Mean field matching and traveling salesman problems in pseudo-dimension 1

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Abstract

Recent work on optimization problems in random link models has verified several conjectures originating in statistical physics and the replica and cavity methods. In particular the numerical value 2.0415 for the limit length of a traveling salesman tour in a complete graph with uniform $[0, 1]$ edge-lengths has been established.

In this paper we show that the crucial integral equation obtained with the cavity method has a unique solution, and that the limit ground state energy obtained from this solution agrees with the rigorously derived value. Moreover, the method by which we establish uniqueness of the solution turns out to yield a new completely rigorous derivation of the limit.

1 Introduction

1.1 Matching and traveling salesman problems

In [14] and the earlier extended version [13], the minimum matching and traveling salesman problems were studied in the pseudo-dimension d mean field (or random link) model for $d \geq 1$. It was shown that certain predictions of [5, 7, 8, 9, 10] based on the replica method are indeed correct. Here we show that the case $d = 1$ allows stronger and more detailed conclusions, and we clarify the relation to the earlier results in [12].

The simplest random model corresponding to $d = 1$ is the complete graph K_n on n vertices, with independent lengths from uniform distribution on the interval $[0, 1]$ associated to the edges. We consider only this model, although the results, ultimately based on the local tree structure of the relatively short edges, remain valid in a number of similar models.

The minimum matching problem asks for a set of $n/2$ edges of minimum total length under the constraint that each vertex must be incident to exactly one edge. This requires n to be even, but for odd n we may allow one vertex to be left out of the pairing. It is known that the asymptotic behavior of the optimum solution remains the same even if we only require $n/2 - o(n)$ disjoint edges, in other words if we allow a small fraction of vertices to remain unmatched.

The traveling salesman problem (TSP) asks for a tour of minimum total length visiting every vertex exactly once. Since the triangle inequality need not hold, there will in general be shorter walks visiting each vertex and returning to the starting point if the same vertex can be visited several times. If such walks are permitted, one may or may not allow the same edge to be traversed more than once. Clearly there are several possible interpretations of the TSP, but we consider the strictest one in which we ask for a cycle of n edges.

The two problems were studied with the replica and cavity methods in [5, 7, 8, 9, 10], and among the results were predictions for the large n limit of the total length of the solution, or in physical language the ground state energy in the thermodynamical limit. As $n \to \infty$, the length L_n of the optimal solution, which is a random quantity for each n, converges to a non-random limit $L^\star$. One may conjecture on fairly general grounds that $E(L_n) \to L^\star$ and that L_n is "self-averaging" so that $L_n \to L^\star$ in probability. Remarkably, methods of physics allow for precise calculation of the limits

$L^\star_M$ and $L^\star_{TSP}$ for matching and TSP respectively.

We can also define the k-factor problem where we ask for a set of $kn/2$ edges of minimum total length under the constraint that each vertex must be incident to exactly k edges. Clearly $k = 1$ is the matching problem, and the case $k = 2$ is a relaxation of the TSP allowing multiple cycles. A nontrivial result proved by A. Frieze [3], tells us that the large n limits of the length of the 2-factor and of the TSP are the same. Frieze's theorem is of interest as the k-factor problem is polynomially solvable, and thus gives an efficiently computable lower bound on the length of the minimum tour. Moreover it confirms an observation based on simulation in [5] that a similar lower bound obtained by the Lagrangian 1-tree relaxation, a stricter but still efficiently solvable relaxation of the TSP, is asymptotically tight. In principle the results presented here can be generalized to the k-factor problem for generic k, but the calculations become less explicit.

1.2 The replica and cavity results

We briefly recall some of the results of [5, 7]. Both problems lead to certain integral equations for the so-called *order parameter function*. Here we consider only the case $r = 0$ (in the notation of [5, 7]), corresponding to $d = 1$ in [13, 14]. For the matching problem the equation is

$$G(x) = \int_{-x}^{\infty} e^{-G(y)}\, dy, \tag{1}$$

and the ground state energy is given by

$$L^\star_M = \frac{1}{2}\int_{-\infty}^{+\infty} G(x)e^{-G(x)}\, dx. \tag{2}$$

For the TSP the equations take a similar form. The order parameter function G has to satisfy

$$G(x) = \int_{-x}^{\infty} (1 + G(y))e^{-G(y)}\, dy, \tag{3}$$

and the ground state energy is

$$L^\star_{TSP} = \frac{1}{2}\int_{-\infty}^{+\infty} G(x)(1 + G(x))e^{-G(x)}\, dx. \tag{4}$$

The equation (1) corresponding to minimum matching has the explicit solution $G(x) = \log(1 + e^x)$, and the ground state energy is $L_M^\star = \pi^2/12$. There does not seem to be an explicit solution to the analogous equation (3) for the TSP, but in [5] a numerical solution led to $L_{TSP}^\star \approx 2.0415$, even though there was no proof that (3) has a solution or that such a solution must be unique.

1.3 Rigorous results

The $\pi^2/12$-limit for matching was established rigorously by David Aldous in 2001 [1, 2]. The method was related to the physics approach, and used the solution to (1). A similar approach to the TSP was indicated in [2], but the main obstacle at the time seems to have been that (3) was not known to have a solution.

In [12] the limit $L_{TSP}^\star$ of the TSP was determined with a different method. The result (conjectured already in the technical report [11]) was

$$L_{TSP}^\star = \frac{1}{2}\int_0^\infty y\,dx, \tag{5}$$

where y as a function of x is defined by $y > 0$ and

$$\left(1 + \frac{x}{2}\right)e^{-x} + \left(1 + \frac{y}{2}\right)e^{-y} = 1. \tag{6}$$

This led to the question whether the numbers given by (4) and (5) are equal, and to the hope that a solution to (3) could somehow be reverse-engineered from (6). Here we answer these questions in the affirmative.

2 Agreement on the TSP

The first new result of this paper is a proof that equation (3) has a unique solution, and that the characterization of $L_{TSP}^\star$ by (4) agrees with (5).

Proposition 2.1. *The integral equation* (3) *has a unique solution.*

Proof. We introduce the auxiliary function T given by $T(g) = (1 + g)e^{-g}$. It follows from (3) that

$$\frac{d}{dx}G(x) = T(G(-x)), \tag{7}$$

and similarly

$$\frac{d}{dx}G(-x) = -T(G(x)).$$

Hence

$$G'(x)T(G(x)) = G'(x)G'(-x) = G'(-x)T(G(-x)). \tag{8}$$

Now let W be the primitive to T for which $W(0) = 0$, or explicitly,

$$W(g) = 2 - 2e^{-g} - ge^{-g}.$$

Then by (8),

$$\frac{d}{dx}W(G(x)) + \frac{d}{dx}W(G(-x)) = 0.$$

Therefore $W(G(x)) + W(G(-x))$ is a constant, which has to be 2 by the boundary conditions. After simplification, the equation is

$$(2 + G(x))\, e^{-G(x)} + (2 + G(-x))\, e^{-G(-x)} = 2. \tag{9}$$

At this point the similarity to (6) becomes apparent. If we let Λ be the function that maps $x > 0$ to the positive solution y to (6), then (9) says that $G(-x) = \Lambda(G(x))$. In particular, $G(0) \approx 1.146$ is the unique positive solution to the equation

$$(2 + G(0))e^{-G(0)} = 1.$$

Replacing $G(-x)$ by $\Lambda(G(x))$ in (7), we obtain

$$G'(x) = T(\Lambda(G(x))),$$

or equivalently

$$\frac{G'(x)}{T(\Lambda(G(x)))} = 1.$$

Although it is not as explicit as one would first hope, we have arrived at a differential equation relating $G'(x)$ to $G(x)$ without involving $G(-x)$. Integrating, we obtain

$$x = \int_{G(0)}^{G(x)} \frac{dx}{T(\Lambda(x))}. \tag{10}$$

Since the integrand is positive and $G(0)$ is known, $G(x)$ is uniquely determined by (10). Conversely, it is clear that the function G defined by (10) is a solution to (3). $\square$

Remarkably, the ground state limit $L^\star_{TSP}$ can be found in terms of Λ directly from (9), without using the uniqueness of the solution:

Proposition 2.2. *The two characterizations of $L^\star_{TSP}$ are consistent. In other words, the right hand side of* (4) *is equal to the right hand side of* (5).

Proof. In view of (7), equation (4) can be written

$$\frac{1}{2}\int_{-\infty}^{\infty} G(x)G'(-x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} G'(x)G(-x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} G'(x)\Lambda(G(x))\,du = \frac{1}{2}\int_{0}^{\infty} \Lambda(t)\,dt, \quad (11)$$

by the substitution $t = G(x)$. This is the same thing as (5). $\square$

If instead we let $T(g) = e^{-g}$, we obtain in the same way the limit $L^\star_M$ for the matching problem. In that case the solution is explicit, with $W(g) = 1 - e^{-g}$ and $\Lambda(t) = -\log(1 - e^{-t})$.

3 Rigorizing the replica results

The proof that the results of [12] are in agreement with the replica and cavity predictions is in itself satisfying as it shows that the inherently nonrigorous approach from statistical mechanics indeed gives a correct result.

Even more interesting is that the trick that transformed the integral equation (3) into an ordinary differential equation can produce an entirely rigorous proof of the TSP ground state limit independently of the results in [12] (in view of the discussion of the TSP in [2] this is perhaps not that surprising). We first consider the technically simpler minimum matching problem, and later return to the TSP.

3.1 Rescaling and diluted relaxation

It is convenient at this point to scale up the edge-lengths by a factor n in order to obtain a *local limit* of the random model. We therefore let the edge-lengths be uniform on the interval $[0, n]$, which means that the total length of the minimum matching will be of order n.

We introduce another parameter λ and consider the *diluted* relaxation of minimum matching, an idea that goes back at least to the study of multi-index matching problems by O. Martin, M. Mézard, and O. Rivoire [6, Section 4.1]. This relaxation consists in allowing any partial matching as a feasible solution, and letting the cost of a solution be the total length of the edges in the matching plus a penalty of $\lambda/2$ for each unmatched vertex. Edges of length greater than λ cannot participate in the optimum solution, since it is less costly to leave the two endpoints unmatched and pay the penalty of $2 \cdot \lambda/2 = \lambda$. Therefore the diluted relaxation is essentially a problem on an Erdös-Rényi random graph, sometimes called a Poisson Bethe lattice in the physics literature [4], where edges are present with probability λ/n and equivalently the average degree is λ.

It was shown in [13, 14] (and in a different setting already in [1]) that in order to find the limit $L_M^\star$ of the minimum length of a perfect matching, it suffices to find the large n limit of the diluted matching problem for fixed λ, and finally to let $\lambda \to \infty$. Therefore in the following we will regard the perfect matching problem only as a large λ limit of the diluted problem.

3.2 A path-forming game

A two-person perfect information zero-sum game called *Exploration* was introduced in [13]. The two players Alice and Bob take turns choosing the edges of a self-avoiding walk starting from a preassigned vertex of a graph with lengths associated to the edges. At every move, the moving player pays an amount equal to the length of the chosen edge to the opponent. Before each move, the moving player also has the option of terminating the game and paying a penalty of $\lambda/2$ to the opponent. Each player is trying to maximize their total payoff (what they receive minus what they pay throughout the game).

As was shown in [13], Exploration is related to the diluted matching problem:

Proposition 3.1. *In a finite graph, Alice's optimal first move is to move along the edge incident to the starting point in the solution to the diluted matching problem if there is such an edge, and otherwise to pay $\lambda/2$ to Bob and terminate the game immediately.*

Hence in order to find the asymptotic total cost of the minimum diluted matching, we can study the probability distribution of the cost of Alice's first move in Exploration starting from an arbitrary vertex. We will do this, but it is worth pointing out already here that in principle it suffices to know the probability (as a function of λ) that an arbitrary vertex is left unmatched, in other words the expected fraction of unmatched vertices. This is because, for quite general reasons, the total cost of the minimum λ_0-diluted matching in an arbitrary edge-weighted graph can be expressed as

$$\frac{1}{2} \cdot \int_{\lambda=0}^{\lambda_0} \#(\text{unmatched vertices in the minimum } \lambda\text{-diluted matching})\, d\lambda. \tag{12}$$

This is essentially just an application of the fundamental theorem of calculus: If the penalty for leaving a vertex unmatched increases by ϵ, then the cost increases by ϵ times the number of unmatched vertices.

3.3 Tree approximation

The Poisson Weighted Infinite Tree (PWIT) was introduced by David Aldous [1, 2]. The PWIT is an infinite rooted tree where each vertex has a countably infinite sequence of children. The edges to the children have lengths given by a rate 1 Poisson point process on the positive real numbers (independent processes for all vertices).

The relevance of the PWIT in this context comes from the fact that it is a *local limit* of K_n (under the rescaled edge lengths). In [1, 2] a concept of *weak* limit was used, but the relaxation to finite λ allows us to work with a stronger and simpler form of local limit.

If only edges of length at most λ are taken into account, the PWIT simplifies to an *edge-weighted Galton-Watson tree* that we denote by $GW(\lambda)$. This is simply a Galton-Watson tree with Poisson(λ) offspring distribution, and where in addition each edge has a weight (or length) from uniform distribution on $[0, \lambda]$. An equivalent way of thinking about the generation of edges is to regard the offspring of a vertex v as governed by a rate 1 Poisson point process on the interval $[0, \lambda]$, where each point of the process is the length of an edge to a child of v.

The tree $GW(\lambda)$ may be finite or infinite (the probability of the process becoming extinct and thus generating a finite tree is 1 for $\lambda \leq 1$, and strictly between 0 and 1 for $\lambda > 1$). For nonnegative integer k, we denote by $GW_k(\lambda)$ the edge-weighted Galton-Watson tree truncated after k generations.

For any graph, we let the (k, λ)-*neighborhood* of a vertex v be the subgraph that can be reached by walking at most k steps from v along edges of length at most λ.

In the language of probability theory, the tree-approximation result says that (under the rescaled edge-lengths, and for fixed k and λ) the (k, λ)-neighborhood of an arbitrarily chosen vertex v of K_n converges in total variation to $GW_k(\lambda)$ as $n \to \infty$. Without reference to the total variation metric, we state this result as:

Proposition 3.2. *For fixed k and λ, an event that depends only on the (k, λ)-neighborhood of an arbitrarily chosen vertex of K_n will have a limit probability as $n \to \infty$, and the limit probability is equal to the probability of the corresponding event on $GW_k(\lambda)$.*

3.4 Exploration on $GW(\lambda)$

In view of Propositions 3.1 and 3.2, it makes sense to study Exploration played on $GW(\lambda)$. A crucial obstacle for drawing conclusions about K_n from $GW(\lambda)$ is that Proposition 3.2 concerns only the truncated tree $G_k(\lambda)$, while there is no bound on the number of moves in the Exploration game. As was shown in [13], this difficulty can be handled by introducing so-called *valuations*, that simply give the best upper and lower bounds on the payoff (under optimal play) based on the first k levels of the tree.

A function f from the vertices of an edge-weighted rooted tree (we will consider $GW(\lambda)$ and $GW_k(\lambda)$) to the interval $[-\lambda/2, \lambda/2]$ is called a *valuation* if for every vertex v it satisfies

$$f(v) = \min(\lambda/2, l_i - f(v_i)), \tag{13}$$

where the minimum is taken over $\lambda/2$ and the children v_i of v, and l_i is the length of the edge from v to v_i. The right hand-side of equation (13) is motivated by the fact that if the tree is finite, it gives a recursive characterization of the payoff (under optimal play) throughout the remaining part of the game, for a player who has just moved to v (notice that if v is a leaf, $f(v) = \lambda/2$, and for a finite tree, equation (13) uniquely determines f).

Let f_k be the unique valuation on $GW_k(\lambda)$. By Proposition 3.2, f_k describes recursively the payoff under optimal play for a truncated Exploration game on K_n where, if the game has not ended before, the player to move after k moves has to terminate the game (and pay the penalty of $\lambda/2$ for doing so). For even k, the truncation rule favors Bob, while for odd k it favors

Alice. Therefore for even k, denoting the root of the Galton-Watson tree by ϕ, $Pr(f_k(\phi) = \lambda/2)$ will be an over-estimate of the large n limit probability that a vertex is unmatched in the optimal λ-diluted matching on K_n, while for odd k it will give an under-estimate.

If $P(f_k(\phi) = \lambda/2)$ has a limit as $k \to \infty$, then that is also the limit probability of a vertex being unmatched in the optimum diluted matching on K_n, in other words the limit fraction of unmatched vertices.

As was shown in [14], the justification of the replica symmetric predictions for the matching problem essentially boils down to showing that this large k limit exists for every λ. In [14], this is proved for the more general pseudo-dimension $d \geq 1$ case. That proof is somewhat non-constructive and consists in showing that (as turns out to be equivalent) even if $GW(\lambda)$ is infinite, it has (with probability 1) only one valuation. Here we show that the case $d = 1$ allows a more direct proof of the convergence of $P(f_k(\phi) = \lambda/2)$. The proof will show that $P(f_k(\phi) = x)$ converges for every x.

Theorem 3.3. *For fixed λ and every x in the interval $-\lambda/2 \leq x \leq \lambda/2$, the sequence $P(f_k(\phi) = x)$ converges as $k \to \infty$.*

To prove Theorem 3.3, we define a sequence of functions by

$$F_k(x) = P(f_k(\phi) \geq x).$$

These are essentially the probability distribution functions for f_k, but upside down due to the inequality going opposite to the standard way. Clearly $F_k(x)$ is equal to 1 for $x < -\lambda/2$, equal to 0 for $x > \lambda/2$, and decreasing with a single discontinuity at $x = \lambda/2$. Pointwise we have

$$0 \leq F_1(x) \leq F_3(x) \leq F_5(x) \leq \cdots \leq F_4(x) \leq F_2(x) \leq F_0(x) \leq 1, \qquad (14)$$

which means that in order to establish Theorem 3.3, it suffices to show that $F_{k+1}(x) - F_k(x) \to 0$ as $k \to \infty$.

Lemma 3.4. *For $-\lambda/2 \leq x \leq \lambda/2$, we have $F_0(x) = 1$ and*

$$F_{k+1}(x) = \exp\left(-\int_{-x}^{\lambda/2} F_k(t)\, dt\right). \qquad (15)$$

Proof. Suppose that $-\lambda/2 \leq x \leq \lambda/2$. Then $F_{k+1}(x)$ is the probability that there is no child v_i of the root such that $l_i - f_{k+1}(v_i) < x$. In other words $F_{k+1}(x)$ is the probability that there is no event in the inhomogeneous Poisson process of l_i's for which $f_{k+1}(v_i) > l_i - x$. Now notice that $f_{k+1}(v_i)$ has the same distribution as $f_k(\phi)$. Therefore

$$F_{k+1}(x) = \exp\left(-\int_0^\infty F_k(l-x)\,dl\right) = \exp\left(-\int_{-x}^{\lambda/2} F_k(t)\,dt\right).$$

□

Proof of Theorem 3.3. It is clear in view of (14) that the sequence $F_k(x)$ for odd k must converge pointwise to a limit function that we denote $F_{odd}(x)$, and similarly that $F_k(x)$ for even k converges to a limit function $F_{even}(x)$. In view of (15), these functions must satisfy

$$F_{odd}(x) = \exp\left(-\int_{-x}^{\lambda/2} F_{even}(t)\,dt\right) \tag{16}$$

and

$$F_{even}(x) = \exp\left(-\int_{-x}^{\lambda/2} F_{odd}(t)\,dt\right). \tag{17}$$

Differentiating, we see that

$$F'_{odd}(x) = -F_{odd}(x)F_{even}(-x),$$

and

$$F'_{even}(x) = -F_{even}(x)F_{odd}(-x).$$

This implies that the quantity

$$F_{odd}(x) + F_{even}(-x)$$

is constant throughout the interval $-\lambda/2 \le x \le \lambda/2$, since its derivative is zero. Consequently,

$$F_{odd}(-\lambda/2) + F_{even}(\lambda/2) = F_{odd}(\lambda/2) + F_{even}(-\lambda/2),$$

and since $F_{odd}(-\lambda/2) = F_{even}(-\lambda/2) = 0$, we conclude that

$$F_{odd}(\lambda/2) = F_{even}(\lambda/2).$$

Carrying the argument a little bit further, we see that $F_k(x)$ converges pointwise not only at $x = \lambda/2$ but throughout the interval $-\lambda/2 \leq x \leq \lambda/2$: From equations (16) and (17) with $x = \lambda/2$ it follows that

$$\int_{-\lambda/2}^{\lambda/2} F_{odd}(t)\, dt = F_{even}(\lambda/2) = F_{odd}(\lambda/2) = \int_{-\lambda/2}^{\lambda/2} F_{even}(t)\, dt.$$

Since $F_{odd}(t) \leq F_{even}(t)$ throughout the interval of integration, the only possibility is that $F_{odd}(x) = F_{even}(x)$ for all x. □

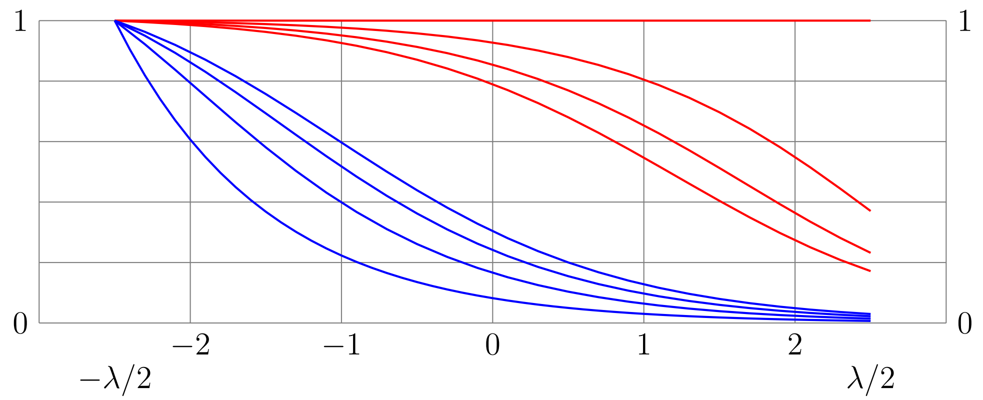

Figure 1: The first few iterations of (15) for $\lambda = 5$. In red, from top down: F_0, F_2, F_4, F_6. In blue, from bottom up: F_1, F_3, F_5, F_7. The crucial issue of "symmetry" is whether the functions F_k have a common limit, or if the limit F_{odd} of the blue functions lies strictly below the limit F_{even} of the red functions. Theorem 3.3 shows that $F_{odd} = F_{even}$ for every λ.

3.5 The limit as $k \to \infty$

It turns out that the limit function, which we denote by

$$F(x) = F_{odd}(x) = F_{even}(x) = \lim_{k\to\infty} F_k(x),$$

can be determined explicitly.

Proposition 3.5. *On the interval $-\lambda/2 \leq x \leq \lambda/2$, the limit function F is given by*

$$F(x) = \frac{1+q}{1+e^{(1+q)x}}, \tag{18}$$

where $q = F(\lambda/2)$, and q is determined by

$$\lambda = \frac{-2\log q}{1+q}. \tag{19}$$

Hence the limit distribution of $f_k(\phi)$ can be regarded as a rescaled and truncated logistic distribution together with a point mass of q at the point $\lambda/2$. If we let $\lambda \to \infty$ and consequently $q \to 0$, then this distribution converges to the logistic distribution, which is what we should expect in view of the results in [2].

Proof of Proposition 3.5. On the interval $-\lambda/2 \leq x \leq \lambda/2$ the limit function F must satisfy

$$F(x) = \exp\left(-\int_{-x}^{\lambda/2} F(t)\,dt\right), \tag{20}$$

and hence

$$F'(x) = -F(x)F(-x).$$

This means that $F'(x) = F'(-x)$, which in turn implies that $F(x) + F(-x)$ is constant. Putting $q = F(\lambda/2)$, we get

$$F(-x) = 1 + q - F(x), \tag{21}$$

and consequently

$$F'(x) = -F(x)(1 + q - F(x)).$$

Writing

$$-\frac{F'(x)}{F(x)(1+q-F(x))} = 1$$

and integrating with respect to x, we obtain

$$\log\left(\frac{1+q-F(x)}{F(x)}\right) = (1+q)x + C,$$

where putting $x = 0$ reveals that $C = 0$. Hence

$$\frac{1+q-F(x)}{F(x)} = e^{(1+q)x},$$

from which we obtain (18). Finally we would like to express q in terms of λ, and either of the equations $F(-\lambda/2) = 1$ or $F(\lambda/2) = q$ gives

$$q = e^{-(1+q)\lambda/2},$$

which in turn yields (19). $\square$

3.6 Limit cost of the minimum matching

As we remarked at the end of Section 3.1, the limit cost of the complete matching is obtained as the large λ limit of the diluted matching problem. In view of equation (12), we obtain this limit cost as

$$\frac{1}{2}\int_0^\infty F(\lambda/2)\, d\lambda = \frac{1}{2}\int_0^\infty q\, d\lambda, \tag{22}$$

where q is given by (19). Notice that in the left-hand side, F depends on λ. Here we have removed a factor n, meaning that we either scale back to edge-lengths from $[0, 1]$ (and penalties $\lambda/(2n)$ for unmatched vertices) or consider the average cost per vertex. The integral in (22), which can be interpreted as the area under the curve given by (19), can equivalently be written as

$$\int_0^1 \lambda\, dq,$$

and therefore the limit cost, or ground state energy, of the matching problem is

$$\frac{1}{2}\int_0^1 \frac{-2\log q}{1+q}\, dq = \frac{\pi^2}{12}.$$

Similarly, for finite λ, the limit cost *of the participating edges* is given by

$$\int_q^1 \frac{-\log t}{1+t}\, dt,$$

where again q is determined by λ from (19). The penalties will clearly be $q\lambda/2$, since q is the fraction of unmatched vertices.

4 The TSP

4.1 Relaxation and comply-constrain game

The finite-λ relaxation of the TSP is obtained by allowing any set of edges for which each vertex has degree at most 2, and where a penalty of $\lambda/2$ is paid for each missing edge at each vertex. Hence a vertex with one edge means a penalty of $\lambda/2$, while a vertex with no edge leads to a penalty of λ. In the case of the TSP the parity of the number n of vertices is not an issue, and therefore equivalently the total penalty is

$$\lambda \cdot (n - \#\text{ edges in the solution}).$$

The fact that the limit cost of the TSP is equal to the large λ limit cost of the diluted relaxation follows from a theorem of Alan Frieze [3].

We show here that the analysis of the minimum matching problem can to a large extent be paralleled. As was described in [13], the TSP is related to a "refusal" or "comply-constrain" version of Exploration: Whenever Alice is about to make a move, Bob has the right to forbid one of her move options, and vice versa. As before, a player can quit the game at cost $\lambda/2$. Some more rules have to be introduced if the game is played on a graph with loops, but on the tree $GW_k(\lambda)$, they reduce to the comply-constrain version.

The comply-constrain game leads to a different definition of valuation. Instead of (13), we require

$$f(v) = \min(\lambda/2, \min_2(l_i - f(v_i))). \tag{23}$$

Here $\min_2$ means second-smallest. We remark that equations equivalent to (23) were derived in [5, 8, 9] and also in [2], although they were not thought of as related to a 2-person game.

Again we study the distributions of $f_k(\phi)$, where f_k is the unique valuation on $GW_k(\lambda)$, and let

$$F_k(x) = P(f_k(\phi) \geq x),$$

now with the new definition of f_k. The pointwise inequalities (14) still hold, but the recursive characterization of F_k becomes different:

For $-\lambda/2 \leq x \leq \lambda/2$, $F_{k+1}(x)$ is now the probability that there is *at most one* child v_i of the root ϕ such that $l_i - f_{k+1}(v_i) < x$. Equivalently, $F_{k+1}(x)$ is the probability that there is *at most one* event in the inhomogeneous Poisson process of l_i's for which $f_{k+1}(v_i) > l_i - x$. Since again $f_{k+1}(v_i)$ has the same distribution as $f_k(\phi)$, we get

$$\begin{aligned} F_{k+1}(x) &= \left(1 + \int_0^\infty F_k(l-x)\,dl\right) \cdot \exp\left(-\int_0^\infty F_k(l-x)\,dl\right) \\ &= \left(1 + \int_{-x}^{\lambda/2} F_k(t)\,dt\right) \cdot \exp\left(-\int_{-x}^{\lambda/2} F_k(t)\,dt\right) \end{aligned} \tag{24}$$

Again there are limit functions F_{odd} and F_{even} that have to satisfy

$$F_{odd}(x) = \left(1 + \int_{-x}^{\lambda/2} F_{even}(t)\,dt\right) \cdot \exp\left(-\int_{-x}^{\lambda/2} F_{even}(t)\,dt\right)$$

and

$$F_{even}(x) = \left(1 + \int_{-x}^{\lambda/2} F_{odd}(t)\,dt\right) \cdot \exp\left(-\int_{-x}^{\lambda/2} F_{odd}(t)\,dt\right)$$

It is convenient to introduce the functions

$$G_{odd}(x) = \int_{-x}^{\lambda/2} F_{odd}(t)\,dt$$

and

$$G_{even}(x) = \int_{-x}^{\lambda/2} F_{even}(t)\,dt$$

Mimicking the trick from Section 2, we let

$$\Delta(x) = \frac{d}{dx}\left[(2 + G_{odd}(x)) \cdot e^{-G_{odd}(x)} + (2 + G_{even}(-x)) \cdot e^{-G_{even}(-x)}\right].$$

Since the derivative of $(2+t)e^{-t}$ is $-(1+t)e^{-t}$, and the inner derivative $G'(x) = F(-x)$, it is easy to check that

$$\Delta(x) = -F_{even}(x)F_{odd}(-x) + F_{odd}(-x)F_{even}(x) = 0. \tag{25}$$

This implies that

$$\begin{aligned} &(2 + G_{odd}(-\lambda/2))e^{-G_{odd}(-\lambda/2)} + (2 + G_{even}(\lambda/2))e^{-G_{even}(\lambda/2)} \\ &\qquad = (2 + G_{odd}(\lambda/2))e^{-G_{odd}(\lambda/2)} + (2 + G_{even}(-\lambda/2))e^{-G_{even}(-\lambda/2)}. \end{aligned} \tag{26}$$

Using the known values $G_{odd}(-\lambda/2) = G_{even}(-\lambda/2) = 0$, this simplifies to

$$(2 + G_{odd}(\lambda/2))e^{-G_{odd}(\lambda/2)} = (2 + G_{even}(\lambda/2))e^{-G_{even}(\lambda/2)}.$$

Since the function $(2+t)e^{-t}$ is monotone decreasing (and therefore injective) for positive t, we conclude that

$$G_{odd}(\lambda/2) = G_{even}(\lambda/2),$$

in other words,

$$\int_{-\lambda/2}^{\lambda/2} F_{odd}(t)\,dt = \int_{-\lambda/2}^{\lambda/2} F_{even}(t)\,dt.$$

Again the conclusion is that since $F_{odd}(t) \leq F_{even}(t)$ pointwise, we must have $F_{odd}(x) = F_{even}(x)$ for all x.

4.2 The finite-λ integral equation

The functions F_k thus converge to a limit that again we denote by F and that now satisfies

$$F(x) = \left(1 + \int_{-x}^{\lambda/2} F(t)\,dt\right) \cdot \exp\left(-\int_{-x}^{\lambda/2} F(t)\,dt\right). \tag{27}$$

If we write G for the common limit of G_{odd} and G_{even}, in other words

$$G(x) = \int_{-x}^{\lambda/2} F(t)\,dt,$$

then

$$G'(x) = (1 + G(-x))e^{-G(-x)}.$$

This equation looks like the Krauth-Mézard-Parisi equation (7), but the difference is that for finite λ, we only require it to hold in the interval $[-\lambda/2, \lambda/2]$. Moreover, the boundary conditions depend on λ. As was established in the previous section (25),

$$(2 + G(x))e^{-G(x)} + (2 + G(-x))e^{-G(-x)} = C \tag{28}$$

for some constant C, which has to be in the interval $2 < C < 4$.

We will return in a moment to the question of how C is related to λ and to the average degree in the solution.

4.3 Length of the minimum tour

For the TSP, we don't have an explicit expression for $F(\lambda/2)$, the probability that Alice should terminate the game immediately. We therefore cannot compute the limit cost of the TSP the same way as we did for matching. Instead we will take an approach used in [2] and going back to [7]. The result, which can also be derived from [12], is the following:

Theorem 4.1. *The limit total length of the participating edges in the diluted 2-factor is the area under the curve*

$$(2+x)e^{-x} + (2+y)e^{-y} = 4 - \alpha, \tag{29}$$

where α is the average degree of a vertex in the solution.

Again this does not include the penalties. This means that in principle, the parameter λ does not enter into the statement of Theorem 4.1.

Proof. The edge-lengths are uniform on $[0, n]$, and therefore the density function for the length of a particular edge is simply $1/n$ on that interval. The expected contribution to the total length of the optimum solution from an arbitrary edge e between vertices v_1 and v_2 of K_n is therefore

$$\frac{1}{n} \cdot \int_0^\lambda z \cdot P(\text{participation given length } z)\, dz. \tag{30}$$

The edge e will participate in the optimum diluted 2-factor if it is the optimal first move for Alice when the game starts at either of v_1 or v_2. We let f_1 and f_2 be the game-theoretical values of playing second if the game would start at v_1 or v_2 respectively, and be played with the edge e deleted from the graph. If the game (with e present) starts at v_1, Alice will go to v_2 in her first move if and only if the length z of the edge e satisfies

$$z \leq f_1 + f_2$$

(for a detailed argument see [14]).

The (k, λ)-neighborhoods of v_1 and v_2 (with e deleted) can be approximated by two independent Galton-Watson trees distributed like $GW_k(\lambda)$. Therefore the edge e will participate in the optimum solution essentially if $z \leq f_1 + f_2$, where f_1 and f_2 are independent and drawn from the distribution given by F in Proposition 3.5.

Hence apart from the scaling factor $1/n$, (30) is equal to

$$\int_0^\lambda z \cdot P(z \le f_1 + f_2)\, dz = \int_0^\infty z \cdot P(z \le f_1 + f_2)\, dz.$$

Without using any particular properties of the probability distribution, we can rewrite this as

$$\int_0^\infty z \int_{-\infty}^\infty (-F'(x)) \cdot P(f_2 \ge z - x)\, dxdz$$
$$= \int_0^\infty z \int_{-\infty}^\infty (-F'(x))F(z-x)\, dxdz. \quad (31)$$

With the substitution $u = z - x$, this becomes

$$\int_{-\infty}^\infty F(u) \int_0^\infty z(-F'(z-u))\, dzdu$$
$$= \int_{-\infty}^\infty F(u) \int_{-u}^\infty (x+u)(-F'(x))\, dxdu,$$

and by partial integration, this is

$$\int_{-\infty}^\infty F(u) \int_{-u}^\infty F(x)\, dxdu. \quad (32)$$

If (as for the matching problem) we had explicit knowledge of F, we could compute (32) directly. Now we don't have a simple expression for F, but there is another method. Recall that

$$G(u) = \int_{-u}^\infty F(x)\, dx.$$

Clearly $G'(-u) = F(u)$, which means that (32) is transformed to

$$\int_{-\infty}^\infty G'(-u)G(u)\, du = \int_{u=-\infty}^{u=\infty} G(u)\, dG(-u). \quad (33)$$

A simple interpretation of (33) is that it is the area under the curve (in the positive quadrant) when $G(u)$ and $G(-u)$ are plotted against each other. In order to find the value of this integral, we therefore only need to know the relation between $G(u)$ and $G(-u)$. Apart from the value of the constant C, this relation is given by (28).

The only remaining issue is therefore the relation between the constant C in (28) and the parameter λ. It turns out that the constant C is more directly related to the average degree of a vertex in the solution, than to λ. If we let $\lambda \to \infty$ (corresponding to the TSP), then $G(\lambda/2) \to \infty$. Since $G(-\lambda/2) = 0$, $C \to 2$. This means that we have rederived the result from [12] that the limit length of the TSP is given by (5) and (6).

To obtain (29) for general α, notice that an edge belongs to the optimum λ-diluted 2-factor if and only if (when the game starts at one of the endpoints of the edge) it would be optimal for Alice to move along the edge (due to the comply-constrain rule, it is possible also in non-degenerate cases that Alice has two optimal moves).

To find the degree distribution of an arbitrary vertex v, we therefore want to know the distribution of the number of edges e incident to v such that if the game starts at v, Alice would be better off playing along the edge e than terminating and paying the penalty. Say that such an edge is *playable*, and let N be the number of playable edges incident to v. Notice that if $N \leq 2$, the playable edges are the optimal moves, while if $N \geq 3$, in general only 2 of them are optimal.

If we go back to the proof of (24), we see that N is Poisson distributed with mean $G(\lambda/2)$. In particular,

$$P(N = 0) = e^{-G(\lambda/2)},$$

$$P(N = 1) = G(\lambda/2)e^{-G(\lambda/2)},$$

and consequently,

$$P(N \geq 2) = 1 - e^{-G(\lambda/2)} - G(\lambda/2)e^{-G(\lambda/2)}.$$

Therefore the average degree α of a vertex is given by

$$\alpha = 0 \cdot e^{-G(\lambda/2)} + 1 \cdot G(\lambda/2)e^{-G(\lambda/2)} + 2 \cdot \left(1 - e^{-G(\lambda/2)} - G(\lambda/2)e^{-G(\lambda/2)}\right) \\ = 2 - (2 + G(\lambda/2))e^{-G(\lambda/2)}. \quad (34)$$

In combination with (28), and the fact that $G(-\lambda/2) = 0$, this means that

$$C = 4 - \alpha,$$

which concludes the proof. □

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