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Model Search Among Multiplicative Models

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Summary

We propose a non-iterative model search technique to find simple patterns of association for several variables. Our selection procedure is restricted to multiplicative models, therefore all patterns under consideration are interpretable in terms of zero partial associations of variable pairs. We illustrate the selection technique on two sets of data, one in a contingency table, one in a covariance matrix.

1. Introduction

The interrelations among several variables can more easily be understood and communicated if they can be characterized by a pattern of association. Multiplicative models form one class of such patterns of association. There are some advantages in considering only multiplicative models in a model search procedure used to find well-fitting patterns for a given set of data: multiplicative models are not tied to a specific form of a distribution function; their interpretations are relatively easy because in each pattern or model one can distinguish variables that belong together from variables that can be separated. Furthermore, in a selection procedure among multiplicative models no iterative fitting algorithms are needed.

Multiplicative models have previously been discussed for contingency tables by Darroch [1962], Bishop [1969], Goodman [1970] and Wermuth [1976], and for covariance matrices by Wermuth [1976]. In the case of a multinomial distribution, multiplicative models are a subclass of log-linear models (Birch [1963]), and in the case of a multivariate normal distribution they are a subclass of covariance selection models (Dempster [1972]). Model search procedures that are not restricted to multiplicative models—hence techniques requiring iterative fitting algorithms—have for instance been proposed by Goodman [1971] for contingency tables and by Dempster [1972] for covariance matrices.

2. Definitions and Notation

We speak of a multiplicative model if the joint distribution of several variables can be factored into the marginal distributions of subgroups of variables (and vice versa). In order to characterize such a model we may either write out how the joint distribution is factored, or we can use some condensed notation. Suppose, for instance, that the joint distribution of five variables can be factored into the marginal distributions of variables (145), (245), (345) and (45), as in

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$$f(x_1x_2x_3x_4x_5) = \frac{f(x_1x_4x_5)f(x_2x_4x_5)f(x_3x_4x_5)}{f(x_4x_5)f(x_4x_5)}, \quad (1)$$

where f denotes probability functions or probability density functions. Then, if we concentrate only on the indices of the five variables, (1) becomes

$$(12345) = \frac{(145)(245)(345)}{(45)(45)}. \quad (2)$$

A condensed notation for this multiplicative model is 145/245/345, that is a list of the variable groups in the numerator of (2) separated by dashes. This list of variable groups shows the variables that belong together, and implicitly the variable pairs that have been separated. In our example the latter are the pairs (1, 2), (1, 3) and (2, 3). Each of these pairs has zero partial association (z.p.a.), that is, each variable pair is conditionally independent given the remaining three variables.

The familiar interpretation for model 145/245/345 is that the variables 1, 2 and 3 are independent given the joint variable 45. They are independent after the influence of variables 4 and 5 has been removed. This interpretation as well as the interpretation of any other multiplicative model may be derived from the independencies that are implied by the number and the constellation of z.p.a.'s (Wermuth [1976]). Generally, the more z.p.a.'s required for a model, the simpler is the interpretation of the resulting pattern of association.

Without giving proofs we now list and illustrate some properties of multiplicative models that make them attractive for a model search procedure:

1) Each multiplicative model can be characterized by its z.p.a.'s, that is by its conditionally independent variable pairs;

2) Each multiplicative model may be conceptually derived by eliminating in a stepwise manner the partial associations among variable pairs;

3) If two multiplicative models differ such that the second model has exactly one additional z.p.a., then in the second model the corresponding variable pair is conditionally independent not only in the joint distribution of all variables but also in some marginal distribution.

For instance, to obtain model 145/245/345 we could successively eliminate the partial associations for the variable pairs (1, 2), (1, 3) and (2, 3). This would yield model 1345/2345 in the first step, model 145/2345 in a second step and model 145/245/345 in a third step. For such a stepwise elimination in general, it is convenient to use the index combination notation introduced in (2) as well as the following rule:

Call (ij) the indices of the variable pair that is to have z.p.a. Now look at the index combinations of the previous step: If (ij) is contained in an index combination of the denominator, then the resulting pattern will not be a multiplicative model. Otherwise, pick out that index combination in the numerator that includes (ij). Call this combination (ijK) such that K denotes all indices in the combination except for i and j. Then replace (ijK) by (iK) (jK) in the numerator and (K) in the denominator and cancel.

Applying this rule to our example we obtain

Step 1

z.p.a. for (1, 2):

$$(12345) \rightarrow \frac{(1345)(2345)}{(345)}$$

Step 2
z.p.a. for (1, 3) after (1, 2):

$$\frac{(1345)(2345)}{(345)} \rightarrow \frac{(145)(2345)}{(45)}$$

Step 3
z.p.a. for (2, 3) after (1, 2), (1, 3):

$$\frac{(145)(2345)}{(45)} \rightarrow \frac{(145)(245)(345)}{(45)(45)}$$

We now look at the changes in index combinations at each step. We have

Step 1
z.p.a. for (1, 2):

$$(12345) \rightarrow \frac{(1345)(2345)}{(345)}$$

Step 2
z.p.a. for (1, 3):

$$(1345) \rightarrow \frac{(145)(345)}{(45)}$$

Step 3
z.p.a. for (2,3):

$$(2345) \rightarrow \frac{(245)(345)}{(45)},$$

and we see that the additional z.p.a. at each step is equivalent to the conditional independence of the variable pair within the joint distribution of a subset of all variables, that is within some marginal distribution. If we define i, j and K as in the "rule," the change in models at two successive steps may be described as

$$\text{z.p.a. for } (i, j)\text{: } (ijK) \rightarrow \frac{(iK)(jK)}{(K)}. \tag{3}$$

Because of this simple relationship a test for an additional z.p.a. will always be a test for conditional independence in some marginal distribution, and likelihood-ratio test statistics for an additional z.p.a. will always be of the same form.

The precise test statistics in the case of a multivariate normal distribution are

$$\chi^2 = -2 \ln \left[\frac{D_{iK}D_{jK}/D_K}{D_{ijK}}\right]^{-n/2} \tag{4}$$

where, for instance, D_{ijK} or D_K denote determinants of observed correlation matrices with variables (ijK) or (K), respectively; n is the sample size. These tests will always have one degree of freedom (d.f.).

The test statistics in the case of a multinomial distribution with observed cell counts n_{ijK}[1] will be

[1] We hope that the the double meaning of i, j, K as indices of variables and as running indices for variable categories is not confusing. We accept this ambiguity in order to emphasize the similarities of the test statistics in (4) and (5).

$$\chi^2 = -2 \ln \pi \left[\frac{n_{i.K} n_{.jK} / n_{..K}}{n_{ijK}} \right]^{n_{ijK}} \tag{5}$$

where $n_{i.K} = \sum_j n_{ijK}$, $n_{..K} = \sum_{ij} n_{ijK}$. These tests have $(I_i - 1)(I_j - 1) \prod_{r \epsilon K} I_r$ d.f. with I_l as the number of categories for the lth variable.

With these definitions and with the "rule" we can formulate the backward selection.

3. *A Backward Selection Procedure*

A model search like the one proposed here is appropriate when little is known about the interrelations among several variables. In such a situation no specific hypotheses that could be tested are available yet. The main interest lies in finding a condensed description of possibly complex interrelations and in generating hypotheses that later on can be tested on a different set of data.

The rationale of the proposed model search is simple: we wish to determine in a stepwise manner how many and which variable pairs can be assumed to have z.p.a. given the evidence of the data. For the reasons stated previously we do not consider all possible models with z.p.a.'s but only the multiplicative ones. The most complex type of pattern for p variables is a model with exactly one z.p.a. The simplest model is the one with $\binom{p}{2}$ z.p.a.'s, that is the model with mutual independence of all p variables. For a systematic description of all possible multiplicative models in the case of four variables compare for instance Bishop [1971] or Wermuth [1976].

In a first step of the selection procedure we try to find a variable pair for which the assumption of z.p.a. is consistent with the data. Thus, for all $\binom{p}{2}$ variable pairs the likelihood-ratio test statistics for exactly one z.p.a. are computed. The test statistic with the highest probability is then selected, that is, the variable pair with the smallest partial association is selected to have zero partial association. If its likelihood-ratio test statistic is significant, then the decision at the first selection step is that no simple pattern fits the data. The selection then stops. In the case of an insignificant test statistic we proceed to find—among all variable pairs still available—that pair with the smallest additional z.p.a. To compute the appropriate test statistics for each additional z.p.a. we use the "rule." Again, the test statistic with the highest probability is selected, and the selection process is terminated if this test gives a significant result.

To evaluate the significance of a result in a model search procedure a number of different rules are available. A survey of such rules and of their effects on the selection results in regression analysis may be found in Dempster, Schatzoff and Wermuth [1976]. In the examples described below we speak of significance whenever the test statistic exceeds the 95 percent quantile of the corresponding chi-square distribution. Experience shows that different rules will frequently lead to the same decisions on well-fitting patterns of association.

Another problem connected with search techniques is well known from regression analysis: in general, it cannot be expected that a backward selection technique will lead to the same result as a forward selection technique. Similarly, the technique described here will not necessarily select the same patterns of association as a model search not restricted to multiplicative models. In any case, a selected pattern of association should be judged on how plausible its interpretation seems to the investigator.

Table 1
OBSERVED CORRELATIONS FOR FIVE MATURITY INDICATORS
(LOWER HALF: MARGINAL; UPPER HALF: PARTIAL)

Variables	1	2	3	4	5
1	1.0000	0.1401	0.1273	0.1345	0.2599
2	0.4314	1.0000	0.3450	0.0925	-0.0263
3	0.5146	0.6263	1.0000	0.6213	0.1507
4	0.4891	0.5466	0.7830	1.0000	0.0232
5	0.4112	0.2604	0.3926	0.3433	1.0000

4. Non-iterative Backward Selection in the Case of a Multivariate Normal Distribution

To illustrate the backward selection in a covariance matrix we use data from the prospective study, "Pregnancy and Child Development," initiated by the Deutsche Forschungsgemeinschaft (German Research Society) in 1964.

We are interested in the question: How many and which indicators are necessary to obtain a good picture of an infant's maturity? Available is information on five indicators for $n = 2{,}473$ male infants. The data contain no outlying or missing values; the five variables are

1 = *length of gestation* (*in days*)
2 = *head circumference* (*in mm.*)
3 = *birth weight* (*in gm.*)
4 = *length at birth* (*in mm.*)
5 = *constructed indicator*

The last variable contains only integer values ranging from 0 to 18. It is constructed from the Apgar score, length of fingernails, the amount of lanugo, and from similar information (Koller [1974]). An inspection of normal probability plots revealed that the data on the first four variables can certainly be regarded as samples from normal distributions, while for the fifth variable this assumption was still roughly justified.

From the marginal correlations in Table 1 it is difficult to detect any simple structure in the data. However, from the partial correlation coefficients (given the remaining three variables), we see that the variable pairs (2, 4), (2, 5) and (4, 5) are most likely to have z.p.a.'s. Table 2 shows the result of several backward selection steps.

At the first step, the smallest non-significant chi-square statistic (on one d.f.) is 1.34: it is the likelihood-ratio test result for z.p.a. between variables 4 and 5. It is computed as

$$-\ 2{,}473\ [\ln D_{12345} - (\ln D_{1234} + \ln D_{1235} - \ln D_{123})]$$

$$= 2{,}473\ [2.0586 - (1.8171 + 1.0736 - .8326)] = 1.34,$$

where, for instance, D_{1234} denotes the determinant of the observed correlation matrix with variables (1234).

Table 2
MODEL SEARCH FOR THE FIVE MATURITY INDICATORS

	Step 1	Step 2		Step 3	
Variable pair	χ^2-statistic for z.p.a.	sub-matrix	χ^2-statistic for z.p.a.	sub-matrix	χ^2-statistic for z.p.a.
(1,2)	55.62	--*)	--	1234	54.33
(1,3)	40.38	--	--	--	--
(1,4)	45.13	1234	52.96	1234	52.96
(1,5)	173.02	1235	180.85	135	180.41
(2,3)	313.39	--	--	1234	313.58
(2,4)	21.26	1234	21.00	1234	21.00
(2,5)	1.71	1235	1.45	X	X
(3,4)	1206.43	1234	1262.04	1234	1262.04
(3,5)	56.82	1235	112.44	135	136.31
(4,5)	1.34	X*)	X	X	X
selected pattern	1234/1235	1234/135		123/134/135	

*) X means that the corresponding variable pair was in a previous step selected to have zero partial association

-- means that the corresponding pattern requires iterative fitting

In the second selection step we test for z.p.a. of an additional variable pair. For instance, to test that $\rho_{14.235} = 0$ given that $\rho_{45.123} = 0$ is the same as testing whether the pair (1, 4) is conditionally independent in the submatrix with variables (1234). In other words, to test $\rho_{14.235} = 0$, the likelihood-ratio test statistic can be computed as

$$52.96 = -2{,}473\,[\ln D_{1234} - (\ln D_{123} + \ln D_{234} - \ln D_{23})].$$

The decision at this second step of the selection is that the partial association for variable pair (2, 5) does not differ significantly from zero. This decision is equivalent to accepting pattern 1234/135. At step three the smallest chi-square statistic has a value of 21.00 for pair (2, 4). Since this result is significant even at a .01 level, the selection stops.

One interpretation for pattern 1234/135 is the following: given information on the length of gestation (1) and on birth weight (3), the constructed indicator (5) is independent of the two indicators length at birth (4) and head circumference (2). In other words, to judge the maturity of infants with the same length of gestation and with the same birth weight, the constructed indicator gives information about the maturity that cannot be obtained from the indicators length at birth and head circumference.

In Table 3 we display the estimated correlations for model 1234/135. Except for the two variable pairs (2, 5) and (4, 5) all marginal correlations agree with the observed correlations in Table 1. For the same two variable pairs the partial correlations have been forced to zero.

Table 3
CORRELATIONS IMPLIED BY PATTERN 1234/135
(LOWER HALF: MARGINAL; UPPER HALF: PARTIAL)

Variables	1	2	3	4	5
1	1.0000	0.1423	0.1261	0.1406	0.2593
2	0.4314	1.0000	0.3409	0.0920	0.0000
3	0.5146	0.6263	1.0000	0.6245	0.1554
4	0.4891	0.5466	0.7830	1.0000	0.0000
5	0.4112	0.2763	0.3926	0.3319	1.0000

5. *Non-iterative Backward Selection in the Case of a Multinomial Distribution*

To illustrate the backward selection in a contingency table, we use data from Coppen [1966], previously analyzed by Lienert [1971]. The type of interrelations among several symptoms on psychiatric patients is of interest. Available is information for 362 patients on the following symptoms:

1 = *validity* (*psychasthenic* −; *energetic* +)
2 = *solidity* (*hysteric* −; *rigid* +)
3 = *stability* (*extroverted* −; *introverted* +)
4 = *acute depression* (*no* −; *yes* +).

Table 4 shows the observed cell counts n_{ijkl} for each of the symptom combinations as well as the expected cell counts for pattern 13/14/24. This pattern was the result of the backward selection procedure displayed in Table 5.

At the first step of the selection, variable pair (2, 3) was selected to have z.p.a., the corresponding chi-square statistic was computed as

$$2[(\sum n_{ijkl} \ln n_{ijkl}) - ((\sum n_{ij.l} \ln n_{ij.l}) + (\sum n_{i.kl} \ln n_{i.kl}) - (\sum n_{i..l} \ln n_{i..l}))] = 3.93.$$

Denote the number of categories for each variable as $I_1 = 2$, $I_2 = 2$, $I_3 = 2$, $I_4 = 2$, then the degrees of freedom for z.p.a. of (2, 3) given variables 1 and 4 are

$$(I_2 - 1)(I_3 - 1)I_1I_4 = 4.$$

At the second step, we test for z.p.a. of an additional variable pair. After pattern 124/134 is already accepted, the test, for instance, for additional independence of variables 1 and 2 given variables 3 and 4 jointly, is the same as the test for conditional independence of variables 1 and 2 given variable 4 alone. Hence, the likelihood-ratio test statistic can be computed from subtables as

$$5.49 = 2[(\sum n_{ij.l} \ln n_{ij.l}) - ((\sum n_{i..l} \ln n_{i..l}) + (\sum n_{.j.l} \ln n_{.j.l}) - (\sum n_{...l} \ln n_{...l}))].$$

The degrees of freedom for this statistic are $(I_1 - 1)(I_2 - 1)I_4 = 2$.

At step four no further zero partial associations can be accepted, since the smallest chi-square statistic is significant even at a .01 level of significance. Therefore, we accept pattern 13/14/24 as the simplest pattern of association consistent with the data.

Table 4

CELL COUNTS ON FOUR SYMPTOMS FOR PATIENTS RECEIVING PSYCHIATRIC TREATMENT

Symptom-combinations 1 2 3 4	Observed cell counts	Expected cell counts for pattern 13/14/24
+ + + +	15	16.785
- + + +	30	36.716
+ - + +	9	13.241
- - + +	32	28.965
+ + - +	23	17.315
- + - +	22	19.185
+ - - +	14	13.659
- - - +	16	15.135
+ + + -	25	22.305
- + + -	22	15.557
+ - + -	46	42.670
- - + -	27	29.762
+ + - -	14	23.009
- + - -	8	8.129
+ - - -	47	44.017
- - - -	12	15.551

This means that we assume the variable pairs (1, 2), (2, 3) and (3, 4) to have z.p.a.'s, and that the marginal associations of the symptoms (1, 3), (1, 4) and (2, 4) are sufficient to explain the interrelations among all four symptoms. One possible interpretation of pattern 13/14/24 is the following: Patients in an acute depression (4+) and convalescents (4−) are two heterogeneous groups with respect to the symptoms validity and solidity, and therefore the two groups should be considered separately (Koller [1964]). When we do this, we recognize that solidity is independent of the symptoms validity and stability, jointly. Furthermore, the association between these latter two symptoms (1, 3) is similar whether we look at it within each of two groups of patients (4+, 4−) or within the collective of all patients.

Table 6 shows the observed marginal associations as well as those implied by pattern 13/14/24.

Again it can be seen that the implied marginal associations coincide with the observed marginal associations except for the selected variable pairs (1, 2), (2, 3) and (3, 4).

6. *A Disadvantage of Multiplicative Models*

We restricted the model search to multiplicative models. The advantages were that the test statistics can readily be computed without evaluating maximum-likelihood estimates, implied marginal associations can be expressed in a closed form (Wermuth [1976]), and the interpretation of the resulting patterns is relatively easy. Nevertheless, there are situations in which these advantages are outweighed by the danger of misinterpreting the evidence in the data.

Table 5
MODEL SEARCH FOR THE FOUR SYMPTOMS

	Step 1		Step 2			Step 3			Step 4		
Variable pair	χ^2-statistic for z.p.a.	d.f.	sub-table	χ^2	d.f.	sub-table	χ^2	d.f.	sub-table	χ^2	d.f.
(1,2)	4.78	4	124	5.49	2	124	5.49	2	X	X	X
(1,3)	12.87	4	134	13.58	2	13	10.02	1	13	10.02	1
(1,4)	33.00	4	-*)	-	-	124	30.80	2	14	28.03	1
(2,3)	3.93	4	X*)	X	X	X	X	X	X	X	X
(2,4)	22.38	4	124	19.73	2	124	19.73	2	24	16.97	1
(3,4)	7.64	4	134	4.99	2	X	X	X	X	X	X
selected pattern	124/134		124/13			13/14/24			3/14/24		

*)X means that the corresponding variable pair was in a previous step selected to have zero partial association

\- means that the corresponding pattern requires iterative fitting

d.f. means degrees of freedom

Consider the following example on the interrelation among only three variables in a $3 \times 2 \times 9$ contingency table: smoking habits of fathers (variable 1), perinatal mortality of the infant (variable 2) and clinics (variable 3). Again the data are taken from the "Pregnancy and Child Development" study.

An increased danger of perinatal mortality in those cases where the fathers are heavy cigarette smokers has been reported by Mau and Netter [1974]. The corresponding data are shown in Table 7. For our purposes we ignore the difficulty of explaining the seemingly protective effect of smoking one to ten cigarettes.

Since both perinatal mortality rates and smoking habits of fathers differ with the nine clinics, it is possible that there is no real association between smoking habits and perinatal mortality if one controls for the clinic heterogeneities.

Table 6
MARGINAL ASSOCIATIONS FOR THE FOUR SYMPYOMS
(LOWER HALF: OBSERVED; UPPER HALF: IMPLIED BY PATTERN 13/14/24)

Variables	1	2	3	4
1	-	0.0598	0.1658	0.2767
2	0.0867	-	0.0099	0.2160
3	0.1658	0.0170	-	0.0458
4	0.2767	0.2160	0.0631	-

Table 7
PERINATAL MORTALITY AND SMOKING HABIT OF THE FATHER

Number of cigarettes per day	Perinatal mortality %	from
none	3.0	1995
1 to 10	2.2	861
more than 10	4.2	1204

A test for the independence of variables 1 and 2 after controlling for a clinics effect is the test for model 13/23. It has 18 d.f. and a chi-square value of 25.95, hence gives an insignificant result. Therefore we could be led to believe that the observed association between variables 1 and 2 in the collective of all patients is due to a mere clinic's effect. But a more careful analysis (compare Bishop [1971]) which requires the fitting of an iterative model (pattern 12/13/23) reveals the contrary. The test for no three-factor interaction, the test for pattern 12/13/23, has 16 d.f. and a chi-square statistic of 18.59. After this pattern has been accepted, the two-factor interaction between smoking habit and perinatal mortality can no longer be assumed to be zero (the conditional test for the interaction between variables 1 and 2 has one d.f. and a value of 7.33).

Table 8
PERINATAL MORTALITY AND SMOKING HABIT OF THE FATHER IN NINE CLINICS

Number of cigarettes per day	Perinatal mortality					
	Clinic 1		Clinic 2		Clinic 3	
	%	from	%	from	%	from
none	3.1	160	3.7	109	2.2	405
1 to 10	0.0	51	1.6	63	1.8	164
more than 10	10.2	44	3.2	95	2.3	270
	Clinic 4		Clinic 5		Clinic 6	
	%	from	%	from	%	from
none	3.7	188	2.5	160	3.9	129
1 to 10	0.0	77	2.1	97	0.0	56
more than 10	6.6	151	3.8	104	5.7	105
	Clinic 7		Clinic 8		Clinic 9	
	%	from	%	from	%	from
none	1.1	176	3.0	538	6.7	120
1 to 10	3.3	60	3.0	231	6.5	62
more than 10	0.9	114	4.5	222	6.4	94

This example shows that in contingency tables it might be necessary to split the overall test for z.p.a. into different components to avoid misinterpretations. Thus, the backward selection procedure described above should not be used if the investigator wants to test a specific hypothesis, such as the question of clinic heterogeneities.

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Recherche D'un Modele Parmi des Modeles Multiplicatifs

Résumé

Nous proposons une technique non-itérative de recherche du modèle pour trouver des structures simples d'association de plusieurs variables. Notre procédé ne s'applique qu'aux modèles multiplicatifs, ce qui nous permet d'interpréter toutes les structures étudiées en termes d'associations partielles de paires de variables. Nous présentons l'application de notre technique à deux ensembles de données, l'un dans une table de contingence, l'autre dans une matrice de covariances.

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