# Sequences of regressions and their dependences

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**ABSTRACT***: In this paper, we define and study the concept of traceable regressions. These are sequences of regressions in joint or single responses for which a corresponding regression graph captures not only an independence structure but represents, in addition, conditional dependences that permit the tracing of pathways of dependence. We give the properties needed for transforming these graphs and graphical criteria to decide whether a path in the graph induces a dependence. The much stronger constraints on distributions that are faithful to a graph are compared to those needed for traceable regressions.*

*Key words:* Chain graphs, Edge-matrix calculus, Faithfulness of graphs, Graphical Markov models, Independence axioms, Regression graphs, Traceable Regressions.

## 1 Introduction and motivation

**Single and joint response regressions.** Sequences of regressions are arguably the most important statistical tool in observational and interventional studies for investigating pathways of dependences and hence development over time. In each regression, one distinguishes ***response*** variables and ***regressor*** variables; with responses depending on the regressors.

In applications, the substantive context determines which variable pairs are modeled by a conditional independence and which are taken to be dependent because they are needed in a generating process of the joint distribution. Suppose one regressor is a risk factor for a response, then quite different sizes of dependence strength will be relevant if this response is the occurrence of a common cold, or the infection with an HIV virus or an accident in a nuclear plant, since the prevention of these risks is judged to be of quite different importance.

There may be single or joint responses, where only the latter permit to model simultaenously occurring effects of an intervention. Components of joint responses may be discrete or continuous random variables or be mixed of both types. Typically, a subset of variables is taken as given, possibly determined by study design, and its components are named ***context variables*** since they describe the context or background or the basic features of individuals under study.

The generated joint density factorizes into an ordered sequence of conditional densities of the responses, which we call shortly regressions, and into a joint marginal density of the context variables. Under mild conditions, estimation of sequences of regressions can be decomposed into separate tasks for each response component of the factorization, using well-developed tools such as linear or logistic regressions or conditional Gaussian regressions, which permit joint responses to be mixed of discrete and continuous component variables; see Lauritzen and Wermuth (1989), Edwards and Lauritzen (2001). Tailored to the requirements in many specific situations, special results are available to estimate the form and parameters of univariate and joint conditional distributions.

However, many consequences of sequences of regressions can already be derived if one does not know or estimate the involved parameters but just uses an associated graph and properties of graph transformations. Relevant, important results concerning independences in sequences of regressions have been obtained only recently; see Sadeghi and Lauritzen (2012) and Wermuth and Sadeghi (2012). The additional properties needed to draw conclusions about induced dependences are set out in this paper.

Sequences of regressions are an essential part of longitudinal studies, named also cohort or panel studies in medical, economic and social science research. Prominent examples are the Framingham heart study, the European Community household panel or the Swiss HIV cohort study. By using regression graphs, it will become possible to simplify analyses and interpretations of sequences of regressions and to directly compare dependences arising in different types of sequences of regressions for the same set of variables, or in sequences of regressions for subsets of variables studied for subpopulations. The results in this paper prepare for these possibilities in applications.

**Independences and dependences given by regression graphs.** Sequences of univariate, that is of single-response regressions, have been represented by ***directed acyclic graphs***. With regression graphs, directed acyclic graphs are extended by including two types of undirected graph, one for joint responses, the other for joint context variables. Nodes of the graph represent random variables. Distinct node pairs are coupled by at most one edge so that a regression graph is one type of what in graph theory are called ***simple graphs***. Each missing edge of a regression graph corresponds to a conditional independence where the conditioning set depends on the type and position of the missing edge, the graph is therefore also one type of **independence graph**.

Properties or axioms for combining independence statements have been studied by Dawid (1979) and Pearl (1988). Their connections to graphs have been discussed and modified in information theory; see Studený (2005) and Lněnička and Matúš (2007). Different types of extensions have been proposed in the computer science literature; see Castillo et al. (1997), Flesch and Lucas (2007). But, for instance, by requiring a property called strong transitivity, one excludes even the whole family of regular joint Gaussian distributions. By contrast, this family forms a subclass of what we introduce here as traceable regressions.

The ***independence structure*** of a graph is the set all independence statements implied by the graph. These are well-studied for regression graphs, but with important results obtained only recently. For instance, a proof by Sadeghi and Lauritzen (2012) implies equivalence of a ***pairwise Markov property***, that is of the set of independences attached to the missing edges of a given regression graph, to the ***global Markov property***, the criterion known to give all independence statements implied by the graph. For two regression graphs with identical node sets and with the same set of coupled node pairs but with different types of edge, there is a simple graphical criterion to decide whether the two graphs define nevertheless the same independence structure, that is whether they are ***Markov equivalent***; see Wermuth and Sadeghi (2012).

**Tracing pathways of dependence.** Much less is known about the dependence structures that can be captured by graphs. Since graphs do not distinguish between additive and interactive effects of regressor variables on responses, nor between linear and nonlinear types of dependences, it has been argued by Wermuth and Lauritzen (1989) that graphs may represent ***research hypotheses about dependent variable pairs needed to generate the joint distribution***. For this, each edge present in the graph indicates a conditional dependence, where the conditioning set depends on the type and position of the edge present, while the form of the dependence is not specified.

For tracing pathways of dependences, dependence-inducing sequences of edges of different type are the focus of interest, while independences just lead to simplified strengthened interpretations of the relevant dependences. In this paper, we set out the properties of traceable regressions and show, in particular that these properties impose mild constraints on the types of generated distribution in contrasts with serious constraints required in general for faithful distributions. This notion was introduced by Spirtes, Scheines and Glymour (1993) for distributions in which all independence statements hold that are implied by a graph and no others.

Tracing pathways of dependence goes back to the geneticist Sewall Wright (1889–1988), who introduced it in 1923 as ***path analysis*** for sequences of univariate linear regressions. He suggested to judge the goodness-of-fit of a research hypothesis, represented by a directed acyclic graph, by comparing observed correlations with those that are expected if the data had been generated over the graph. His rules for computing expected marginal correlations, trace all pathways that induce a dependence by marginalizing.

The extension of tracing pathways of dependences, when there is conditioning on variables in addition to marginalizing, became feasible after a first ***separation criterion*** had been formulated by Pearl (1988) and proven by Geiger, Pearl and Verma (1990) to give the global Markov property of directed acyclic graphs. When separation fails then there is at least one path in the directed acyclic graph that may induce a dependence by marginalizing over one subset of variables and conditioning on another set. Here, such a path is said to be edge-inducing since it leads to a transformed graph.

**Structure of the paper.** In Section 2, we introduce and discuss dependence base regression graphs and traceable regressions. Section 3 contains examples of tracing paths and of planning future follow-up studies on the same topic so that there are no paths distorting a generating dependence of interest. Small Gaussian families of distributions are used to illustrate independence properties of traceable regressions. In Section 4, several discrete families of distributions are given to show how the properties of traceable regressions can be violated. In Section 5, the known properties of an edge matrix calculus to transform graphs are collected first. These are used to derive new properties of transforming regression graphs and to distinguish traceable regressions from distributions that are faithful to regression graphs. A short discussion ends the paper.

## 2 Definitions and terminology

**Some terminology for graphs.** Most of the following definitions are standard or evocative and listed for completeness. A graph consist of a ***node set*** $\boldsymbol{N} = \{1, \dots d_N\}$ and of ***edges*** that couple node pairs. In simple graphs, edges couple exclusively distinct node pairs by at most one edge so that the endpoints $i$ and $k$ of an ***ik-edge*** never coincide. For an $ik$-arrow, $i \prec\!\!\!-\!\!\!- k$, node $k$ is commonly named the ***parent*** of node $i$.

For a regression graph, $\boldsymbol{G}^{\boldsymbol{N}}_{\mathbf{reg}}$, there is an ordered partitioning of the node set as $N = (u, v)$ where $u$ contains the response nodes, each having possibly several parent nodes and $v$ contains context nodes, none of which has a parent node; see for instance Figure 1 below. There are ***three types of edge sets***, $\boldsymbol{E}_{\prec\!-}$ for directed dependences of responses on their regressors, $\boldsymbol{E}_{--}$ for undirected dependences among components of a joint response, and $\boldsymbol{E}_{\text{—}}$ for undirected dependences among context variables.

An ***ik-path*** connects the ***path endpoints*** $i$ and $k$ by a sequence of edges. An $ik$-path can be an edge, otherwise it has ***distinct inner nodes*** such that each edge visits an inner node once. There is is an ***a-line path***, if all its inner nodes are in subset $a$ of $N$. A path of arrows is *direction-preserving* if all its arrows point in the same direction.

For $a, b$ arbitrary disjoint subsets of $N$, one says there is a ***path between b and a*** if one endpoint is in $a$ and the other endpoint is in $b$, while we say there is a ***path from b to a*** if the subsets are ordered as $(a, b)$ that is if direction-preserving paths may start in $b$ and point to $a$, but not vice-versa. In a direction-preserving $ik$-path, node $k$ is named an ***ancestor*** of $i$ and node $i$ a ***descendent*** of $k$.

A ***subgraph, induced by a subset a*** of the node set $N$, consists of the nodes within $a$ and of the edges present in the graph within $a$. A special type of induced subgraph, needed here, consists of three nodes and two edges. It is named a $\mathsf{V}$-***configuration*** or just a $\mathsf{V}$. Thus, a three-node path forms a $\mathsf{V}$ if its induced subgraph has two edges.

In a ***complete graph***, every node pair is coupled by an edge. In a ***connected***

***subgraph***, every node can be reached by a path. The ***connected component*** of a regression graph are the disjoint connected graphs that remain when all its arrows are removed. Nodes within a connected component are said to be ***concurrent nodes***.

**Generating sequences of regressions and graphs.** We consider $d_N$ random variables $Y_i$, which may be discrete or continuous or a mixture of both types. For a more formal definition of the measure spaces, asked for by a referee, see for instance Lauritzen and Wermuth (1989). The variables have labels in node set $N$ and form a vector variable, denoted by $Y_N$. In the following, an element $i$ of $N$ is not distinguished from the singleton $\{i\}$ and the union sign for combining subsets of $N$ is often omitted.

For $i, k$ a node pair and $c \subset N \setminus \{i, k\}$, we write $\boldsymbol{i \perp\!\!\!\perp k|c}$ for $Y_i, Y_k$ conditionally independent given $Y_c$. If such an independence constraint is satisfied by a density $f_{ikc}$,

$$i \perp\!\!\!\perp k|c \iff (f_{i|kc} = f_{i|c}) \iff f_{ik|c} = (f_{i|c} f_{k|c}).$$

It has become common to say that a joint family of densities $f_N$ can be ***generated over a chain graph*** if it factorizes according to a set ordering of the nodes, called a chain, and $f_N$ satisfies all independences implied by the graph. Different types of chain graph and corresponding models for discrete variables are discussed by Drton (2009).

When independence structures are the focus of interest, one starts traditionally with the graph. Regression graphs in node set $N$ have three types of edge sets, $E_{\prec\!\!-\!\!-}$, $E_{- -}$, and $E_{\relbar\!\!\relbar}$. For a regression graph, denoted by $G^N_{\text{reg}}$, there is a split of the node set as $N = (u, v)$, so that concurrent response nodes are in $u$ and concurrent context nodes in $v$, sets of ordered concurrent nodes are denoted by $g_j$ for $j = 1, \dots, J$. Subgraphs induced by any $g_j$ are undirected. Undirected subgraphs induced by $g_j$ within $v$ have edges $i \relbar\!\!\relbar k$ and are commonly called ***concentration graphs***. Undirected subgraphs induced by $g_j$ within $u$ have edges $i - - - k$ and are called ***covariance graphs***.

For $g_j$ in $u$, nodes in $g_{>j} = g_{j+1} \cup g_{j+2}, \dots, \cup g_J$ are said to be in the ***past of*** $\boldsymbol{g_j}$. Arrows may start from any node, except from those in $g_1$, but never point to a node in $g_{>j}$. With $g_{>J} = \emptyset$, the basic factorization of $f_N$ generated over a regression graph is

$$f_N = f_{u|v} f_v \text{ with } f_{u|v} = \prod\nolimits_{j \in u} f_{g_j|g_{>j}} \text{ and } f_v = \prod\nolimits_{j \in v} f_{g_j}. \tag{1}$$

Several orderings of $g_j$ may give the same factorization as explained below for Figure 1.

Here, tracing of pathways is of main interest, hence we start instead with a **stepwise generating process of** $\boldsymbol{f_N}$ for which $N = (u, v)$ and ***one ordering of*** $\boldsymbol{g_j}$ ***is fixed***. In this process, the density of variables in $g_J$ is generated first, the one of $g_{J-1}$ given $g_J$ next, up to the density of $g_1$ given $g_{>1}$. Then, variable pairs needed to generate $f_N$ give the edge set of $G^N_{\text{reg}}$ with Definition 1 below and the factorization of equation (1) results.

For a ***variable pair*** $Y_i, Y_k$ ***needed in the generating process of*** $f_N$, we say it is conditionally dependent given $Y_c$ for some $c \subset N \setminus \{i, k\}$ and write $\boldsymbol{i \pitchfork k|c}$ and a graph is ***edge-minimal*** for a distribution generated over it, if every missing edge in the graph corresponds to a conditional independence statement and every edge present to a dependence. A family of densities $f_N$ generated over an edge-minimal graph changes if any one edge is removed from the graph.

**Definition 1. Defining pairwise dependences of $G^N_{\text{reg}}$.** *An edge-minimal regression graph specifies with $g_1 < \dots < g_J$ a generating process for $f_N$, where the dependences*

$$\begin{array}{lll} i\,{-}{-}{-}\,k: \; i \pitchfork k | g_{>j} & \textit{for } i, k \textit{ concurrent response nodes in } g_j \textit{ of } u, & \\ i \prec\!\!\!-\!\!\!-\, k: \; i \pitchfork k | g_{>j} \setminus \{k\} & \textit{for response node } i \textit{ in } g_j \textit{ of } u \textit{ and node } k \textit{ in } g_{>j}, & (2) \\ i \,-\!\!\!-\!\!\!-\, k: \; i \pitchfork k | v \setminus \{i, k\} & \textit{for } i, k \textit{ concurrent context nodes in } g_j \textit{ of } v, & \end{array}$$

*define the* **edges present** *in $G^N_{\text{reg}}$. The meaning of each* **edge missing** *in $G^N_{\text{reg}}$ results with the dependence sign $\pitchfork$ replaced by the independence sign $\perp\!\!\!\perp$.*

Thus, for the given order of the components $g_j$, the graph implies for each variable pair $i, k$ either conditional dependence or independence given the same type of conditioning set, with $i\,{-}{-}{-}\,k$ for two response nodes, with $i \prec\!\!\!-\!\!\!-\, k$ for $i$ a response node in $g_j$ and $k$ a node in the past of $g_j$, with $i \,-\!\!\!-\!\!\!-\, k$ for two context nodes. Notice that except for context nodes, each pair of variables is exclusively conditioned on variables that are in the past of the $g_j$ that contains node $i$. This permits to model simulateanously occurring effects of an intervention while this is not possible if the graph is directed acyclic or if it is another type of chain graph.

Different generating processes may lead to the same regression graph and hence to the same implied independence structure. Then, some components, $g_j, g_{j+1}, \dots, g_t$, say of $G^N_{\text{reg}}$, have an interchangeable labeling because they induce disconnected undirected subgraphs. Such components are displayed in Figure 1 within stacked boxes. In a connected $G^N_{\text{reg}}$, connected stacked components $g_j, \dots, g_t$ have the nodes in $g_{>t}$ as their ***common past*** and nodes in $g_{<j} = g_1 \cup g_2, \dots, \cup g_{j-1}$ as their ***common future***. For a generating process, one of the possibly many ***compatible orderings*** is fixed. In each, arrows point to response nodes in the common future but never to a node in the common past.

In Figure 1 below, $g_6$ and $g_7$ are in $v$, all other connected components are in $u$. The order implied by the arrows in $E_{\prec\!-}$ of $G^N_{\text{reg}}$ remains unchanged if, for instance, the two disconnected subgraphs induced by $g_3$ and $g_4$ are interchanged or if they are replaced by a single dashed-line complete graph in nodes $\{4, 5, 6, 7\}$.

Recall that connected components of $G^N_{\text{reg}}$ are uniquely obtained as the connected subgraphs that remain after deleting all arrows from the regression graph and keeping the undirected edges and all nodes. Thus, for any given graph, it is not necessary to show stacked boxes, but they are sometimes included to reflect the first ordering, the prior knowledge about possibly joint responses and joint context variables. By convention, we number nodes and components $g_j$ of $G^N_{\mathrm{reg}}$ first from top to bottom, then from left to right. In Figure 1, $g_3 = \{4, 5, 6\}$ and $g_8 = \{12, 13, 14\}$ contain three nodes, each of $g_2$ and $g_6$ contain two nodes, all others contain a single node.

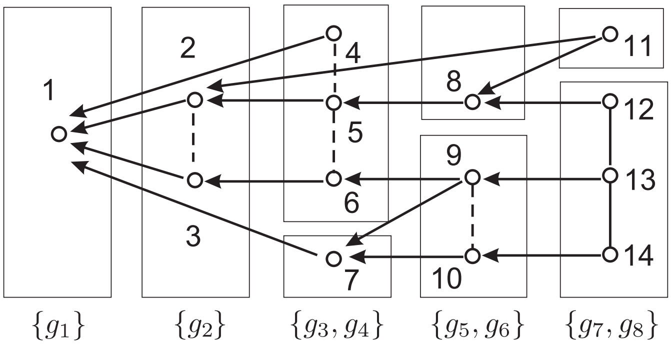


Figure 1: A regression graph in 14 nodes and node set partitioned into 8 connected components; single responses in $g_1, g_4, g_5$ and joint responses in $g_2, g_3, g_6$; context variables in $g_7, g_8$.

Single responses correspond in the statistical model to univariate regressions, joint responses to multivariate regressions, including the seemingly unrelated regressions of Zellner (1962). In Figure 1, seemingly unrelated regressions belong to the subgraphs induced by each of the three node sets $\{2, 3, 5, 6\}, \{5, 6, 8, 9\}, \{9, 10, 13, 14\}$.

**General and special properties of probability distributions.** For $i, h, k$ single, distinct indices and $a, b, c, d$ disjoint subsets of index set $N$, where only $d$ may be empty, there are the common independence properties $(i)$ to $(iv)$ which are satisfied by all probability distributions. The discussed properties $(v)$ to $(viii)$ constrain distributions, but they permit the use of just the graph to derive different types of consequences for families of distributions $f_N$ generated over $G^N_{\mathrm{reg}}$.

$(i)$ **symmetry:** $a \perp\!\!\!\perp b|c \iff b \perp\!\!\!\perp a|c$,

$(ii)$ **contraction:** $(a \perp\!\!\!\perp b|cd \text{ and } b \perp\!\!\!\perp c|d) \iff ac \perp\!\!\!\perp b|d$,

$(iii)$ **decomposition:** $a \perp\!\!\!\perp bc|d \implies (a \perp\!\!\!\perp b|d \text{ and } a \perp\!\!\!\perp c|d)$,

$(iv)$ **weak union:** $a \perp\!\!\!\perp bc|d \implies (a \perp\!\!\!\perp b|cd \text{ and } a \perp\!\!\!\perp c|bd)$.

Joint distributions, for which the reverse implications of $(iii)$ and of $(iv)$ hold, have as additional properties, respectively,

$(v)$ **composition:** $(a \perp\!\!\!\perp b|d \text{ and } a \perp\!\!\!\perp c|d) \implies a \perp\!\!\!\perp bc|d$,

$(vi)$ **intersection:** $(a \perp\!\!\!\perp b|cd \text{ and } a \perp\!\!\!\perp c|bd) \implies a \perp\!\!\!\perp bc|d$.

Properties $(v)$ and $(vi)$ are needed to derive the independence structure implied by $G^N_{\mathrm{reg}}$. Two further types of properties are to be considered for tracing pathways of dependence,

$(vii)$ **set transitivity:** $(a \perp\!\!\!\perp b|d \text{ and } a \perp\!\!\!\perp b|cd) \implies (a \perp\!\!\!\perp c|d \text{ or } b \perp\!\!\!\perp c|d)$, *or*

$(viii)$ **singleton transitivity:** $(i \perp\!\!\!\perp k|d$ and $i \perp\!\!\!\perp k|hd) \implies (i \perp\!\!\!\perp h|d$ or $k \perp\!\!\!\perp h|d)$.

Thus, distributions that satisfy set transitivity are also singleton-transitive, since $c$ may contain only one element. Singleton transitivity requires for a conditional independence of $Y_i, Y_k$ given $Y_d$ and given $Y_h, Y_d$ to hold both, there has to be at least one additional independence given $Y_c$ involving $Y_h$, the additional variable in the conditioning set. It is unfortunate that, in the literature, the term weak transitivity has sometimes been used for property $(vii)$ and sometimes for $(viii)$.

We shall show that set transitivity, $(vii)$, is used in addition to $(i)$ to $(vi)$ in transformations of $G^N_{\text{reg}}$ by which no edge of the starting graph gets removed and by which an edge criterion for the global Markov property is obtained, while only singleton transitivity, $(viii)$, is needed in addition to $(i)$ to $(vi)$ to decide with a given edge-minimal $G^N_{\text{reg}}$, whether a path is inducing a dependence for its path endpoints or not.

Singleton transitivity is a feature of what we define below as traceable regressions. So far, it had been known to be common to all positive binary distributions where, for instance, for $(1 \pitchfork 2$ and $1 \pitchfork 3)$ either $2 \perp\!\!\!\perp 3$ can hold or $2 \perp\!\!\!\perp 3|1$ but not both; see Simpson (1951). It also holds in all regular Gaussian distributions; see for instance Studený (2005), Corollary 2.5 in Section 2.3.6.

On the other hand, set transitivity imposes stronger constraints on any specific distribution; see for instance the discussion of Figure 1 for trivariate binary distributions in Wermuth, Marchetti and Cox (2009)It also excludes some regular Gaussian families of distribution such as the following.

**A regular Gaussian family violating set transitivity.** For $N = (u, v)$, let $Y_u$ and $Y_v$ be mean-centered vector variables with a joint Gaussian distribution. Let them have equal dimension, $dim$, the components of $Y_u$ and of $Y_u$ be mutually independent and all elements in the covariance matrix $\text{cov}(Y_u, Y_v) = \Sigma_{uv}$ be nonzero, then every component of $Y_u$ is dependent on every component in $Y_v$ and

$$\text{cov}(Y_u) = \Sigma_{uu} \text{ diagonal}, \quad \text{cov}(Y_v) = \Sigma_{vv} \text{ diagonal}.$$

Let further the components of $Y_v$ have equal variances $\omega > 1$ and the equal variances of the components $Y_u$ be $\kappa > \omega + 1$. Whenever in the described situation $\Sigma_{uv}$ is orthogonal, the joint covariance matrix is invertible so that the joint distribution is regular and the marginal independences carry over to conditional independences so that also

$$\text{cov}(Y_u|Y_v) = \Sigma_{uu|v} \text{ diagonal}, \quad \text{cov}(Y_v|Y_u) = \Sigma_{vv|u} \text{ diagonal}.$$

Set transitivity is always violated, for a split $v = (a, b)$, $c = \{1, \ldots, dim\}$ and $d = \emptyset$. This family extends the example in equation (8) of Cox and Wermuth (1993).

**Some important properties of $G^N_{\text{reg}}$ and $f_N$.** Two basic types of Vs in $G^N_{\text{reg}}$ need to be distinguished. There are ***collision*** **V*s***:

$$i\,\text{- - -}\circ \prec\!\!\text{—}\, k, \qquad i\, \text{—}\!\!\succ \circ \prec\!\!\text{—}\, k, \qquad i\,\text{- - -}\circ\text{- - -}\, k,$$

and ***transmitting*** **V*s***:

$$i \prec\!\!\text{—} \circ \prec\!\!\text{—}\, k, \qquad i \prec\!\!\text{—} \circ \text{ —— } k, \qquad i \text{ —— } \circ \text{ —— } k, \qquad i \prec\!\!\text{—} \circ \text{—}\!\!\succ k, \qquad i \prec\!\!\text{—} \circ\text{- - -}\, k\,.$$

Recall that two different graphs in the same node set are Markov equivalent if they define nevertheless the same independence structure, that is the set of all independences implied by the graph. The skeleton of a graph consists of its nodes and its set of edges, irrespective of the type of edge. It results by replacing each edge present by a full line.

**Lemma 1. Markov equivalence**. (Wermuth and Sadeghi, 2012). *Two regression graphs with the same skeleton are Markov equivalent if and only if their sets of collision* V*s are identical.*

A more compact characterization of the pairwise independences in Definition 1 is based on the notion of anterior paths. Recall first that with $N = (u, v)$, there are only undirected full-line paths within $v$ and there are only arrows pointing from $v$ to $u$. An ***anterior ik-path*** is either a descendant-ancestor $ik$-path, or a context nodes $ik$-path, or a descendant-ancestor $iq$-path with a context-nodes $qk$-path attached to it,

$$i \prec\!\!\text{—} \underbrace{\overbrace{\circ \prec\!\!\text{—} \circ, \dots, \circ \prec\!\!\text{—} q}^{\text{ancestors of } i} \text{ —— } \circ, \dots, \circ \text{ —— } k}_{\text{anteriors of } i}\,.$$

We denote the joint set of anteriors of nodes $i, k$ by $\text{ant}_{ik} = \{\text{ant}_i \cup \text{ant}_k\} \setminus \{i, k\}$. Similarly, for any subset $c$ of $N$, the anterior set of nodes within $c$ is denoted by $\text{ant}_c$.

The intersection ($vi$) and the composition property ($v$) are needed for Lemmas 2 and 3. By using them, the independences attached to the missing edges of $G^N_{\text{reg}}$ in Definition 1 reduce to the more compact statements $i \perp\!\!\!\perp k | \text{ant}_{ik}$ and this leads to the definition of an active path in $G^N_{\text{reg}}$ due to Sadeghi (2009) for a more general class of graphs.

Let $\{a, b, c, m\}$ partition $N$, where $c$ denotes a conditioning set of interest for $a, b$ and $m$ the set of nodes to be ignored that is to be marginalized over. Only $c$, $m$ or both may be empty sets. A ***path in $G^N_{\text{reg}}$ is active given c*** if of its inner nodes, every collision node is in $c \cup \text{ant}_c$ and every transmitting node is in $m$. For graph transformation, such a path is also said to be ***edge-inducing***.

**Lemma 2. Global Markov property of $G^N_{\text{reg}}$.** (Sadeghi, 2009). *The regression graph $G^N_{\text{reg}}$ implies $a \perp\!\!\!\perp b | c$ if and only if there is no active path in $G^N_{\text{reg}}$ between $a$ and $b$ given $c$.*

**Lemma 3. Equivalence of the pairwise and the global Markov property**. (Sadeghi and Lauritzen, 2012). *The independence structure of* $G^N_{\text{reg}}$ *is equivalently defined by its lists of the three types of missing edges and by its global Markov property.*

To make $\mathsf{V}$s dependence inducing, we take an edge-minimal regression graph for $f_N$, assume properties $(i)$ to $(vi)$ and, in addition property $(viii)$, that is singleton transitivity. We then say $G^N_{\text{reg}}$ is a ***dependence base*** for $f_N$ since the implications of this type of graph can be derived with respect to both independences and dependences. We note first that by enumeration in Definition 1, the inner node of each collision $\mathsf{V}$ is excluded from the defining conditioning set for its endpoints, while the inner node of each transmitting $\mathsf{V}$ is included in it. This observation is generalized with Lemma 4.

**Lemma 4.** *The conditioning set of any independence statement implied by* $G^N_{\text{reg}}$ *for the endpoints of any of its* $\mathsf{V}$*s, includes the inner node if it is a transmitting* $\mathsf{V}$ *and excludes the inner node if it is collision* $\mathsf{V}$ *.*

*Proof.* The statement results directly with Lemma 2. □

Let now a $\mathsf{V}$ in a dependence base $G^N_{\text{reg}}$ have endpoints $i, k$ and inner node o. Then by Definition 1 und Lemma 4, there is at least one $c$ with $c \subseteq N \setminus \{i, k, \mathrm{o}\}$ such that $i \perp\!\!\!\perp k|c$ is implied if $(i, \mathrm{o}, k)$ is a collision $\mathsf{V}$ and $i \perp\!\!\!\perp k|\mathrm{o}c$ if $(i, \mathrm{o}, k)$ is a transmitting $\mathsf{V}$.

**Proposition 1. Dependence inducing** $\mathsf{V}$**s**. *For* $(i, \mathrm{o}, k)$ *any* $\mathsf{V}$ *of a dependence base* $G^N_{\text{reg}}$ *and each* $c \subseteq N \setminus \{i, k, \mathrm{o}\}$ *such that this regression graph implies one of* $i \perp\!\!\!\perp k|c$ *or* $i \perp\!\!\!\perp k|\mathrm{o}c$*, the following two equivalent statements hold:*

$$- \quad (i, \mathrm{o}, k) \textit{ forms a collision } \mathsf{V} \iff (i \perp\!\!\!\perp k|c \implies i \pitchfork k|\mathrm{o}c),$$
$$- \quad (i, \mathrm{o}, k) \textit{ forms a transmitting } \mathsf{V} \iff (i \perp\!\!\!\perp k|\mathrm{o}c \implies i \pitchfork k|c)\,.$$

*Proof.* For $c = \emptyset$, collision $\mathsf{V}$s are Markov equivalent and transmitting $\mathsf{V}$s are Markov equivalent by Lemma 1. By edge-minimality, both edges of any $\mathsf{V}$ indicate conditional dependence for pairs $i$,o and $k$,o and by Definition 1, $i \perp\!\!\!\perp k$ holds for an inner collision node and $i \perp\!\!\!\perp k|\mathrm{o}$ for an inner transmitting node. Including the inner node of a collision $\mathsf{V}$ into the conditioning set, or excluding the inner node of a transmitting $\mathsf{V}$ from the conditioning set, generates an active path by Lemma 2. Such a path induces a dependence unless singleton transitivity is violated which contradicts an assumption. Similarly for $c \neq \emptyset$, an independence is implied by $G^N_{\text{reg}}$ if there is no active path between $i$ and $k$ given $c$ by Lemma 4, but an active path is generated just as for $c = \emptyset$. □

We can now define sequences of regressions that permit the tracing of pathways of dependence for $f_N$ when $a, b, c, d$ denote disjoint subsets of $N$ and only $d$ may be empty.

**Definition 2. Traceable regressions.** *We say* $f_N$ *results from traceable regressions if*

1. *it could have been generated over a dependence base regression graph,* $G^N_{\text{reg}}$ *,*
2. *it has three equivalent decompositions of the joint independence* $b \perp\!\!\!\perp ac|d$

   $$(i)(b \perp\!\!\!\perp a|cd \text{ and } b \perp\!\!\!\perp c|d), \quad (ii)(b \perp\!\!\!\perp a|d \text{ and } b \perp\!\!\!\perp c|d), \quad (iii)(b \perp\!\!\!\perp a|cd \text{ and } b \perp\!\!\!\perp c|ad),$$

3. *dependence-inducing* $\mathsf{V}$*'s of* $G^N_{\text{reg}}$ *are also dependence-inducing for* $f_N$*.*

Decompositions $(i)$ to $(iii)$ in Definition 2 combine the previously discussed properties $(ii)$ to $(vi)$. Symmetry of independences, that is property $(i)$, holds trivially as in all probability distributions. Undirected edges correspond to symmetric dependence statements. For each arrow $i \prec\!\!\!- k$ in $G^N_{\text{reg}}$ , symmetry of dependence holds only in the following weak sense. From Definition 1 for $i$ in $g_j$, there is some $c \subseteq g_{>j} \setminus k$ with $f_{i|kc} \neq f_{i|c}$ used in the generating process. Then, for $Y_k$ regressed instead on $Y_i, Y_c$, also $f_{k|ic} \neq f_{k|c}$.

Notice that traceable regression behave like regular Gaussian families generated over an edge-minimal $G^N_{\text{reg}}$ . Therefore, for traceable regressions, a violation of set transitivity can occur only when there are at least two paths connecting the same node pair; see the family of regular Gaussian distributions given above that violates set transitivity and for further examples Wermuth and Cox (1998). We call these special types of parametric constellations ***path cancellations*** as they result for a pair $i, k$ after combining dependences induced by active $ik$-paths in such a way that the joint contributions of all paths cancel.

## 3 Applications and illustrations of traceable regressions

**Tracing paths**. Whenever a pathway of dependence is traced in terms of a graph, one uses implicitly that every edge present is a strong enough dependence to be of interest in the given substantive context and that every $\mathsf{V}$ along a path is dependence-inducing for its endpoints, since otherwise, no dependence is implied for the path endpoints.

Figure 2 shows a well-fitting regression graph for nine features observed for patients. The regression graph represents a research hypothesis on the sets of regressors needed for each response to generate the joint distribution. In this example, we use data of Kappesser (1997) on 201 chronic pain patients, where variable descriptions and detailed statistical analyses are given in Wermuth and Sadeghi (2012), not in this paper.

The graph does not contain any information on the types of the dependence, but supplemented by estimates for the dependences, one can use the graph to interpret pathways of dependences. For instance the path $Y, Z_a, A, B$ leads, together with the parameters estimated with linear and logistic models, to the following interpretations.

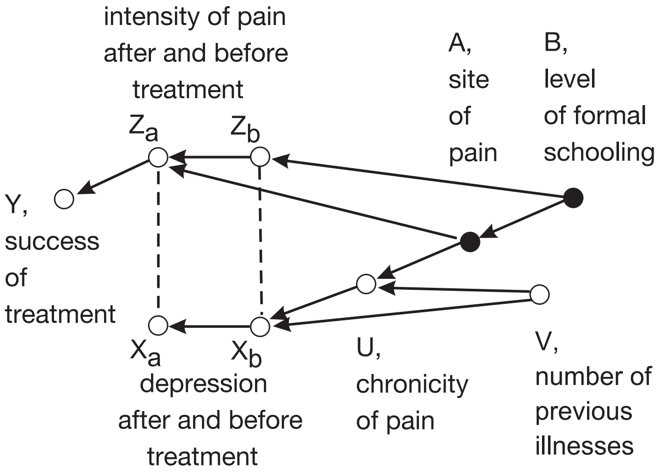


Figure 2: Regression graph, well compatible with the data and resulting from statistical analyses. Binary variables are indicated by dots, variables treated as continuous by circles.

Patients with a higher level of formal schooling are more likely to have head or neck pain than back pain. For patients with head or neck pain, the intensity of pain is better reduced after treatment than for the back pain patients. For lower pain intensity scores after treatment, treatment is the more successful the lower the pain intensity. For higher pain intensity scores after treatment, there are no systematic changes in $Y$.

The graphs in Figure 3 are consequences of the generating graph in Figure 2. Figure 3a) implies that site of pain, $A$, would show a direct effect on $Y$ if the two symptoms of chronic pain before and after treatment were either not measured or just omitted from the list of potentially important regressors. Similarly, chronicity of pain, $U$, would show a direct effect on $Y$ if, in addition, site of pain, $A$, is omitted in 3b).

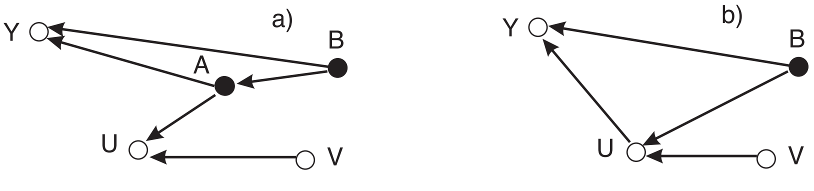


Figure 3: The graph of Figure 2 transformed, preserving the original ordering for the remaining variables by a) marginalizing over symptoms before and after treatment, $X_a$, $Z_a$, $Z_b$, $X_b$; b) marginalizing over symptoms before and after treatment and, in addition, over site of pain, $A$.

To derive and interpret transformed graphs well, such as those in Figure 3, and more complex graphs involving both marginalizing and conditioning, one needs to know the general properties of transforming regression graphs and realize that in general, induced dependences may not be reflected in significant statistical test results, in particular for small sample sizes or weak dependences attached to edges in the generating graph.

**Planning future follow-up studies**. To show how tracing of active paths may lead to an improved planning of follow-up studies, we use the generating process, represented by the graph in Figure 3, adapted from Robins and Wasserman (1997), and assume that all those dependences are strong that correspond to edges present in the graph.

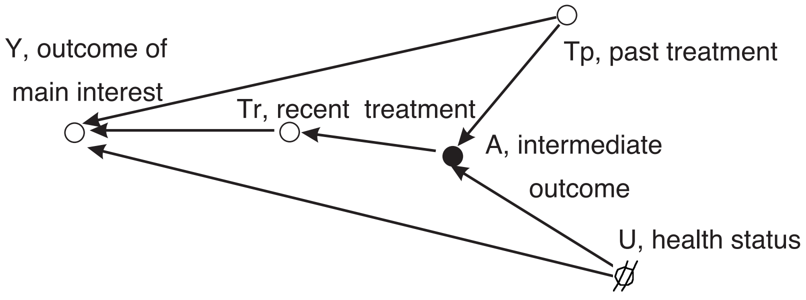


Figure 4: Generating process in five variables, missing edge for $(T_\mathrm{p}, U)$ due to full randomized allocation of individuals to treatments, and missing edges for $(T_\mathrm{r}, U)$ and $(T_\mathrm{r}, T_\mathrm{p})$ due to randomization conditionally on $A$; $U$ expected to be unobserved in a follow-up study.

Suppose that in the planned study, it will be possible to observe all variables of Figure 4 except for $U$, because the tools needed to diagnose the health status, $U$, before treatment will not be available. Marginalizing over $U$ is indicated in Figure 4 by a crossed out node, ⌀. Then $U$ is excluded from any conditioning set for $Y$, the main response of interest. In general, whenever no active path is generated, one may proceed safely with estimating an effect, a dependence of main interest, in the follow-up study.

With $U$ unobserved, the dependence of $Y$ on the past treatment $T_p$ will always be modified, since by excluding also the intermediate outcome, $A$, and recent treatment, $T_r$ from the list of regressors, one generates the active path $Y, T_r, A, T_p$, while by including either $T_r$ or $A$ or both as regressors for $Y$, one generates the active path $Y, U, A, T_p$; see Lemma 2. The former is an example of an overall effect deviating from a conditional effect and the latter is an ***example of indirect confounding***.

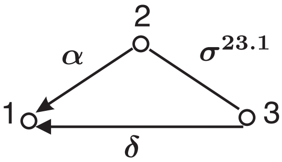
If on the other hand, the dependences of $Y$ on the recent treatment, $T_r$, is of main interest, then $T_p$ is a common ancestor and the path $Y, T_p, A, T_r$ becomes active by marginalizing over the inner nodes; an ***example of direct confounding***. But no active path is generated between $Y$ and $T_r$ when $A$ and $T_p$ are regressors in addition to $T_r$, so that the conditional dependence of $Y$ on $T_r$ given $A, T_p$ can be estimated.

Even though it may in principle be possible to recover the generating dependence given some distributional assumptions; see e.g. Wermuth and Cox (2008), one needs to obtain very precise estimates to make any correction worthwhile since poorly estimated parameters may also lead to bad corrections.

Both types of confounding can also be detected using graphical criteria on transformed graphs in reduced node sets, named summary graphs; see Wermuth (2011). For constructing summary graphs by removing repeatedly single nodes, one needs to take into account that any given node can be a collision node on one path and a transition node on another path. This contrasts with the graph transformations in this paper, where different types of active paths are closed in sequence.

**Examples of small Gaussian regression graph models.** We illustrate next the intersection and the composition property by describing two different types of complete regression graphs in three nodes and the associated saturated models in the special case of regular families of Gaussian distributions for variables standardized to have zero mean and unit variance. Parameters are attached to the edges of the graphs. Example I shows that the intersection property is implicitly used with backward selections of important regressors in multiple regressions and Example II how the composition property is relevant for selecting important regressors in multivariate regressions.

**Example I: a complete single response graph with two context variables.** The following complete graph in nodes $1, 2, 3$

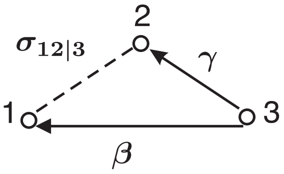


defines implicitly for standardized Gaussian variables, $Y_1, Y_2, Y_3$ three nonzero parameters measuring dependence in

$$\mathrm{E}(Y_1|Y_2, Y_3) = \alpha Y_2 + \delta Y_3 \quad \mathrm{E}(Y_2 Y_3) = \rho_{23} \quad \sigma^{23.1} = -\rho_{23}/(1-\rho_{23}^2)\,,$$

where $\rho_{23}$ denotes the marginal correlation of $Y_2, Y_3$ and $\sigma^{23.1}$ the concentration in their bivariate distribution, that is after marginalizing over $Y_1$. For this complete graph, $\alpha \neq 0$ means $1 \pitchfork 2|3$, $\delta \neq 0$ means $1 \pitchfork 3|2$, and $\sigma^{23.1} \neq 0$ means $2 \pitchfork 3$. With $\alpha = \delta = 0$, one requires $1 \perp\!\!\!\perp 2|3$ and $1 \perp\!\!\!\perp 3|2$ and removes the 12-edge and the 13-edge from the complete graph so that node 1 remains isolated from 2 —— 3. For the resulting graph, the seemingly obvious interpretation $1 \perp\!\!\!\perp (2, 3)$ requires the intersection property.

**Example II: a complete joint response graph with a single regressor.** The following complete graph



defines for standardized Gaussian variables three non-vanishing parameters, $\beta, \gamma, \sigma_{12|3}$, in

$$\mathrm{E}(Y_1|Y_3) = \beta Y_3 \quad \mathrm{E}(Y_2|Y_3) = \gamma Y_3 \quad \mathrm{cov}(Y_1 Y_2|Y_3) = \sigma_{12|3}\,.$$

Here, $\sigma_{12|3} \neq 0$ means $1 \pitchfork 2|3$, $\beta \neq 0$ means $1 \pitchfork 3$, and $\gamma \neq 0$ means $2 \pitchfork 3$. With $\beta = \gamma = 0$, one requires $1 \perp\!\!\!\perp 3$ and $2 \perp\!\!\!\perp 3$ and removes the 13-edge and the 23-edge from the complete graph so that node 3 remains isolated from $1 {-}{-}{-} 2$. For the resulting graph, the interpretation $(1,2) \perp\!\!\!\perp 3$ requires the composition property.

**Standard properties for combining independences.** Properties (*ii*) to (*iv*) that are common to all probability distributions with a given density, are illustrated next by using the directed acyclic graphs in the three ordered nodes $(1,2,3)$ shown in Figure 5, again for standardized Gaussian distributions.

**Example III: a complete directed acyclic graph.** The complete graph in nodes $1,2,3$ of Figure 5a) gives for standardized Gaussian variables three nonzero parameters, $\alpha, \delta, \gamma$, measuring dependence in

$$\mathrm{E}(Y_1|Y_2,Y_3) = \alpha Y_2 + \delta Y_3, \qquad \mathrm{E}(Y_2|Y_3) = \gamma Y_3, \qquad \mathrm{E}(Y_3) = 0\,,$$

where $\alpha \neq 0$ means $1 \pitchfork 2|3$, $\delta \neq 0$ means $1 \pitchfork 3|2$, and $\gamma \neq 0$ means $2 \pitchfork 3$.

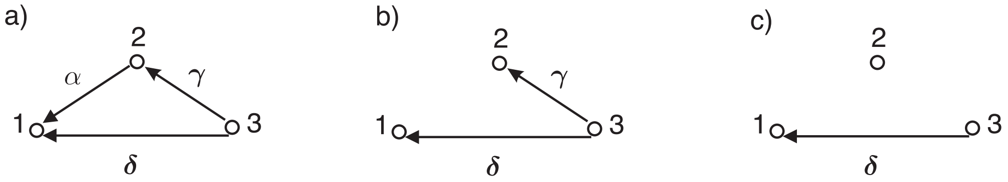


Figure 5: Directed acyclic graphs in 3 nodes with parameters in standardized Gaussian distributions attached to the edges; a) the complete graph, b) the graph implying $1 \perp\!\!\!\perp 2|3$, c) the graph implying $2 \perp\!\!\!\perp (1,3)$.

The interpretation of $\delta$ changes to $\delta \neq 0$ means $1 \pitchfork 3$ in Figure 5b) where $1 \perp\!\!\!\perp 2|3$ is implied by the graph. This reflects that a different family of distributions is generated when the 12-edge is removed. The graphs define implicitly the factorizations of $f_N$ in equation (1), respectively, as

$$f_{123} = f_{1|23} f_{2|3} f_3, \qquad (f_{123} = f_{1|3} f_{2|3} f_3) \implies 1 \perp\!\!\!\perp 2|3, \qquad (f_{123} = f_{1|3} f_2 f_3) \implies 2 \perp\!\!\!\perp (1,3)\,.$$

The factorization of a joint density as specified with a complete directed acyclic graph is formally always possible. Independence constraints imposed in sequence on two consecutive factors of $f_{123}$ generated as in Figure 5a), such as $1 \perp\!\!\!\perp 2|3$ constraining $f_{1|23} = f_{1|3}$ changes the triangle in the graph of Figure 5a) to a V in Figure 5b) and $2 \perp\!\!\!\perp 3$ constraining $f_{2|3} = f_2$ creates next an isolated node 2 and $1 \prec\!\!\!-\!\!\!- 3$, in Figure 5c).

The removal of the two arrows gives one direction of the contraction property, starting from the factorization to Figure 5c) gives the other direction. Given the factorization of any density to Figure 5c), marginalizing over $Y_3$ leaves $f_{12} = f_1 f_2$ and marginalizing over $Y_1$ gives directly $f_{23} = f_2 f_3$ that is decomposition, while conditioning on $Y_2, Y_3$ leaves directly $f_{1|23} = f_{1|3}$ and conditioning on $Y_1, Y_2$ gives $f_{3|12} = f_{3|1}$ that is weak union. Also in more complex situations, these three properties, (*ii*), (*iii*), (*iv*), common to all probability distributions, can be derived by transforming factorized densities.

## 4 Violating properties of traceable regressions.

Some small discrete families of distribution are given that are not traceable regressions. These may be extended and many similar families may be constructed.

**Violation of singleton transitivity.** As mentioned before, singleton transitivity is satisfied in all regular Gaussian distributions and in all binary distributions. But the following discrete family of distributions for a $2 \times 2 \times 3$ contingency table violates singleton transitivity. It is adapted from Birch (1963), equation (5.4). We write $\pi_{ijk}$ for the joint probabilities of variables $A, B, C$ at levels $i, j, k$ and e.g. $\pi_{+jk} = \sum_i \pi_{ijk}$. Conditional probabilities e.g. for $A$ given $B, C$ are $\pi_{i|jk} = \pi_{ijk}/\pi_{+jk}$.

Table 1: A family of distributions that violates singleton transitivity

$4\pi_{ijk}(1+\alpha+\alpha^2)$, $\alpha > 1$

|  | $C:\ k=1$ |  | $k=2$ |  | $k=3$ |  |
|---|---|---|---|---|---|---|
| $A/B:$ | $j=1$ | $j=2$ | $j=1$ | $j=2$ | $j=1$ | $j=2$ |
| $i=1$ | $\alpha^2$ | $\alpha$ | $\alpha$ | 1 | 1 | $\alpha^2$ |
| $i=2$ | $\alpha$ | 1 | $\alpha^2$ | $\alpha$ | 1 | $\alpha^2$ |
| odds-ratio | 1 |  | 1 |  | 1 |  |

Here, the conditional odds ratios being 1 imply that $A \perp\!\!\!\perp B|C$ and the marginal probabilities of $A, C$ and of $B, C$ show that $A \pitchfork C$ and $B \pitchfork C$. Nevertheless, also $A \perp\!\!\!\perp B$ since

$$\textstyle\sum_k \pi_{i+k}\, \pi_{+jk}/\pi_{++k} = \pi_{i++}\pi_{+j+},$$

a very special constellation discussed first by Darroch (1962) and generalized by Wermuth and Cox (2004), section 7, to general types of distributions that are also not dependence inducing. Though one can construct families of distributions with such peculiar parametric constraints, it is difficult to imagine that they could capture a structure of interest in any substantive context when studying sequences of regressions.

In a generating process of $f_N$, singleton transitivity can be achieved when the individual regressions are permitted to vary independently of the other response components and of their common past. This is reached, in particular, when the family to a complete graph has a rich enough parametrisation and only the independence constraints of Definition 1 are imposed on $G^N_{\text{reg}}$.

**Violation of the intersection property.** The intersection property is always satisfied by positive distributions and in all in regular Gaussian distributions, even though the known necessary and sufficient conditions are less restrictive; see San Martin, Mouchart and Rolin (2005). The following discrete family of distributions for a $2 \times 2 \times 3$ contingency table violates the intersection property. This happens whenever a pair of variables shares some common information. For three binary variables, violation of the intersection property coincides with the degenerate case of two variables being identical.

Table 2: A family of distributions that violates the intersection property

$3\pi_{ijk}$, $0 < \alpha \neq \beta < 1$, $2\alpha + \beta < 1$

| | $C:\ \ k=1$ | | $k=2$ | | $k=3$ | |
|---|---|---|---|---|---|---|
| $A/B:$ | $j=1$ | $j=2$ | $j=1$ | $j=2$ | $j=1$ | $j=2$ |
| $i=1$ | $\alpha$ | 0 | $\alpha$ | 0 | 0 | $\beta$ |
| $i=2$ | $1-\alpha$ | 0 | $1-\alpha$ | 0 | 0 | $1-\beta$ |

In the family shown in Table 2, $A \perp\!\!\!\perp B|C$ and $A \perp\!\!\!\perp C|B$, since

$$\pi_{i|jk} = \pi_{i|k} \text{ and } \pi_{i|jk} = \pi_{i|j}$$

but $A \pitchfork BC$. More precisely, $A \pitchfork B$ since $\pi_{i|j} \neq \pi_i$ and $A \pitchfork C$ since $\pi_{i|k} \neq \pi_i$. The marginal joint distribution of $B, C$ shows the type of common information shared by variables $B$ and $C$. Variable $B$ taking on level 1 coincides with $C$ taking on value 1 or 2 and $B$ being at level 2 coincides with $C$ being at level 3.

Thus, when the joint distribution of $B, C$ had been generated by first knowing the distribution of variable $C$ and then generating the conditional distribution of $B$ given $C$, the levels of variable $B$ are not permitted to vary freely and thereby lead to the violation of the intersection property.

**Violation of the composition property.** The composition property is always satisfied in regular Gaussian distributions and in multivariate symmetric binary distributions generated over directed acyclic graphs; see Wermuth, Marchetti and Cox (2009). On the other hand, it is always violated when pairwise independences do not imply mutual independence.

The following binary family of distributions for a $2 \times 2 \times 2$ contingency table also violates the composition property. In this family, there is a log-linear three-factor interaction since the conditional odd-ratios for $A, B$ differ at the two levels of $C$.

Table 3: A family of distributions that violates the composition property

$8\pi_{ijk}$, $0 < 2\alpha < 1$

| | $C:\ \ k=1$ | | $k=2$ | |
|---|---|---|---|---|
| $A/B:$ | $j=1$ | $j=2$ | $j=1$ | $j=2$ |
| $i=1$ | $1+2\alpha$ | $1-2\alpha$ | 1 | 1 |
| $i=2$ | $1-2\alpha$ | $1+2\alpha$ | 1 | 1 |
| odds-ratio | $\{(1+2\alpha)/(1-2\alpha)\}^2$ | | 1 | |

More precisely, at level 2 of $C$, the variables $A, B$ are independent while the dependence of this pair is strong at level 1 of $C$ whenever $\alpha$ is large. At the same time, the marginal $AC$ and $BC$ tables reveal that $A \perp\!\!\!\perp C$ and $B \perp\!\!\!\perp C$.

Thus, when regressing the two components of a joint response $AB$ separately on $C$, one sees no separate effects, but the conditional dependence of $A$ on $B$ changes with the levels of $C$. This type of structure could in particular not be generated by a single unobserved common explanatory variable or if all sets of variable with higher-order effects also have main effects in the regressions, that is lowest order interactions.

With a pragmatic strategy for model selection in which one checks for higher order interactions only when there are also main effects, one may overlook such structures that could be of substantlve interest. For sequences of discrete joint responses, the violation will be detected when using the parametrization suggested by Marchetti and Lupparelli (2011). In general, the graphical checks for nonlinearities and interactions, as proposed by Cox and Wermuth (1994), provide some protection, but only for effects that are detectable also in marginal trivariate distributions.

## 5 Transforming regression graphs

The transformations of regression graphs to be introduced, are based on binary matrix representations of $G^N_{\text{reg}}$ . Our notation for these edge matrices mimics the one for parameter matrices in Gaussian sequences of regressions generated over the graph. There are one-to-one correspondences between a zero in an edge matrix, a vanishing parameter in the regular Gaussian family of distributions and a conditional independence statement.

**Linear sequences of regressions.** For a mean-centered vector variable $Y_N$ with a regular Gaussian distribution generated over $G^N_{\text{reg}}$ with a split $N = (u, v)$, the matrix of equation parameters, denoted by $H_{NN}$, is upper block-triangular and

$$H_{NN} Y_N = \eta_N \text{ with } W_{NN} = \text{cov}(\eta_N) \text{ block-diagonal in the sizes of } g_j,$$

where the submatrix of $H_{uu}$ in rows $g_j$ and columns $g_{>j}$ is $-\Pi_{g_j|g_{>j}}$, the negative of the population least-squares coefficient matrix obtained when regressing $Y_{g_j}$ on $Y_{>g_j}$. The square diagonal submatrices in the sizes of $g_j$ are identity matrices. The submatrix $H_{vv}$ is the marginal concentration matrix of $Y_v$, denoted by $\Sigma^{vv.u}$. This implies $W_{vv} = \Sigma^{vv.u}$. The square submatrices of $W_{uu}$ are $\Sigma_{g_j g_j|g_{>j}}$, the conditional covariance matrices of $Y_{g_j}$ given $Y_{>g_j}$. For just two connected components $a, b$ the parameter matrices are

$$H_{NN} = \begin{pmatrix} I_{aa} & -\Pi_{a|b.v} & -\Pi_{a|v.b} \\ 0_{ba} & I_{bb} & -\Pi_{b|v} \\ 0_{va} & 0_{vb} & \Sigma^{vv.ab} \end{pmatrix} \qquad W_{NN} = \begin{pmatrix} \Sigma_{aa|bv} & 0_{ab} & 0_{av} \\ 0_{ba} & \Sigma_{bb|v} & 0_{bv} \\ 0_{va} & 0_{vb} & \Sigma^{vv.ab} \end{pmatrix},$$

where we use a Yule-Cochran notation for $\Pi_{a|bv}$, the regression coefficient matrix of $Y_a$ regressed on $Y_b$, $Y_u$, for instance $0_{ba}$ denotes a matrix of zeros, and $I_{bb}$ an identity matrix.

For any split of $N = (a, b)$, to obtain $f_{a|b} f_b$ we let $c = a \cap u$, $d = b \cap u$, and get

$$K_{NN} = \begin{pmatrix} H^{-1}_{aa} & -H^{-1}_{aa} H_{ab} \\ H_{ba} H^{-1}_{aa} & H_{bb} - H_{ba} H^{-1}_{aa} H_{ab} \end{pmatrix} \qquad Q_{uu} = \begin{pmatrix} W_{cc} - W_{cd} W^{-1}_{dd} W_{dc} & W^{-1}_{dd} W_{dc} \\ -W^{-1}_{dd} W_{dc} & W^{-1}_{dd} \end{pmatrix},$$

by partial inversion of $H_{NN}$ with respect to $a$ and by partial inversion of $W_{uu}$ with respect to $b$; see for instance Marchetti and Wermuth (2009), Appendix 1.

**Lemma 5. Orthogonalised linear equations.** (Wermuth and Cox (2004), Thm 1.) *The Gaussian density* $f_N = f_{u|v} f_v$ *generated over* $G^N_{\mathrm{reg}}$ *is for any split* $N = (a, b)$ *transformed into* $f_N = f_{a|b} f_b$ *with* $E(Y_a|Y_b) = \Pi_{a|b}$, $\mathrm{cov}(Y_a|Y_b) = \Sigma_{aa|b}$, $\mathrm{con}(Y_b) = \Sigma^{bb.a}$ *as*

$$\Pi_{a|b} = \mathrm{In}[K_{ab} + K_{aa} Q_{ab} K_{bb}], \tag{3}$$

$$\Sigma_{aa|b} = \mathrm{In}[K_{aa} Q_{aa} \mathcal{K}^{\mathrm{T}}_{aa}], \qquad \Sigma^{bb.a} = \mathrm{In}[H^{\mathrm{T}}_{bb} Q_{bb} H_{bb}]. \tag{4}$$

**The edge matrices of regression graphs.** Edge matrices are binary matrix representations of graphs. They are symmetric for undirected graphs, upper block-triangular for arrows in a generating $G^N_{\mathrm{reg}}$ and upper-triangular for directed acyclic graphs. The essential change compared to the more traditionally used adjacency matrices is that ones are added along the diagonal of each square matrix. This has the effect that sums of matrix products are well-defined and can represent the closing of special types of path in graphs; such as in equations (8) and (9) below.

Regression graphs have three types of edge sets, $E_{\prec\!\!\!\relbar\!\!\relbar}$, $E_{- -}$, and $E_{\relbar\!\!\relbar}$. The edge matrix components of $G^N_{\mathrm{reg}}$ are a $d_N \times d_N$ upper block-triangular matrix $\mathcal{H}_{NN} = (\mathcal{H}_{ik})$ such that

$$\mathcal{H}_{ik} = \begin{cases} 1 & \text{if and only if } i \prec\!\!\!\relbar\!\!\relbar k \text{ or } i \relbar\!\!\relbar k \text{ in } G^N_{\mathrm{reg}} \text{ or } i = k, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

and a $d_u \times d_u$ symmetric matrix $\mathcal{W}_{uu} = (\mathcal{W}_{ik})$ such that

$$\mathcal{W}_{ik} = \begin{cases} 1 & \text{if and only if } i\,{-}{-}{-}\,k \text{ in } G^N_{\mathrm{reg}} \text{ or } i = k, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where, $E_{- -}$ corresponds to $\mathcal{W}_{uu}$, $E_{\relbar\!\!\relbar}$ to $\mathcal{H}_{vv}$, and $E_{\prec\!\!\!\relbar\!\!\relbar}$ to $\mathcal{H}_{uN}$.

Every regression graph $G^N_{\mathrm{reg}}$ can be represented by its edge matrices given in equations (5) and (6). Every dependence base $G^N_{\mathrm{reg}}$ defines in particular a corresponding family of Gaussian regressions in which each edge present can be identified by a single non-vanishing parameter, an off-diagonal element of $H_{NN}$ or $W_{u,u}$.

**Partial closure of paths.** Partial closure, introduced by Wermuth, Wiedenbeck and Cox (2006), is a matrix operator, denoted by $\mathrm{zer}_a(\cdot)$ which acts on row and collums $a$ of a binary matrix. It is applied to edge matrix representations of a starting graph in node set $N$ to give the edge matrix representations of a new graph in which there is an additional $ik$-edge for a pair $i, k$ that is in the starting graph uncoupled but connected by a specific type of edge-inducing $a$-line path.

With partial closure, the set of nodes, node labels, and edges present in the starting graph, are preserved in the transformed graph so that the mappings are graph homomorphisms; for this notion see Hell and Nešetřil (2004), for corresponding reparametrizations of exponential families see Wiedenbeck and Wermuth (2010).

**Lemma 6. Basic properties of partial closure.** (Wermuth, Wiedenbeck and Cox, (2006)). *Partial closure is* ($i$) *commutative,* ($ii$) *cannot be undone and* ($iii$) *is exchangeable with selecting a submatrix.*

By property ($i$), it is enough, for some purposes, to show how the operator acts on a single node. By property ($ii$), independences can be removed but never reintroduced so that these transformations satisfy set transitivity. Property ($iii$) justifies node and edge reductions since closing edge-inducing $a$-line paths in a large graph and then selecting a square submatrix for a subset containing $a$, gives the same result as selecting the square submatrix first and then closing the $a$-line paths.

Because of property ($i$), one can always permute the matrix $\mathcal{F}$ into $\tilde{\mathcal{F}}$ and start partial closure with node $i$ corresponding to position (1,1) of $\tilde{\mathcal{F}}$. Then for $b = N \setminus \{i\}$,

$$\mathrm{zer}_i\, \tilde{\mathcal{F}} = \mathrm{In}[\begin{pmatrix} 1 & \mathcal{F}_{ib} \\ \mathcal{F}_{bi} & \mathcal{F}_{bb} + \mathcal{F}_{bi}\mathcal{F}_{ib} \end{pmatrix}], \tag{7}$$

which says that particular Vs in the graph are closed which have node $i$ as inner node. In the three small examples of Figure 6, an edge for node pair $1, 3$ is induced with $i = 2$.

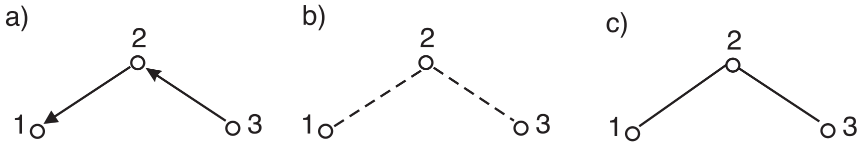


Figure 6: Dependence base, 3-node graphs: a V in a a) directed acyclic, b) covariance, c) concentration graph; an active path (1,2,3) induces in a) $1 \pitchfork 3$, in b) $1 \pitchfork 3|2$, and in c) $1 \pitchfork 3$

Applying $\mathrm{zer}_i$ to the edge matrix of a directed acyclic graph, covariance graph or concentration graph mimics, respectively, the recursion relation for regression coefficients, covariances and concentrations; discussed for instance in Wermuth and Cox (1998).

By letting the edge induced by the three V 's in Figure 6, 'remember the type of edge at the path endpoints', the induced edges become, respectively,

$$\text{a) } 1 \prec\!\!\!-\!\!\!- 3, \qquad \text{b) } 1\text{---}3, \qquad \text{c) } 1 \text{ —— } 3.$$

The transformation $\mathrm{zer}_a(\mathcal{F})$ means that all Vs along $a$-line paths represented by the edge matrix $\mathcal{F}$ are closed by an edge. The basic property ($i$) implies that the nodes in $a$ may be chosen for this in any order. This requires in particular that the inner nodes of the paths of $\mathcal{F}$ are of the same type, either all are collision nodes to form ***collision paths***, or are all transmitting nodes.

**Lemma 7. Partial closure applied to $G^N_{\text{reg}}$** . *The transformation* $\mathcal{K}_{NN} = \text{zer}_a(\mathcal{H}_{NN})$ *closes each a-line anterior path and* $\mathcal{Q}_{uu} = \text{zer}_b(\mathcal{W}_{uu})$ *each dashed, b-line collision path.*

*Proof.* Each anterior path in $G^N_{\text{reg}}$ and no other type of path is represented by $\mathcal{H}_{NN}$ and each dashed-line path in $G^N_{\text{reg}}$ and no other type of path is represented by $\mathcal{W}_{uu}$. By Proposition 1, a $\mathsf{V}$ along the former is edge-inducing by marginalizing over its inner node and of the latter by conditioning on its inner node. Remembering the type of edge at the endpoints of each $\mathsf{V}$ on an $a$-line path of $\mathcal{H}_{NN}$ leads to the same induced edge for the endpoints of the path irrespective of the order in choosing single nodes of $a$. $\square$

**Closing active paths in regression graphs.** For directed acyclic graphs, it is known that the path criterion on the starting graph for separation of $\alpha$ from $\beta$ given $c$ can be reduced to an edge criterion after transforming first the generating graph in terms of partial closure and closing next the remaining paths that are relevant for deciding whether $\alpha \perp\!\!\!\perp \beta | c$ is implied; see Marchetti and Wermuth (2009). This approach is now extended to regression graphs and to dependences in traceable regressions. For this, we take the partitioning $N = \{\alpha, \beta, c, m\}$ of the node set of $G^N_{\text{reg}}$ , $a = \alpha \cup m$, $b = \beta \cup c$, and

$$\mathcal{K}_{NN} = \text{zer}_a \mathcal{H}_{NN}, \quad \mathcal{Q}_{uu} = \text{zer}_b \mathcal{W}_{uu}, \quad \mathcal{Q}_{uv} = 0, \quad \mathcal{Q}_{vv} = \mathcal{K}_{vv}.$$

**Proposition 2. Induced edge matrices for $f_{a|b} f_b$.** *Sequences of regressions with graph* $G^N_{\text{reg}}$ *in node set* $N = (u, v)$ *and generating edge matrices* $H_{NN}$ *and* $W_{uu}$ *imply for* $f_{a|b} f_b$, *with the induced regression graph* $G^{N-a|b}_{\text{reg}}$ *for* $Y_a$ *regressed on* $Y_b$, *as edge matrices*

$$\mathcal{P}_{a|b} = \text{In}[\mathcal{K}_{ab} + \mathcal{K}_{aa} \mathcal{Q}_{ab} \mathcal{K}_{bb}], \tag{8}$$

$$\mathcal{S}_{aa|b} = \text{In}[\mathcal{K}_{aa} \mathcal{Q}_{aa} \mathcal{K}^{\text{T}}_{aa}], \qquad \mathcal{S}^{bb.a} = \text{In}[\mathcal{H}^{\text{T}}_{bb} \mathcal{Q}_{bb} \mathcal{H}_{bb}]. \tag{9}$$

*Proof.* Partial closure mimics transformations of partial inversion such that all elements of the induced matrices are non-negative. The zero entries in equations (3), (4) coincide with those in (8), (9), nonzero entries in the former correspond to ones in the latter; see Lemma 3 of Marchetti and Wermuth (2009) for more detail. $\square$

Of the active paths, defined for Lemma 2 and needed to decide for uncoupled pairs $i, k$ of $G^N_{\text{reg}}$ whether they are coupled in $G^{N-a|b}_{\text{reg}}$, some remain uncoupled after applying $zer_a \mathcal{H}_{NN}$ and $\text{zer}_b \mathcal{W}_{uu}$ but get closed with the non-negative sums of edge matrix products in (8), (9). Thus, as with partial closure, no edges get ever removed with the latter types of graph transformations so that set transitivity is used implicitly.

For the $N = (a, b)$ as for Proposition 2, let $\text{o}_a$ denote nodes in $a$ and $\text{o}_b$ nodes in $b$.

**Corollary 1.** *For $i, k$ the endpoints of paths that are edge-inducing for $G_{\text{reg}}^{N-a|b}$, there are three types of ik-path uncoupled in the graph having edge matrices $\mathcal{K}_{NN}$ and $\mathcal{Q}_{uu}$,*

$$i \prec\!\!\relbar\!\!\relbar\, \circ_a - - - \circ_b \prec\!\!\relbar\!\!\relbar\, k, \qquad i \prec\!\!\relbar\!\!\relbar\, \circ_a - - - \circ_a \relbar\!\!\relbar\!\!\succ k, \qquad i \relbar\!\!\relbar\!\!\succ \circ_b - - - \circ_b \prec\!\!\relbar\!\!\relbar\, k,$$

*which are closed with the induced edge matrices $\mathcal{P}_{a|b}$, $\mathcal{S}_{aa|b}$, $\mathcal{S}^{bb}$, respectively, in* (8), (9).

After remembering the types of edge at the path endpoints, we have with $\mathcal{P}_{a|b}$ an induced bipartite graph of arrows pointing from $b$ to $a$, with $\mathcal{S}_{aa|b}$ an induced covariance graph, and with $\mathcal{S}^{bb.a}$ an induced concentration graph.

**Lemma 8. Edge matrices induced by $\boldsymbol{G_{\text{reg}}^N}$ for $\boldsymbol{f_{\alpha\beta|c}}$.** *The subgraph induced by nodes $\alpha \cup \beta$ in $G_{\text{reg}}^{N-a|b}$ captures the independence implications of $G_{\text{reg}}^N$ for $f_{\alpha|\beta c} f_{\beta|c}$.*

*Proof.* By the interpretation of the edge matrix components $\mathcal{P}_{a|b}, \mathcal{S}_{aa|b}, \mathcal{S}^{bb.a}$, no edges are induced by taking

$$\mathcal{P}_{\alpha|\beta.c} = [\mathcal{P}_{a|b}]_{\alpha,\beta}, \qquad \mathcal{S}_{\alpha\alpha|b} = [\mathcal{S}_{aa|b}]_{\alpha,\alpha}, \qquad \mathcal{S}_{\beta\beta.a} = [\mathcal{S}^{bb.a}]_{\beta\beta}.$$

Jointly, these edge submatrices define the subgraph induced by $\alpha \cup \beta$ in $G_{\text{reg}}^{N-a|b}$. ☐

The induced graphs in node set $\alpha \cup \beta$ and $G_{\text{reg}}^{N-a|b}$ in node set $N$, are examples of independence-predicting graphs in contrast to independence-preserving graphs such as the ribbonless graphs of Sadeghi and Lauritzen (2012) and the different types of Markov-equivalent graphs, such as summary graphs. With ***independence-preserving graphs***, one can derive effects of additional marginalizing and conditioning in the starting graph while ***independence-predicting graphs*** can, in general, only be used to decide on edges present or missing in the induced graph.

**Proposition 3. Edge criteria for implied independences and dependences.** *A dependence base $G_{\text{reg}}^N$ implies $\alpha \perp\!\!\!\perp \beta|c$ if $\mathcal{P}_{\alpha|\beta.c} = 0$ and it implies $\alpha \pitchfork \beta|c$ if $\mathcal{P}_{\alpha|\beta.c} \neq 0$.*

*Proof.* The statement results with Lemma 7, equation (8) and Lemma 8. ☐

**Distributions satisfying all and only the independences captured by $\boldsymbol{G_{\text{reg}}^N}$.** A given distribution is said to be faithful to a graph if every of its independence constraints is captured by a given independence graph; see Spirtes, Glymour and Scheines (1993). For a distribution to be faithful to $G_{\text{reg}}^N$, it has to satisfy the properties needed for the graph transformations of Proposition 3, that is properties ($i$) to ($vii$).

**Corollary 2. Distributions that are faithful to $\boldsymbol{G_{\text{reg}}^N}$.** *For a distributions with density $f_N$ generated over a dependence base $G_{\text{reg}}^N$, the following statements are equivalent*
($i$) *the distribution is faithful to $G_{\text{reg}}^N$,*
($ii$) *every independence and every dependence statement implied by $G_{\text{reg}}^N$ holds for $f_N$,*
($iii$) *$f_N$ satisfies as additional properties: composition, intersection and set transitivity,*
($iv$) *$f_N$ can be generated as a traceable regression without any path cancellations.*

Thus, faithfulness imposes in general an additional strong condition on traceable sequences of regressions. Exceptions are, for instance, directed acyclic graphs in which each response has only one parent. But the most common situation in observational and in interventional studies is to have two or more regressors influencing a response. Thus, for using regression graphs to interpret such structures or to plan future studies with a subset of the variables in a subpopulation, it is not sensible to assume that a given distribution is faithful to a regression graph. One needs to have traceable regressions though and should investigate reasons for path cancellations if they happen to occur.

## 6 Discussion

Sequences of regressions in joint responses permit to model changes in several response components occurring at the same time when there is an intervention. This contrasts with interventions in sequences of regressions in single responses and in other types of chain graph models.

We have identified properties of sequences of regressions in essentially arbitrary joint and single response variables and named them traceable regressions. A corresponding regression graph, $G^N_{\text{reg}}$ is a dependence base of the joint distribution in addition to capturing the independences in the regressions. One knows now that the independence structure of such traceable regressions can differ from the implications derived in terms of its generating regression graph only when there are path cancellations.

The consequences derivable with a graph give changes in structure that result in families of distributions generated over the graph while one may not be able to generalize to this family from the structure that one can see for a distribution with one given set of parameters, for instance as estimated in a sample.

Sequences of traceable regressions and a regression graph $G^N_{\text{reg}}$ have implications for a regression of $Y_a$ on $Y_b$ and dependences of $Y_b$ alone when these are based on a reordered node set $N = (a, b)$ that can be expressed with transformed edge matrix components of $G^N_{\text{reg}}$. When marginalizing over $m$ in $a = \alpha \cup m$ and conditioning on $c$ in $b = \beta \cup c$, the specific implications of $G^N_{\text{reg}}$ for the conditional densities $f_{\alpha|\beta c}$ and $f_{\beta|c}$ can now be derived with a subgraph induced by $\alpha \cup \beta$ in this transformed graph. An edge matrix criterion instead of a path criterion gives the global Markov property of $G^N_{\text{reg}}$ and detects, in addition, induced dependences when $G^N_{\text{reg}}$ is a dependence base for $f_N$.

Many new questions have opened up. These include types of conditions on a given distribution under which it represents a traceable regression, conditions on independence-predicting graphs which assure that they are also independence-preserving, applications such as the special details needed to improve existing methods for meta-analyses, or computational aspects, such as conditions under which one type of several equivalent graph transformations becomes computationally much less intensive than others.

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## References.

Birch, M.W. (1963). Maximum likelihood in three-way contingency tables. *J. Roy. Statist. Soc. B* **25**, 220–233.

Castillo, E., Gutiérrez, J. M., and Hadi, A. S. (1997). *Expert Systems and Probabilistic Network Models.* Springer, New York.

Cox, D.R. and Wermuth, N. (1993). Linear dependencies represented by chain graphs (with discussion). *Statist. Science* **8**, 204–218; 247–277.

Cox, D.R. and Wermuth, N. (1994). Tests of linearity, multivariate normality and adequacy of linear scores. *J. Roy. Statist. Soc. C* **43**, 347–355.

Darroch, J.N. (1962). Interactions in multi-factor contingency tables. *J. Roy. Statist. Soc. B* **24**, 251–263.

Dawid, A.P. (1979). Conditional independence in statistical theory (with discussion). *J. Roy. Statist. Soc. B* **41**, 1–31.

Drton, M. (2009). Discrete chain graph models. *Bernoulli* **15**, 736–753.

Edwards, D., Lauritzen, S.L. (2001). The TM algorithm for maximising a conditional likelihood function. *Biometrika* **88**, 961–972.

Flesch, I. and Lucas, P. (2006). Markov equivalence in Bayesian networks. in: *A*dvances in probabilistic graphical models.— (eds. Lucas, P., Gámez, J. and Samerón, E.) Springer, Berlin, 3–38.

Geiger, D., Verma, T.S. and Pearl, J. (1990). Identifying independence in Bayesian networks. *Networks* **20**, 507–534.

Hell, P. and Nešetřil , J. (2004). *Graphs and homomorphisms.* Oxford University Press, Oxford.

Kappesser, J. (1997). *Bedeutung der Lokalisation für die Entwicklung und Behandlung chronischer Schmerzen.* Thesis, Department of Psychology, University of Mainz.

Lauritzen, S. L. and Wermuth, N. (1989). Graphical models for association between variables, some of which are qualitative and some quantitative. *Ann. Statist.* **17**, 31–57.

Lněnička, R. and Matúš, F. (2007). On Gaussian conditional independence structures. *Kybernetika* **43**, 323–342.

Marchetti, G.M. and Lupparelli, M. (2011). Chain graph models of multivariate regression type for categorical data. *Bernoulli*, **17**, 827–844.

Marchetti, G.M. and Wermuth, N. (2009). Matrix representations and independencies in directed acyclic graphs. *Ann. Statist.* **47**, 961–978.

Pearl, J. (1988). *Probabilistic reasoning in intelligent systems.* Morgan Kaufmann, San Mateo.

Robins, J. and Wasserman, L. (1997). Estimation of effects of sequential treatments by reparametrizing directed acyclic graphs. In: *Proc. 13th Ann. Conf., UAI* (eds. D. Geiger and O. Shenoy) Morgan and Kaufmann, San Mateo, 409–420.

Sadeghi, K. (2009). Representing modified independence structures. *Transfer thesis, Oxford University.*

Sadeghi, K. and Lauritzen, S. L. (2012). Markov properties of mixed graphs. *Submitted.*

San Martin E., Mochart M. and Rolin, J.M. (2005). Ignorable common information, null sets and Basu's first theorem. *Sankhya* **67**, 674–698.

Spirtes, P., Glymour C. and Scheines R. (1993). *Causation, prediction and search.* Springer, New York.

Studený, M. (2005). *Probabilistic conditional independence structures.* Springer, London.

Wermuth, N. (2011). Probability models with summary graph structure. *Bernoulli*, **17**, 845–879.

Wermuth, N. and Cox, D.R. (1998). On association models defined over independence graphs. *Bernoulli* **4**, 477–495.

Wermuth, N. and Cox, D.R. (2004). Joint response graphs and separation induced by triangular systems. *J.Roy. Stat. Soc. B* **66**, 687-717.

Wermuth, N. and Cox, D.R. (2008). Distortions of effects caused by indirect confounding. *Biometrika* **95**, 17–33.

Wermuth, N., Cox, D.R. and Marchetti, G.M. (2006). Covariance chains. *Bernoulli* **12**, 841-862.

Wermuth, N. and Lauritzen, S.L. (1990). On substantive research hypotheses, conditional independence graphs and graphical chain models (with discusssion). *J. Roy. Statist. Soc. B* **52**, 21–75.

Wermuth, N., Marchetti, G.M. and Cox, D.R. (2009). Triangular systems for symmetric binary variables. *Electr. J. Statist.* **3**, 932–955.

Wermuth N. and Sadeghi, K. (2012). Sequences of regressions and their independences. To appear as invited discussion paper in *TEST*.

Wermuth, N., Wiedenbeck, M. and Cox, D.R. (2006). Partial inversion for linear systems and partial closure of independence graphs. *BIT, Numerical Mathematics* **46**, 883–901.

Wiedenbeck, M. and Wermuth, N. (2010). Changing parameters by partial mappings. *Statistica Sinica* **20**, 823–836.