The Angel Problem

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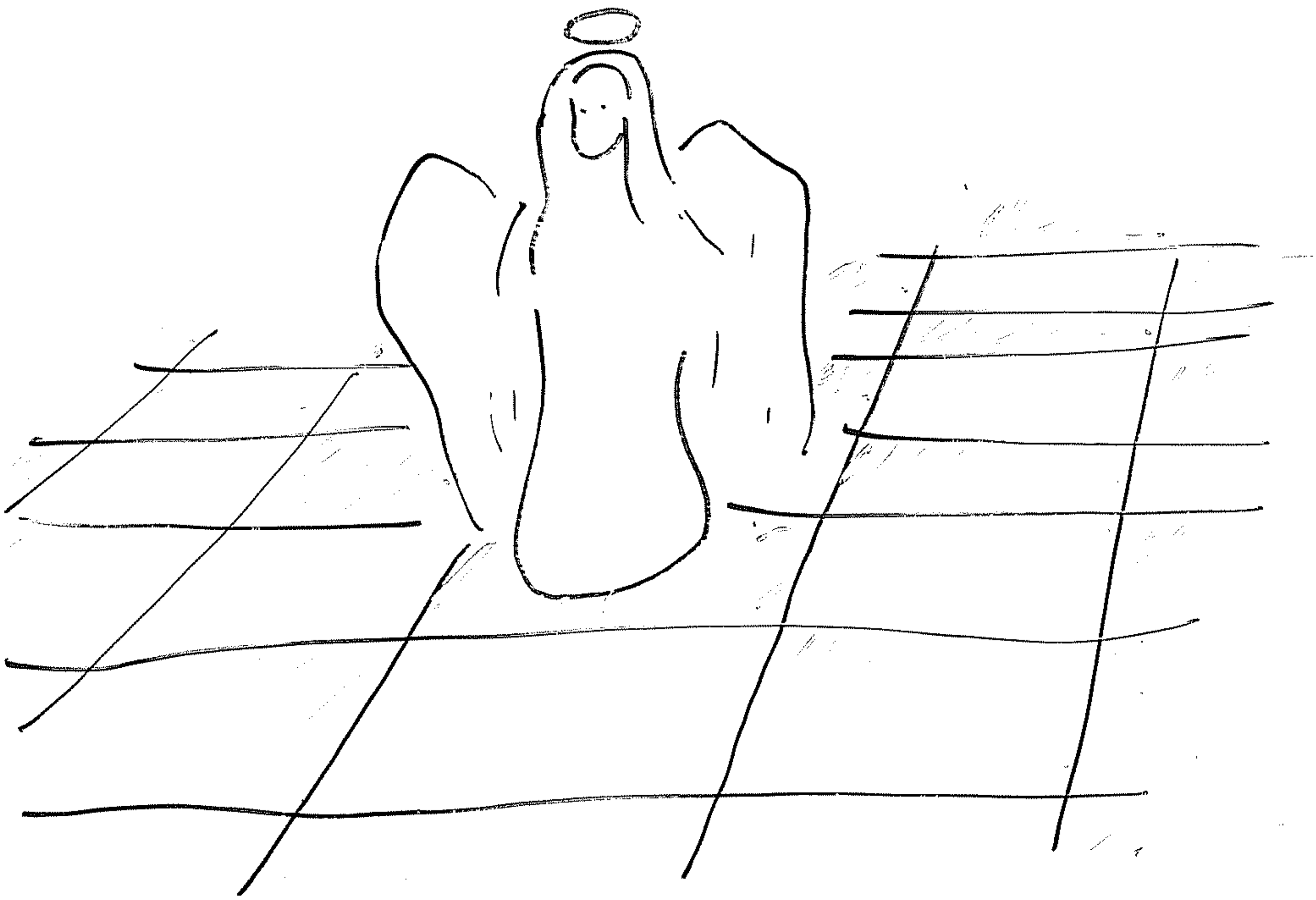
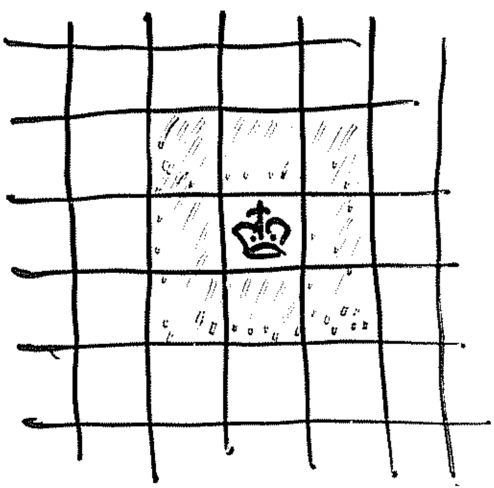
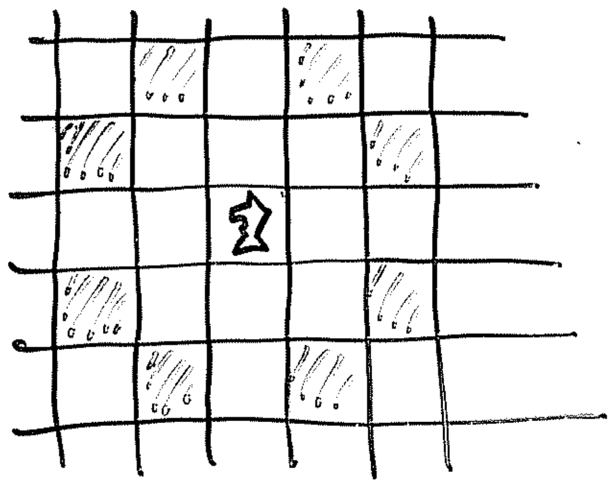


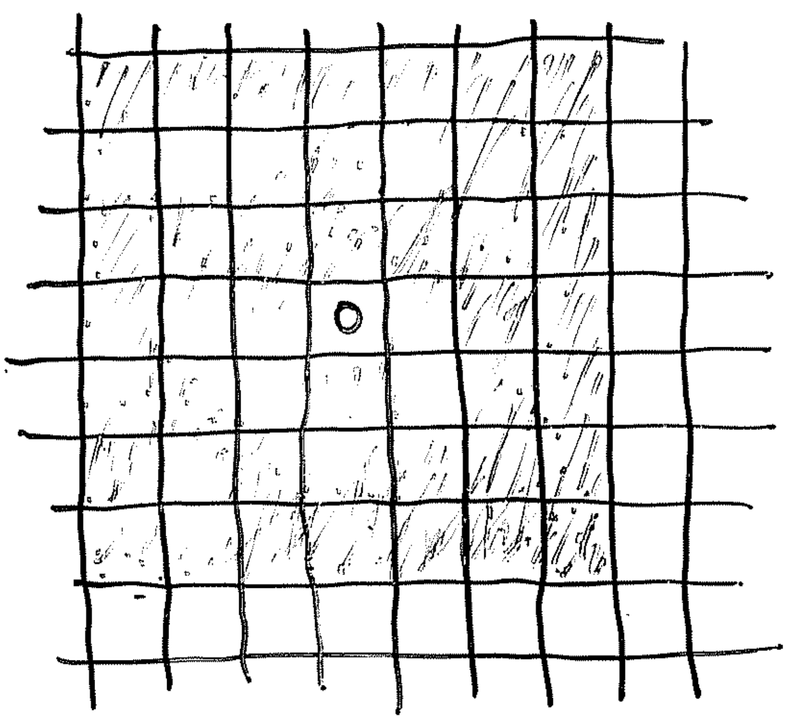
Described in Winning Ways
(Berlekamp, Conway, Guy 1982)

Solved in 2006

Angel = Finite range chess piece







Angel-and-devil game

On each turn:

D eats a square

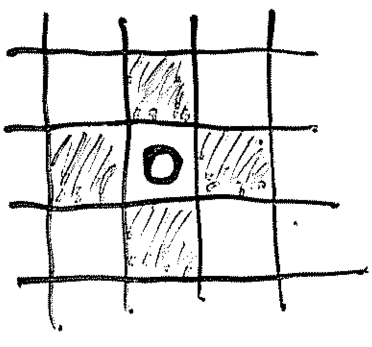
A moves to an uneaten square

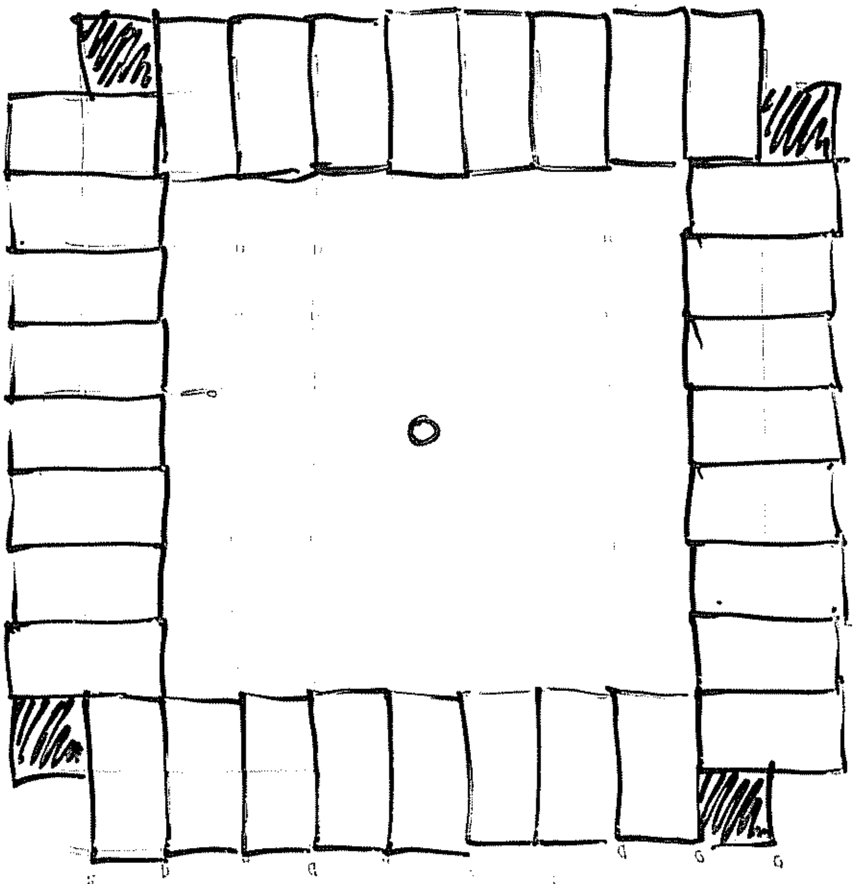
D wins if A cannot move.

A wins by escaping forever.

For a given pattern of movement, either D or A has a winning strategy.

Example: A "duke" loses





Angel Problem:

Is there a p such that an angel of power p has a winning strategy?

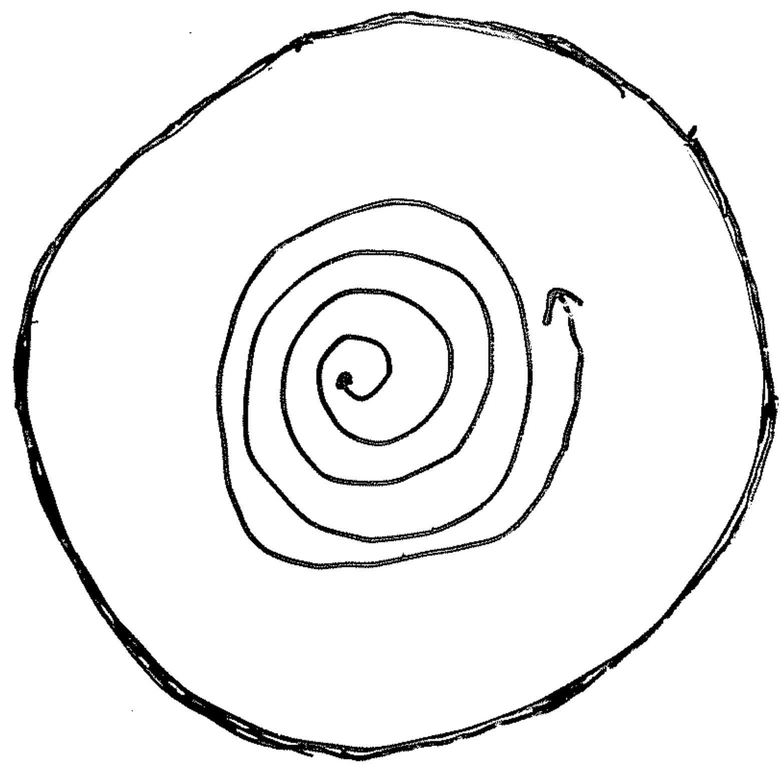
Conway 1994: Proof that

A wins:	\$100
D wins	\$1000

Why hard?

No locally defined winning strategy

Running angel caught by an ambush



Slow angel caught by distant wall

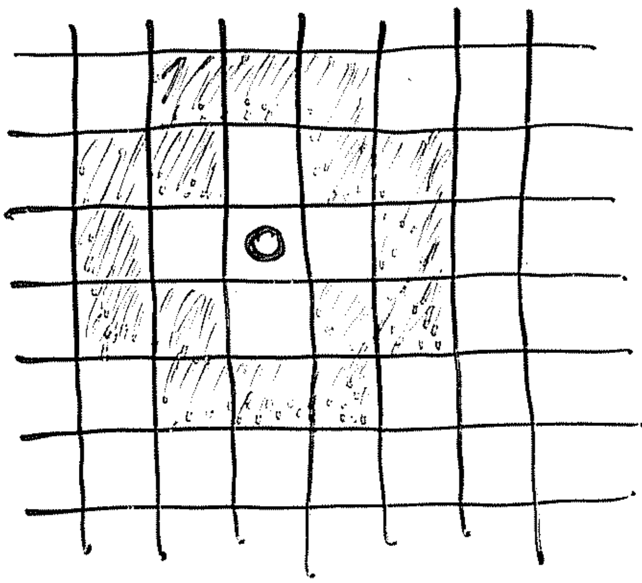
Solved in 2006 by

B. Bowditch $p = 4$

P. Gács $p < \infty$

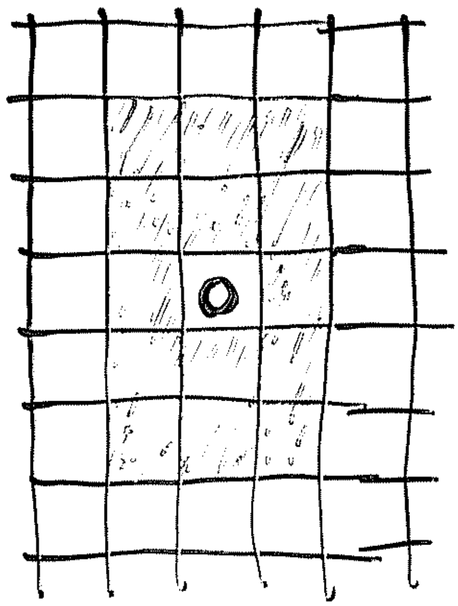
O. Kloster $p = 2$

A. Máthé $p = 2$



Kloster's winning 16-angel

I will give a simple proof that the 14-angel



is winning.

Generalize:

- Infinite set of "cells"
- Some rule for moving from one cell to another

Máthé's lemma

Ambush = D eats a cell that A has visited or could have moved to in an earlier move.

Let S be a finite set of cells. A strategy for D is S-winning if it guarantees that A never leaves S.

If D has an S-winning strategy, then D has an S-winning strategy that never makes an ambush

Proof Suppose D has an S-winning strategy that doesn't make an ambush in the first n moves.

If this strategy requires D to make an ambush in a cell a at move $n+1$, then instead eat some other cell.

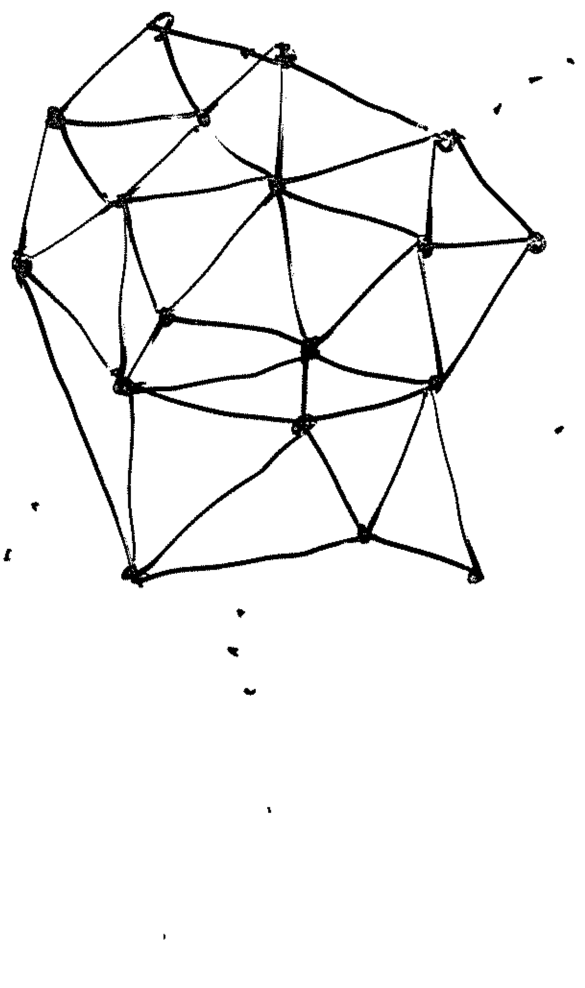
If A ever moves to a, play as would have been required if A had moved to a the first time she could.

By induction, D never has to make an ambush. The lemma follows by compactness.

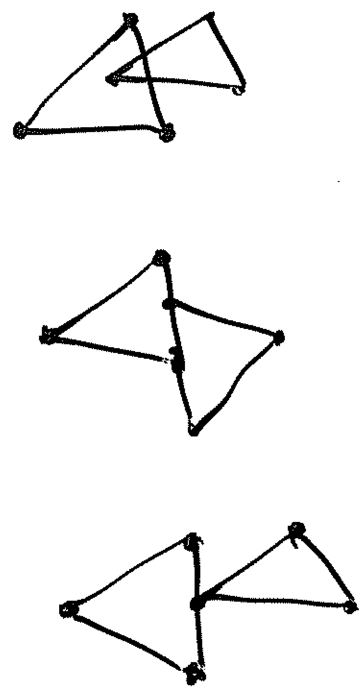
Nice devil = D that never makes an ambush

By compactness, if A wins against a nice D, then A can win against any D.

Triangulation:

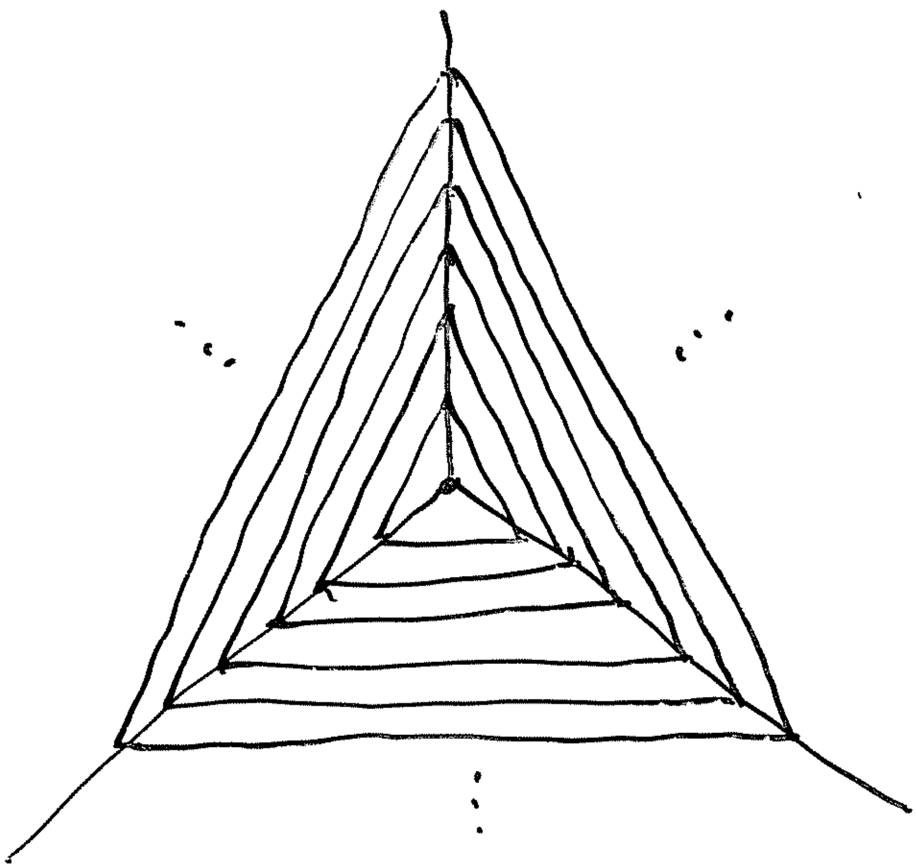


Not:

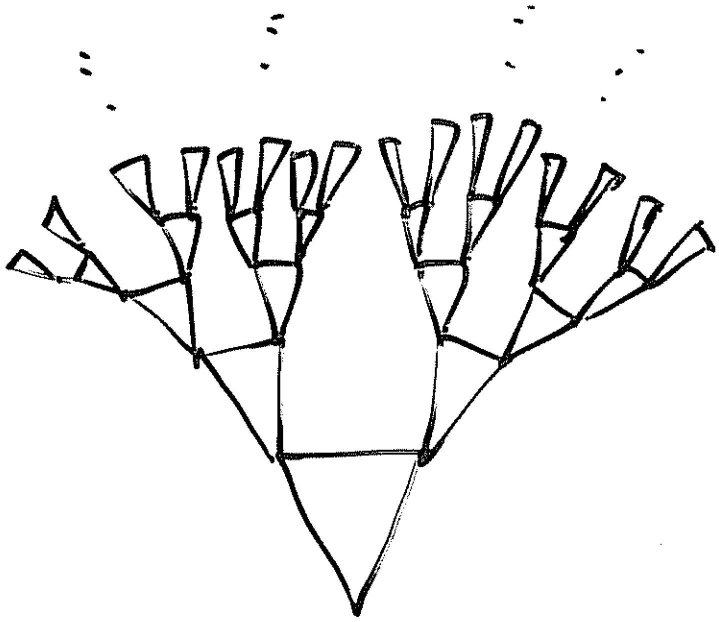


King = angel that can move between weakly adjacent cells.

Example: Sometimes D wins



Sometimes A wins:



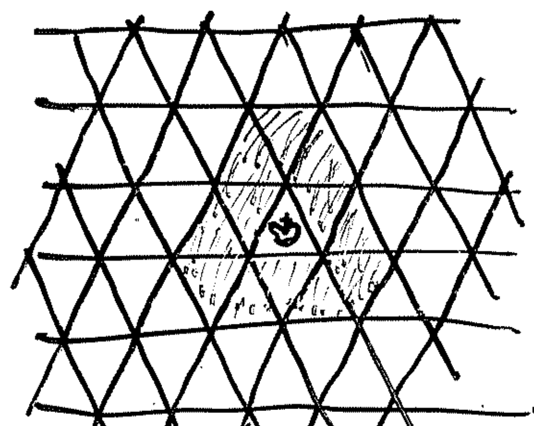
binary tree

K reaches $\geq c^N$ cells in N moves

Cannot map to the original problem

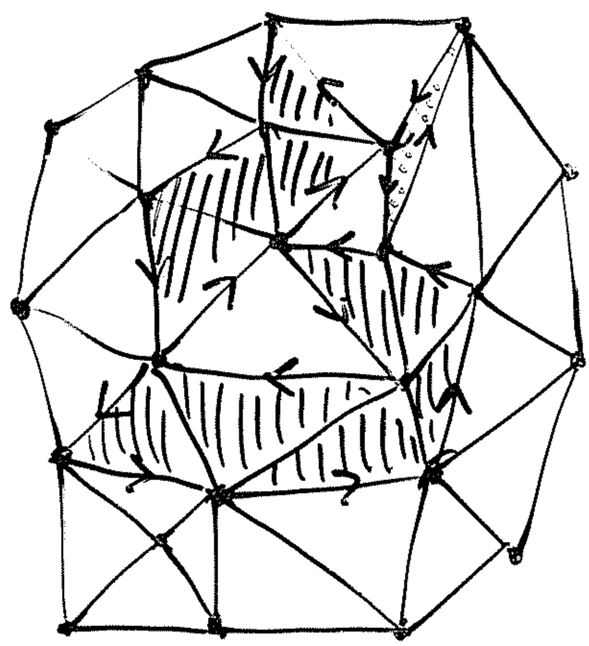
On the regular triangular lattice,

the king wins



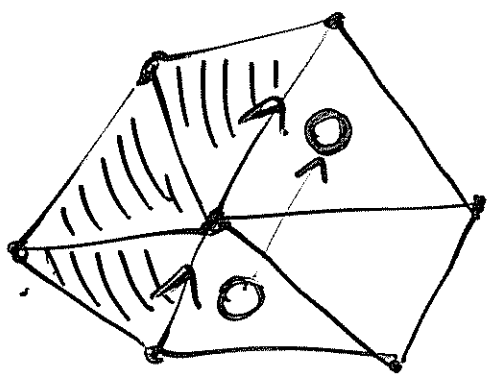
A set of cells is said to be connected if it is connected w.r.t. strong adjacency

Counter-clockwise orientation of boundary of connected components of eaten cells:



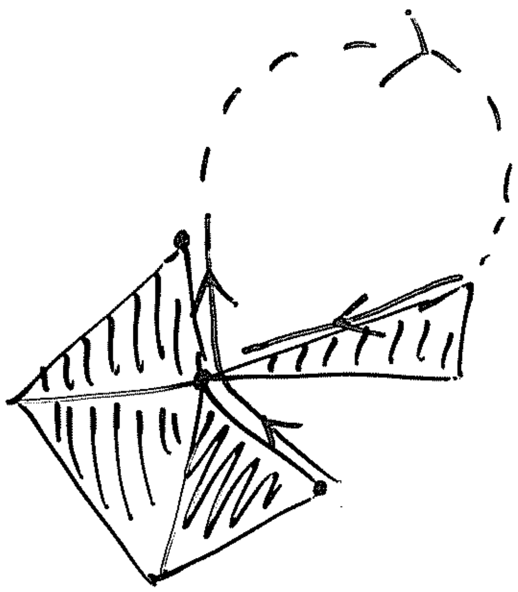
Position and future strategy of the king determined by one of these edges.

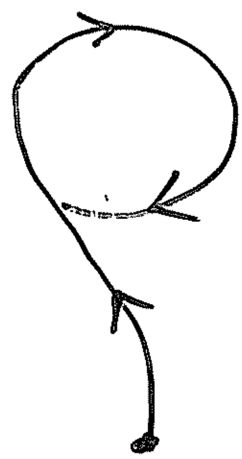
Runner strategy: Go to the next edge of the oriented boundary



If a nice D traps the king in a finite region, then the king will eventually return to the starting point, facing in the same direction

<u>Proof</u> Suppose this is not true





↯

The perimeter of a finite set of cells = number of edges that separate it from its complement

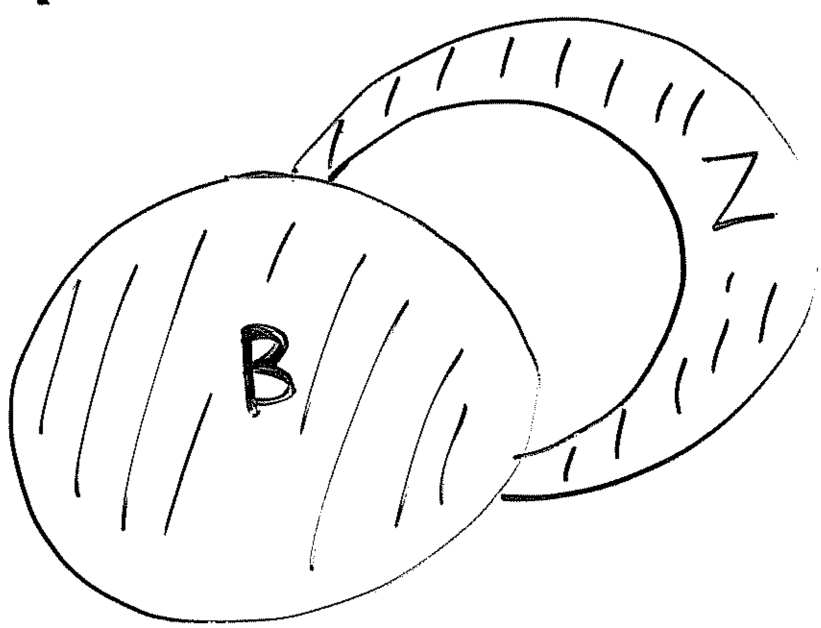
Lemma A connected set of n cells has perimeter $\leq n+2$.

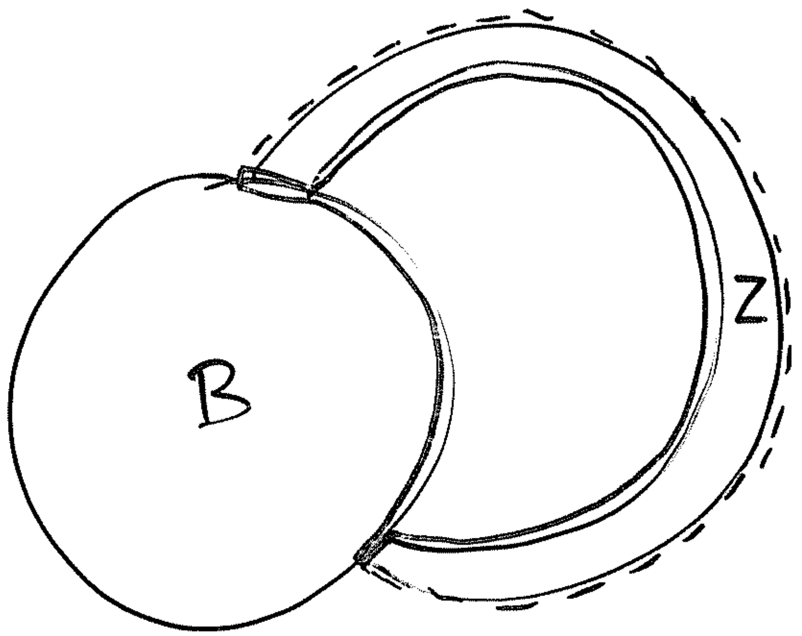
Proof Induction.

Main Lemma Let A be a finite set of cells. Against a nice devil there is some starting point from which the king can either escape to infinity, or walk around A and return to the starting point

<u>Proof</u> Suppose w.l.o.g. A connected. Let $B \supseteq A$ have minimal perimeter, and let K start in a cell adjacent to B.

If the nice D stops K from walking around B, then it has to eat a connected set Z of cells forming a "handle" attached to B.





Suppose K makes n moves before returning to the starting point.

By the minimality of B, the perimeter of Z has to be at least $n+4$.

D must have made at least $n+2$ moves. $\lightning$

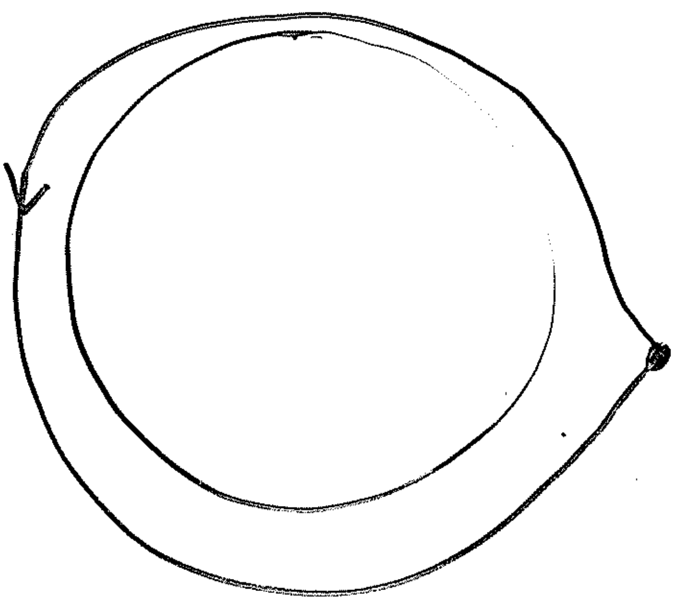
Let distance between cells
= maximal point to point distance

For each x, there is some starting point from which the king can reach some cell at distance $\geq x$ from where it started

<u>Proof</u> Suffices to consider nice D.

Let A be a disk of diameter x.

the set of cells that intersect

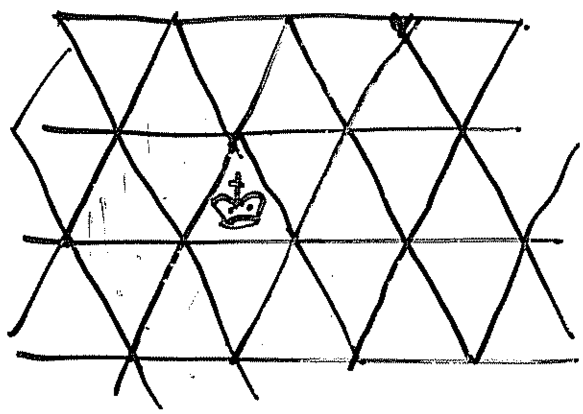


<u>Cor</u> If there is an upper bound on the side-lengths of the cells, then for each N, there is some starting point from which a king can make at least N moves before being trapped by the D.

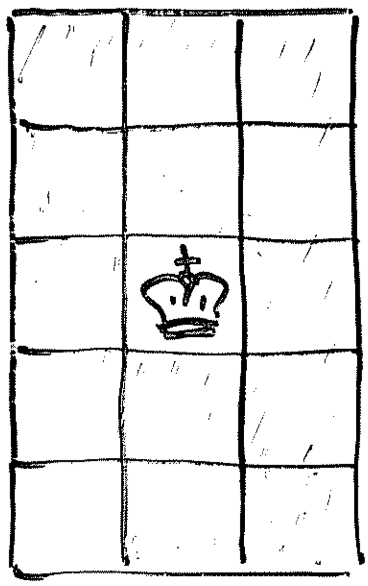
<u>Note</u> Cannot conclude that K wins from some starting point

<u>Cor</u> If the cells can be finitely colored so that the symmetries of the triangulation act transitively on each color class, then some color class consists of winning starting positions for the king

Cor The king wins the angel-and-devil game on the regular triangular lattice.



Cor On the chess board, the 14-angel wins



Map to chess board: