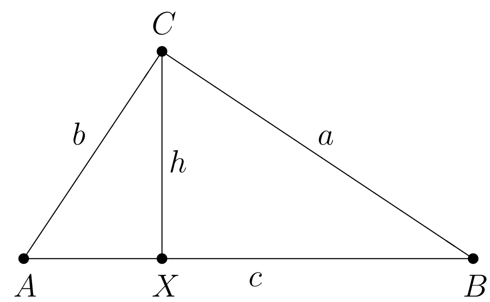

Figure 3: Inverse Pythagorean theorem: $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{h^2}$.

Proof. Let a, b and c be the sides of the triangle ABC, and let h be the distance from C to X. The area of the triangle ABC can be found either as $ab/2$ or as $ch/2$. Therefore

$$ab = ch.$$

Moreover, by the Pythagorean theorem,

$$a^2 + b^2 = c^2.$$

The amount of light received at C from A and B together is therefore equal to

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{a^2 + b^2}{a^2 b^2} = \frac{c^2}{(ch)^2} = \frac{1}{h^2},$$

which is the amount of light received at C from a star at X. □

Returning to the circular model of stars, we apply Proposition 3 to prove that for fixed x, the amount of light received at P in the 2-star model is the same as in the 1-star model!

Lemma 4.

$$f_1(x) = f_2(x).$$

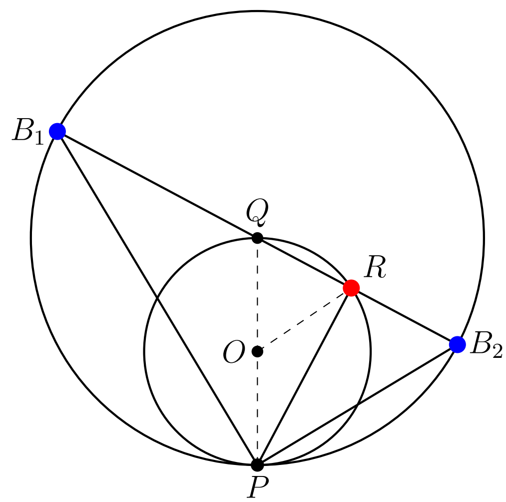

Figure 4: Replacing one star by two.

Proof. Let the 1-star model with displacement x be the red star R on the smaller circle in Figure 4, and let Q be the point opposite to P on that circle. Draw a new circle with center Q and going through P. On the new circle, and on the diameter going through R, put two blue stars B_1 and B_2.

Obviously the new circle has perimeter 2, and thus the distance between the blue stars along that circle is 1. Moreover, the angular displacement from P to B_2 in the blue system is equal to the angle $PQB_2 = PQR$ which in turn, by the central angle theorem, is half of the angular displacement POR in the smaller circle. Therefore the displacement in the blue system is the same as in the red system, and $f_2(x)$ is equal the total amount of light received from the blue stars at P.

Since PQ is a diameter in the smaller circle, the angle PRQ is right, and similarly since B_1B_2 is a diameter in the larger circle, the angle B_1PB_2 is right. Therefore Proposition 3 applies, showing that at P, the light received from the blue stars together is equal to the light received from the red star. □

The argument easily generalizes.

Lemma 5. *For all N and x,*

$$f_N(x) = f_{2N}(x).$$

Proof. We replace each star in the N-star model by a pair of opposite stars in the $2N$-star model as in Figure 5. Again the new circle has twice the perimeter of the old one, and therefore the distances between the blue stars along the new circle is 1. By considering only the closest red star, it follows in the same way as in the case $N = 1$ that the displacement in the new model is the same as in the old one. Again by applying Proposition 3, it follows that each pair of opposite blue stars give the same total amount of light at P as the red star they replace. It follows that at P, the blue stars together have the same apparent brightness as the red stars. □

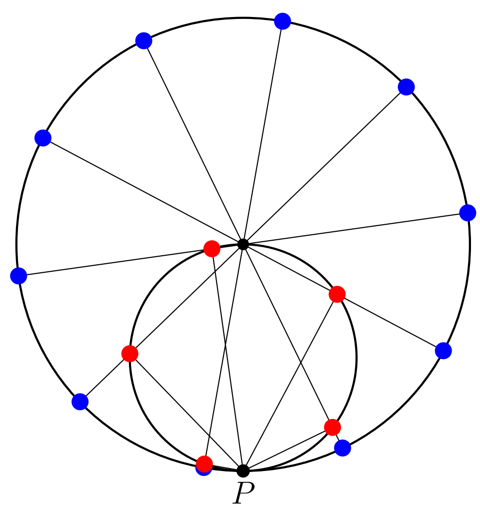

Figure 5: Replacing the red N-star model by the blue $2N$-star model.

By repeatedly applying Lemma 5, starting from $N = 1$, we conclude that $f_N(x)$ is equal to the right hand-side of (5) whenever N is a power of 2:

Proposition 6. *If the number N of stars is a power of 2, then*

$$f_N(x) = \left(\frac{\pi}{\sin \pi x}\right)^2.$$

This raises the question whether the limitation to powers of 2 is an artifact of the proof or if the 2-powers really are special. The answer is that the identity holds for all N, and this will follow from the analysis in the next section.

Asymptotical solution

Having identified $f_N(x)$ with the right hand-side of (5), what we do next is to "estimate" $f_N(x)$ in order to show that it is close to the sum in the left hand-side. It may seem backwards to try to estimate something when we already know the exact value, but remember that we are trying to evaluate the sum in (5).

We start with the $2N$-star system for a fixed x, and proceed with a sequence of operations leading to an approximate value of $f_{2N}(x)$.

The circle of the $2N$-star system has radius $r = N/\pi$. We now delete the N stars on the half of the circle which is most distant from P, continuing the calculation with the remaining N stars. The error in the estimate of $f_{2N}(x)$ occurring from this deletion is at most

$$N \cdot \frac{1}{(N/\pi)^2} = \frac{\pi^2}{N},$$

since we have deleted N stars, each of which was at distance at least equal to the radius N/π (in fact the distance is at least $\sqrt{2}$ times the radius, but we only need a rough bound).

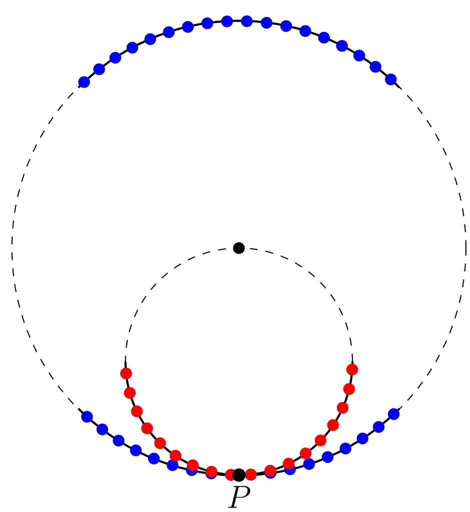

Figure 6: After deleting the N most distant red stars, we replace the remaining ones by blue stars in two sectors.

We now replace each of the remaining N stars by two stars on the circle of twice the radius as in Figure 6. What we then have is part of the $4N$-star system, but with only the N stars in the closest quarter and the N stars in the most distant quarter remaining. Now we repeat the procedure: We discard the N distant stars, this time causing an error bounded by

$$N \cdot \frac{1}{(2N/\pi)^2} = \frac{\pi^2}{4N},$$

since the new radius is $2N/\pi$. Then we replace the remaining N stars by $2N$ stars on a circle twice as large. Continuing this process, we have in each step $2N$ stars from the $2^k(2N)$-star system, the N closest to P and the N most distant from P. The deletion of the N distant stars will cause an error which is bounded by

$$\frac{\pi^2}{4^k N},$$

since the radius of the current circle is $2^k N/\pi$.

As $k \to \infty$, the set of N closest stars will approach the points at coordinates $n - x$ measured from P along a straight line tangent to all the circles at P. Therefore in the limit $k \to \infty$, the total radiation at P from the N remaining stars will approach

$$\sum_{|n-x|<N/2} \frac{1}{(n-x)^2}.$$

The total error from all the deletions of stars will be at most

$$\frac{\pi^2}{N} + \frac{\pi^2}{4N} + \frac{\pi^2}{16N} + \cdots = \frac{\pi^2}{N} \cdot \frac{1}{1-1/4} = \frac{4\pi^2/3}{N}.$$

Since we already know that $f_N(x) = f_{2N}(x)$, we conclude that for every N,

$$f_N(x) = \sum_{|n-x|<N/2} \frac{1}{(n-x)^2} + \frac{\theta}{N}, \tag{6}$$

where θ may depend on N but satisfies $0 \le \theta \le 4\pi^2/3$.

In the final twist we apply equation (6) with N replaced by $2^k N$, although we aready know that $f_{2^k N}(x) = f_N(x)$. We conclude that for every N,

$$f_N(x) = \lim_{k\to\infty} f_{2^k N}(x) = \lim_{k\to\infty} \sum_{|n-x|<2^k N/2} \frac{1}{(n-x)^2} + \frac{\theta}{2^k N} = \sum_{n=-\infty}^{\infty} \frac{1}{(n-x)^2}. \tag{7}$$

This calculation might seem backwards at first, since we start from a known quantity (at least if N is a power of 2), and express it as a limit of something we know to be constant, but the right hand-side is what we wish to evaluate.

Since we know that

$$f_1(x) = \left(\frac{\pi}{\sin \pi x}\right)^2,$$

we establish (5) and thereby Euler's identity by putting $N = 1$ in (7).

Finally, the fact that (7) holds for every N shows that indeed $f_N(x)$ is independent of N. This ties up the loose end left in the previous section, showing that Proposition 6 holds without the assumption that N is a power of 2.

Final remarks

There are quite a few proofs of the identity (1) in the literature, and references can be found in [1]. The idea of proving (1) through exact trigonometric identities goes back at least to Yaglom and Yaglom [2]. These identities can be established from de Moivre's identity and the binomial theorem. The observation that instead they can be proved for powers of 2 by repeated application of addition formulas for trigonometric functions has been made by several mathematicians independently, but seems to have been published first by J. Hofbauer [1], who also remarks that in the end it follows that they must hold for all N. Hofbauer also remarks that the extension to general x can be proved with the same method.

What is new in our presentation is the physical interpretation which allows us to replace all trigonometric identities by classical geometry, and the final twist where we obtain the necessary bounds on $f_N(x)$ without referring to inequalities for trigonometric functions.

References

[1] Hofbauer, Josef, *A Simple Proof of* $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ *and Related Identities*, The American Mathematical Monthly **109** (2002), 196–200.

[2] Yaglom, A. M. and Yaglom, I. M., *An elementary derivation of the formulas of Wallis, Leibnitz and Euler for the number* π (in Russian), Uspechi matematiceskich nauk. (N. S.) **57** (1953) 181–187.