A weaker winning angel

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Abstract

In 2006 the so-called angel problem was solved independently by four authors. The solution shows that a sufficiently strong chess piece can escape a "devil" who removes one square on each move from an infinite chess board. We show that on the regular triangular lattice, a chess piece that moves to weakly adjacent triangles can escape the devil. A solution to the original problem is derived from this result. Compared to earlier solutions, our approach gives a simpler proof as well as a weaker winning angel. We show that an angel moving from the middle square to any other square of a 3 by 5 rectangle is winning.

1 Introduction

The angel problem was introduced by Berlekamp, Conway and Guy [1] in 1982. An angel is a finite range chess piece. On an infinite two-dimensional chess board the angel plays the following game against the devil: On each turn, the devil eats a square of his choice anywhere on the board. Then the angel makes a move. The restriction is that the angel cannot move to a square that has been eaten by the devil. The devil wins if at some point the angel cannot move. Otherwise the angel wins.

Once the legal moves of the angel are specified, we can ask whether it is the angel or the devil who has a winning strategy. Specifically, an *angel of*

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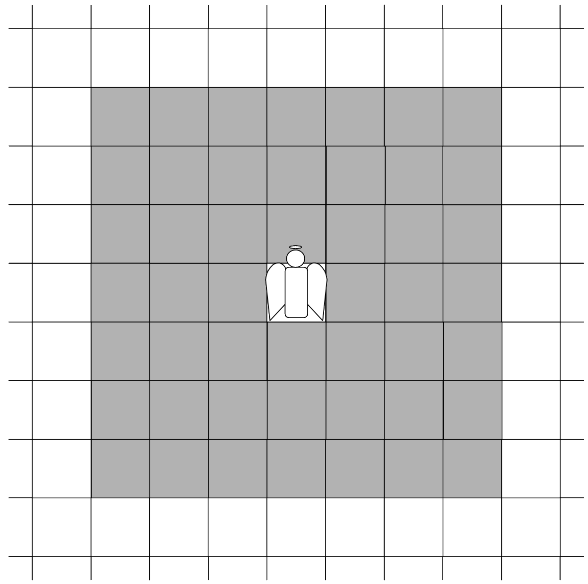

Figure 1: An angel of power 3 can move to the shaded squares.

power p is a chess piece that can jump in one move to any square at distance at most p king's moves (see Figure 1).

It was shown in [1] that the devil can trap an angel of power 1, in other words a chess king. This version of the game, also known as "quadraphage", had been studied prior to [1] in [5, 7, 14]. The question known as the *angel problem*, first posed in [1], is whether there is some p such that an angel of power p can win the game.

A strange feature of this game is that the angel only wins by escaping forever. One way to obtain a finite version of the game is to specify a finite region, say an n by n square for some large odd n, let the angel start at the middle, and declare the angel to be the winner if she reaches a square outside this region. If the devil has a winning strategy for some n, then clearly he has a winning strategy for every larger n, and also in the original game. There is a standard compactness argument that shows that the converse is also true. If the angel can win the finite game for every n, then since she has only finitely many legal moves, there has to be a first move after which she still wins the finite game for every n. By consistently choosing such moves she escapes forever, which means she wins the original game. Therefore if the power p of the angel is specified, then either the angel or the devil has a winning strategy.

Another unusual feature of the game is that however badly the devil plays, he will never be worse than in the initial position. This means that if the devil is winning in the first place, he can never blunder and reach a position where the angel is winning. In fact he can play his first million moves completely at random. If he later discovers a way to trap the angel, those million eaten squares will not harm him. It was pointed out to me by Olle Häggström that there is an explicit strategy for the devil which is optimal in the sense that if there exist winning strategies for the devil, then this is one of them. Moreover, this strategy is independent of the angel's pattern of movement and therefore wins against any angel that can be beaten! At first this sounds exciting, since it means that in order to exhibit a winning angel it suffices to find one that wins against this particular strategy. Unfortunately the strategy for the devil simply consists in trying, for each n, the finitely many strategies in the n by n game, one after the other, each time translating the starting point to the angel's current position.

It is hard to imagine that the devil would be able to make any progress against an angel of power 1000 for instance, but in [4], several simple-minded strategies are quite surprisingly proved to be losing regardless of p. In 1994, John Conway [4] offered $100 for a proof that some angel can win, and $1000 for a proof that the devil can beat any angel. The problem remained unsolved until 2006 when it was solved independently by four authors, Brian Bowditch [3], Péter Gács [6], Oddvar Kloster [8] and András Máthé [13]. They showed that as expected, a sufficiently powerful angel has a winning strategy. In dimensions 3 and higher, the existence of winning angels had already been established [2, 10, 12]. Further information can be found on Kloster's webpage [9].

Of the four proofs, the two by Kloster and Máthé are the simplest. At the same time they obtain considerably weaker winning angels than Bowditch and Gács. Kloster's and Máthé's proofs are closely related, and both show that an angel of power 2 wins. Máthé's proof is conceptually simpler, but on the other hand Kloster's proof has the advantage of giving an explicit winning strategy for the angel.

As was pointed out by Máthé [13], the angel of power 2 never has to make a diagonal move of length 2, and hence an angel with the pattern of movement shown in Figure 2 can escape the devil. Although it is not stated, it is implicit from Kloster's proof [8] that the angel never has to make a

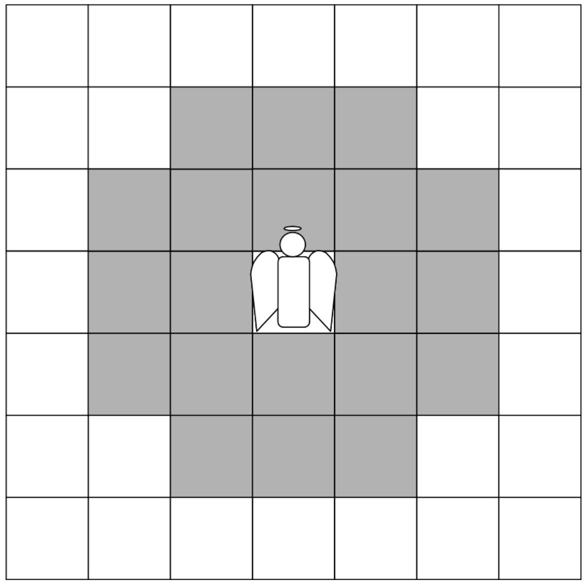

Figure 2: Máthé's winning angel.

horizontal or vertical move of length 1 either. This shows that even the angel of Figure 3 is winning.

Naturally we may ask for the weakest winning angel, and there are several ways of making this question precise. Let us say that an angel has *strength* k if its maximal number of legal moves from any position is k. First note that the angel-and-devil game can be defined for any directed graph. The devil eats vertices of the graph, and the angel moves along the edges in the specified direction. In this form, the strength of the angel is equal to the maximum out-degree of any vertex in the graph. In order for the angel to win, its strength obviously has to be at least 2. On the other hand the angel wins on an infinite binary tree. Hence in this more general form it is almost trivial that the minimum strength of a winning angel is 2.

In order for the question to be relevant to the original problem, we have to impose some geometric restriction on the angel's pattern of movement. There are at least two ways of doing this:

- *Translation invariance*: If we identify the two-dimensional chess board with the abelian group $\mathbf{Z} \times \mathbf{Z}$, then translation invariance means that if the angel can move from a to b, then it can also move from $a + c$ to

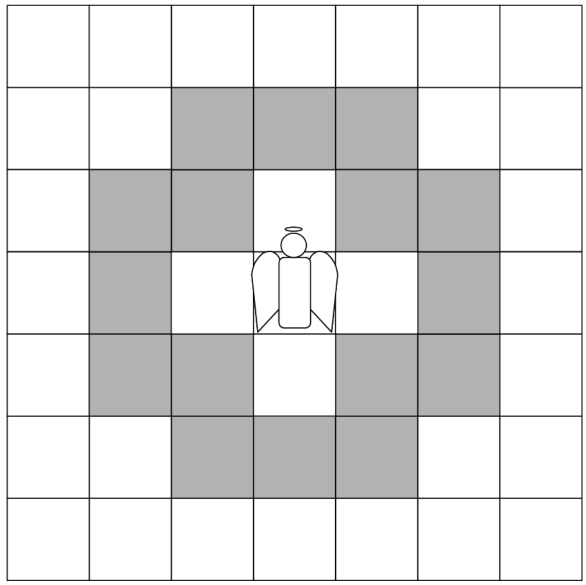

Figure 3: Kloster's winning angel.

$b + c$ for every c.

- *Boundedness*: The angel's pattern of movement is bounded if there is some x such that if the angel can move from a to b, then the euclidean distance from a to b is at most x. This is equivalent to saying that there is some p for which the angel's legal moves constitute a subset of the legal moves of an angel of power p.

If the strength of the angel is finite, translation invariance implies boundedness, so the first condition is more restrictive. As was pointed out by Bowditch, the translation invariance condition means that the angel problem can be formulated for arbitrary groups. In [3], a group is said to be *diabolic* if the devil wins for any finite pattern of movement, and there is even a conjectured characterization of diabolic groups.

The angel of Figure 3 is translation invariant and has strength 16. In this paper we show that the angel of strength 14 in Figure 4 is winning. We do this by first constructing a winning angel of strength 12 whose pattern of movement is bounded but not translation invariant (see Figures 12 and 13).

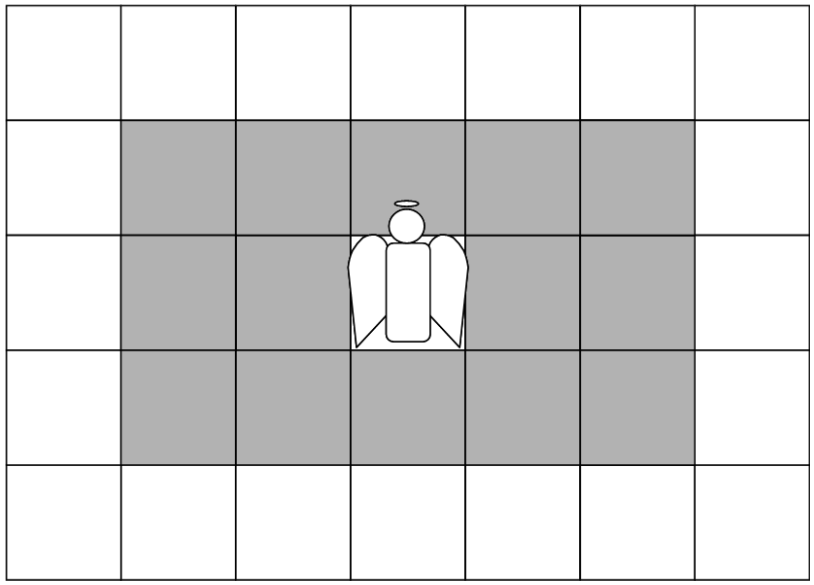

Figure 4: A winning 14-angel.

2 Máthé's lemma

Máthé's proof relies on a clever strategy-stealing argument that shows that if the devil has a winning strategy, then there has to be a *nice* winning strategy (to be defined below). Here we give a simpler proof of Máthé's lemma. This lemma is valid in the more general setting of directed graphs, and is therefore independent of the geometry of the board.

In the following, we shall consider a certain angel-and-devil game on a triangulation of the plane, and therefore we refer to the "squares" of the board as *cells*.

Definition 2.1. A devil's move on a cell that the angel has visited (including the starting point), or could have moved to in an earlier move is called an *ambush*.

If S is a finite set of cells, then a devil's strategy that guarantees that the angel never leaves S is called S*-winning*. By the compactness argument given in the introduction it follows that the devil has a winning strategy if and only if for some finite S he has an S-winning strategy.

Lemma 2.2 (Máthé [13]). *Let S be a finite set of cells. If the devil has an S-winning strategy, then there is an S-winning strategy that never makes an ambush.*

Proof. We prove this by induction. Suppose that the devil has an S-winning strategy Θ that makes no ambush in the first n moves. Under this assumption we prove that there is an S-winning strategy that makes no ambush in the first $n+1$ moves.

Let the devil play according to Θ for the first n moves. Suppose that in move $n+1$, Θ requires the devil to play in a cell a that the angel has visited or could have visited on an earlier move (otherwise there is nothing to prove). Then we modify the devil's strategy. In move $n+1$, the devil plays an arbitrary move which is not an ambush. In the following, the devil plays according to Θ as long as the angel does not go to cell a. Whenever the angel goes to cell a, the devil switches to playing as he would have played according to Θ if the angel had moved to cell a the first time she could.

We conclude that there is no n for which the devil is forced to make an ambush during the first n moves in order to keep the angel inside S. Since the devil has only finitely many "reasonable" moves (the cells in S and those that the angel can reach in one move from S), the statement in the lemma follows by the standard compactness argument. $\square$

Definition 2.3. A devil that never makes an ambush is called a *nice devil.*

3 Triangulations and the chess king

Both Kloster's and Máthé's proofs rely geometrically on a certain inverse isoperimetric inequality stating that a connected set of n cells cannot have arbitrarily large perimeter. On the square lattice, the upper bound on the perimeter is $2n+2$. Comparing with the hexagonal and the triangular lattices, one realizes that the coefficient 2 is the number of sides of a cell minus 2. On the hexagonal lattice the corresponding upper bound is $4n+2$ while on the triangular lattice it is $n+2$. The fact that the bound is better for triangles suggests that one may find a weaker winning angel on a triangular lattice. With Máthé's method we will show that on the regular triangular lattice, the angel in Figure 5 is winning.

We shall consider an even more general form of chess board. By a *triangulation* of the plane we mean a dissection of the plane into triangles such

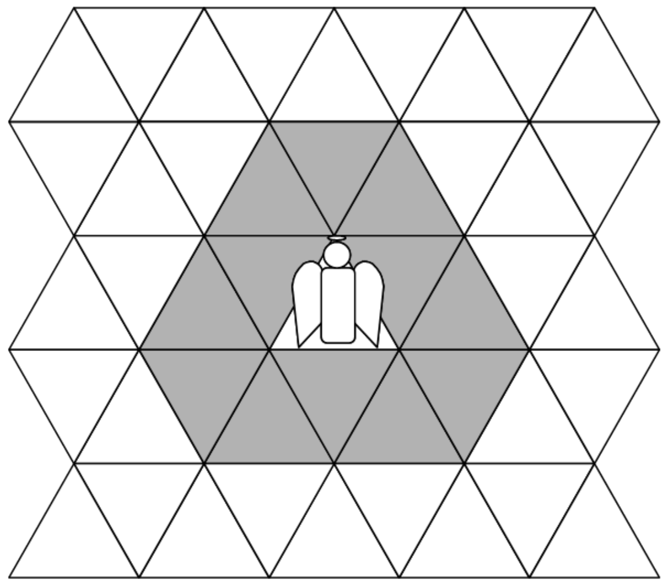

Figure 5: A king wins on the triangular lattice.

that the intersection of any two distinct triangles is either empty, a common vertex of the two, or a common side. We also require it to be locally finite in the sense that a bounded subset of the plane intersects only finitely many of the triangles (see Figure 6).

In the following, the triangles of a given triangulation are called *cells*. Two cells are *weakly adjacent* if they share at least a vertex. A *king* is defined as an angel that can move from a cell to any weakly adjacent cell (which is how kings move in chess). Hence the angel of Figure 5 is a king. In the angel-and-devil game, the king moves between cells, and on each turn, the devil eats a cell of the triangulation. In principle, a king is allowed at a move to remain in its present cell, although a simple strategy stealing argument shows that this is never necessary.

Example 3.1. It is easy to construct a triangulation of the plane such that the devil wins regardless of the king's starting position. Take three straight lines extending from the origin at 120° angles. Let $p_0 = q_0 = r_0$ be the origin, and let p_i, q_i and r_i for positive integer i be the points at distance i from the origin on the three lines. Now connect p_i to p_{i+1}, q_i to q_{i+1} and r_i to r_{i+1}. Moreover, connect p_i, q_i and r_i pairwise. Finally complete the triangulation

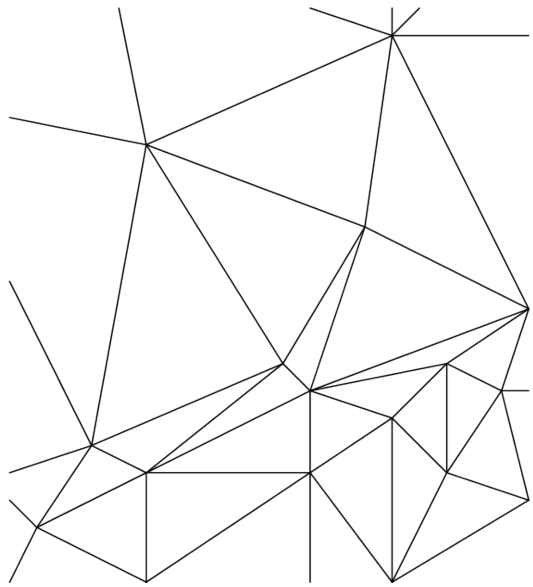

Figure 6: A triangulation of the plane.

by drawing an arbitrarily chosen diagonal in each of the quadrilaterals (see Figure 7).

It is clear that a king can only increase its distance to the origin by 1 in each move, and no matter how far away its starts, the devil can shut a layer in 6 moves, trapping the king in a finite region.

In fact there is a uniform bound on the number of moves it takes for the devil to kill the king. It takes the devil 6 moves to fill a layer. This means that the king can always be trapped in a region containing only 13 layers. When the devil has filled these and the two adjacent layers, a total of 15 layers, the king cannot move. Regardless of starting point, the king can never make more than 90 moves against this strategy.

Notice that the triangulation of Figure 7 has some peculiar properties compared to the regular triangulation. For instance there is no upper bound on the lengths of the sides of the cells.

Example 3.2. It is also easy to construct an example where the king wins. As is indicated in Figure 8, it is possible to embed an infinite binary tree in a locally finite triangulation of the plane.

Hence for general triangulations of the plane, sometimes the devil wins

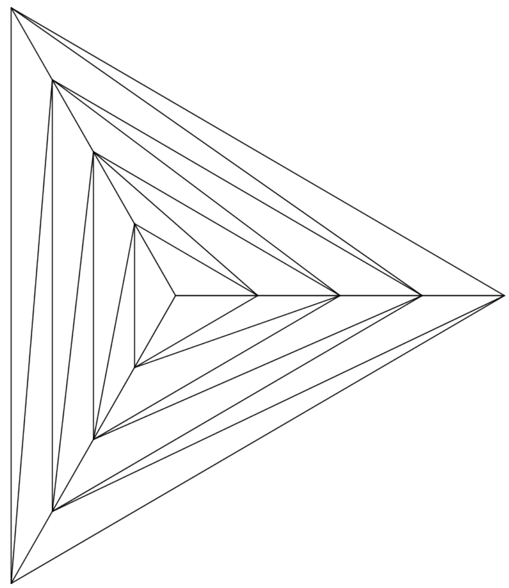

Figure 7: A triangulation where the devil wins.

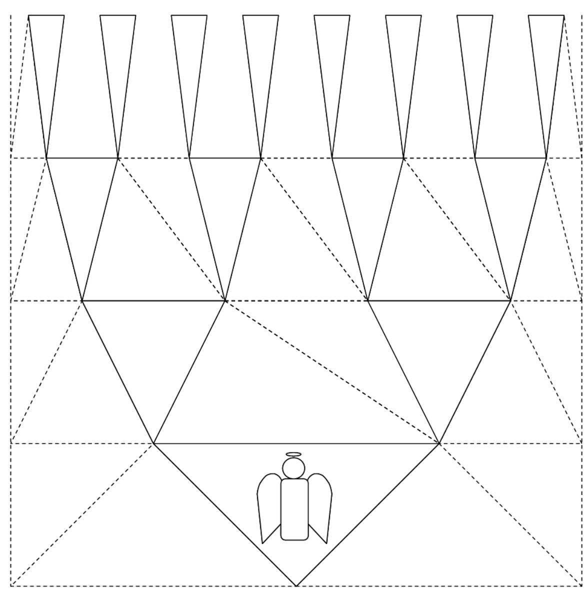

Figure 8: A triangulation where the king wins.

and sometimes the angel wins. In a binary tree, the number of cells that the angel can reach in n moves grows exponentially in n, while in the original problem, this number is only quadratic in n. Therefore a mapping of the binary tree to the original chess board will not show anything relevant to the original problem.

4 The runner strategy

We describe a strategy for the king on an arbitrary triangulation of the plane. We call it the *runner* strategy since it is similar to the strategy of [13].

Definition 4.1. A set A of cells is *connected* if for any two cells b and c in A there is a sequence $b = a_0, a_1, \ldots, a_n = c$ of cells in A such that for $0 \leq i < n$, a_i and a_{i+1} have an edge in common.

The runner strategy requires that at every moment, the king is in a cell that has an eaten cell next to it. The position and future strategy of the king is determined by the edge that separates the cell where the king stands from the particular eaten cell next to it. Notice that since we assume that the devil is nice, the cell where the king is standing will never be eaten.

At a given moment, consider the set of all edges that separate an eaten cell from an uneaten one. Let us give these edges an orientation such that a connected group of eaten cells is surrounded by a cycle oriented counter-clockwise. The king's strategy consists in choosing the next edge in this cycle, see Figure 9.

Notice that since the two oriented boundary edges have a point in common, such a move is always a legal king's move. Also notice that the cell to which the king moves may have several eaten cells next to it, and in any case the devil eats another cell before the next move, so the king has to remember which boundary edge it is moving along. Also notice that this strategy may require the king to stay in its present cell (although if we want to avoid this, we can modify the strategy by instead letting the king move along the boundary until it reaches another cell, or simply as long as possible).

In order for the runner strategy to be well-defined, we must modify the game so that there is an eaten cell next to the starting position. When the runner strategy is applied, we shall always assume that some set of cells are already eaten at the start of the game. Obviously this cannot help the king to escape.

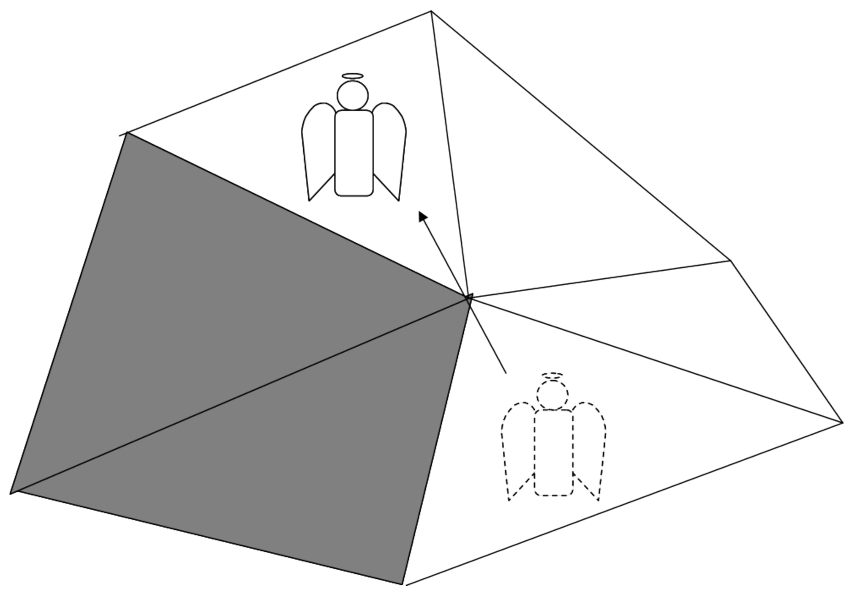

Figure 9: The runner strategy. Dark cells have been eaten by the devil.

Lemma 4.2. *Suppose the king plays according to the runner strategy. If a nice devil captures the king in a finite region, then the king will eventually return to the starting point, facing in the same direction.*

Proof. If the devil traps the king in a finite region, then at some point the king has to return to where it has previously been, facing in the same direction (a nice devil never actually kills the king, it can always go back to where it has previously been). Consider the moment when this happens for the first time, and call this position a. Suppose that the statement in the lemma is not true. Then the king must have arrived to position a from a position b, and later from another position c (see Figure 10).

In order for this to be possible, the devil must have eaten the cell at position b before the king moved from position c. This means that the devil has made an ambush, contradicting the hypothesis. $\square$

Definition 4.3. The *perimeter* of a finite set A of cells is the number of edges that separate a cell in A from a cell not in A.

Lemma 4.4. *A connected set of n triangular cells has perimeter at most $n + 2$.*

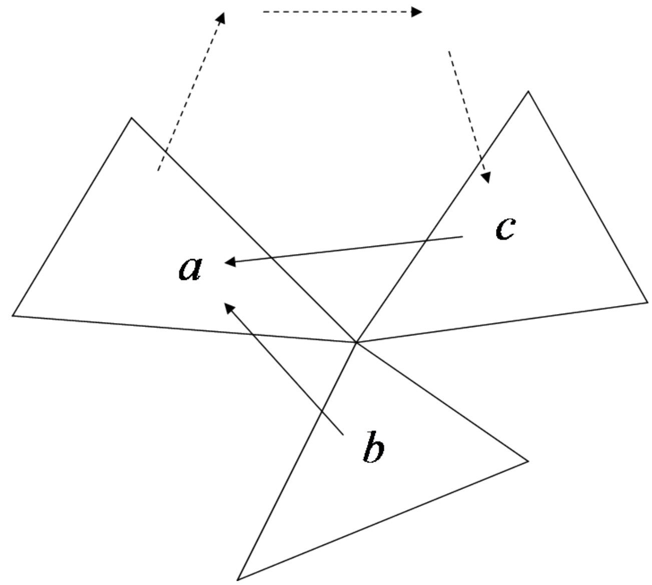

Figure 10: The running king cannot reach a from two distinct positions.

Proof. This follows by induction. A connected set can be built by adding cells one by one, retaining connectivity. Each time a cell is added, the perimieter increases by at most 1. □

Theorem 4.5. *Let A be a finite set of cells. Then there is some starting point from which the king is guaranteed against a nice devil either to escape to infinity, or trace around the set A in counter clockwise direction, returning to its starting point.*

Proof. Without loss of generality we can assume that A is connected. Let B be a superset of A of minimal perimeter. Obviously B too is connected. Suppose that all the cells in B are eaten at the start of the game, and let the king start in a cell adjacent to B and apply the runner strategy. Suppose, for a contradiction, that the nice devil traps the king in a finite region, and that it stops the king from enclosing B. Then the devil must have eaten a connected set Z of cells forming a "handle" attached to B, see Figure 11 (possibly the devil has also eaten some other cells).

If the king makes n moves before returning to its starting point, then the inner component of the boundary of $B \cup Z$ must have length n. By the minimality of B, the outer component of the boundary of $B \cup Z$ has length

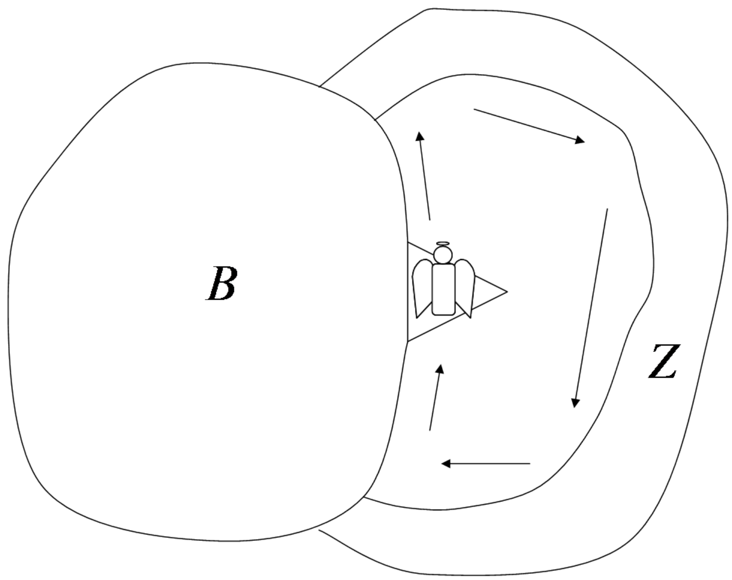

Figure 11: An impossible situation. The king either escapes to infinity or walks around B.

at least equal to the perimeter of B. This implies that the perimeter of Z is at least $n+4$. By Lemma 4.4, the devil must have eaten at least $n+2$ cells. This is cheating, since the king has made only n moves. □

Let us define the distance between two distinct cells as the maximum euclidean distance between their points.

Theorem 4.6. *For each triangulation and each real number x, there is some starting point from which the king is guaranteed to reach some cell at distance at least x from where it started.*

Notice that if we would define the distance between two cells as the minimum distance between their points, then Theorem 4.6 would be false. The counter-example is again given by Example 3.1. On the other hand the definition of distance between cells is somehow not important: If there is an upper bound on the sides of the cells, then the conclusion holds regardless of how distance between cells is defined, while if there is no such upper bound, the statement is trivial.

Proof of Theorem 4.6. By Lemma 2.2, it is sufficient to prove the statement assuming that the devil is nice. Take a disk of diameter x and let A be the set of cells that intersect this disk. if the king's path encloses A, then at some point the king has to be at distance at least x from where it started. Hence Theorem 4.6 follows from Theorem 4.5. □

Corollary 4.7. If there is an upper bound on the side lengths of the cells, then for each n there is a starting point from which the king can make at least n moves without being trapped by the devil.

This means that a triangulation with the properties of Example 3.1 is possible only if the side lengths of the cells can be arbitrarily large. Also notice that even if the side lengths are bounded we cannot conclude that there is a starting point from which the king can win the angel-and-devil game. A counter-example is constructed as follows: Draw the lines $|x| = |y|$, for each n the square $\max(|x|, |y|) = n$, and for every n the set of eight rays given by $\min(|x|, |y|) = 17n$. Then put an arbitrary diagonal in each of the quadrilaterals. Let *layer* n be the points (x, y) in the plane for which $n - 1 \leq \max(|x|, |y|) \leq n$. Layer $17n$ is partitioned into $16n$ triangles. If the king starts in a cell in layer n, then in $16n$ moves, the devil eats the entire layer $17n$. When this is finished, the king has made only $16n - 1$ moves, and is therefore still inside layer $17n$.

On the other hand we can conclude the following:

Corollary 4.8. If the cells can be finitely colored so that the symmetries of the triangulation act transitively on each color class, then some color class consists of winning starting positions for the king in the angel-and-devil game.

Proof. This follows from Corollary 4.7 by compactness. □

By tiling the plane with a finite part of Example 3.1 we see that the conclusion of Corollary 4.8 need not hold for all color classes.

Corollary 4.9. On the regular triangular lattice, the king wins the angel-and-devil game.

5 A winning angel on the square lattice

By the 1-1 mapping between the regular triangular and the regular square lattice shown in Figure 12, the king of the triangular lattice is transformed to

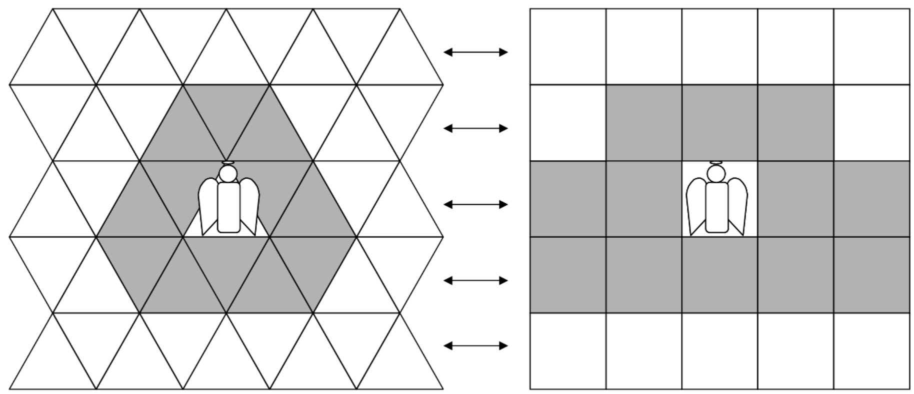

Figure 12: A 1-1 mapping.

a strength 12 winning angel on the square lattice. The pattern of movement of this angel is bounded but not translation invariant. With the standard coloring of the chess board, this angel will move as in Figure 13(a) from squares of one color, and as in Figure 13(b) from squares of the other color. It follows that the angel of Figure 13(c) that can move from the middle square to any other square of a 5 by 3 rectangle can escape from the devil. This angel of strength 14 is the weakest known translation invariant winning angel, but we believe that there exist considerably weaker winning angels.

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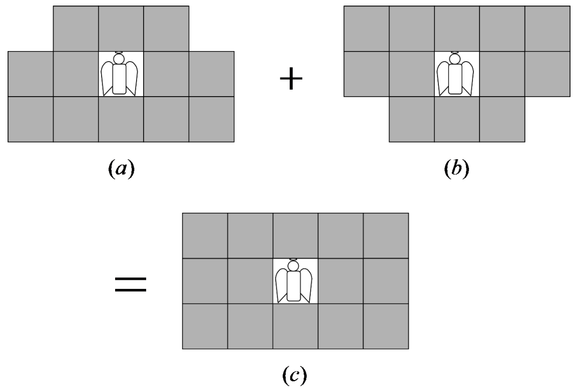

Figure 13: A winning angel on the square lattice.

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