THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Residue Currents and their Annihilator Ideals

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Abstract

This thesis presents results in multidimensional residue theory. From a generically exact complex of locally free analytic sheaves $\mathcal{C}$ we construct a vector valued residue current $R^{\mathcal{C}}$, which in a sense measures the exactness of $\mathcal{C}$.

If $\mathcal{C}$ is a locally free resolution of the ideal (sheaf) J the annihilator ideal of $R^{\mathcal{C}}$ is precisely J. This generalizes the Duality Theorem for Coleff-Herrera products of complete intersection ideals and can be used to extend several results, previously known for complete intersections.

We compute $R^{\mathcal{C}}$ explicitly if $\mathcal{C}$ is a so called cellular resolution of an Artinian monomial ideal J, and relate the structure of $R^{\mathcal{C}}$ to irreducible decompositions of J.

If $\mathcal{C}$ is the Koszul complex associated with a set of generators f of the ideal J the entries of $R^{\mathcal{C}}$ are the residue currents of Bochner-Martinelli type of f, which were introduced by Passare, Tsikh and Yger. We compute these in case J is an Artinian monomial ideal and conclude that the corresponding annihilator ideal is strictly included in J, unless J is a complete intersection.

We also define products of residue currents of Bochner-Martinelli type, generalizing the classical Coleff-Herrera product, and show that if f defines a complete intersection the product of the residue currents of Bochner-Martinelli type of subtuples of f coincides with the residue current of Bochner-Martinelli type of f.

Keywords: residue currents, Bochner-Martinelli formula, ideals of holomorphic functions, monomial ideals, coherent sheaves, free resolutions of modules, cellular resolutions

AMS 2000 Subject Classification: 32A26, 32A27, 32C30, 32C35, 13D02

This thesis consists of an introduction and the following papers:

Paper I:. Elizabeth Wulcan. Products of residue currents of Cauchy-Fantappiè-Leray type. *Arkiv för Matematik*, to appear

Paper II:. Elizabeth Wulcan. Residue currents of monomial ideals. *Indiana Univ. Math. J.* **56** (2007) 365 – 388

Paper III:. Mats Andersson and Elizabeth Wulcan. Noetherian residue currents. *Preprint*

Paper IV:. Elizabeth Wulcan. Residue currents constructed from resolutions of monomial ideals. *Preprint*

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RESIDUE CURRENTS AND THEIR ANNIHILATOR IDEALS

ELIZABETH WULCAN

1. Introduction

Residue calculus was introduced by Cauchy in 1825 as a tool for computing integrals and univariate series, and has since then developed to become a powerful tool in the study of many problems in algebra, geometry and analysis. For historical accounts and some recent applications, including effective versions of Hilbert's Nullstellensatz, generalizations of the Jacobi vanishing theorem, and explicit versions of the Fundamental principle, we refer to the surveys and books [13], [23], [24], [56], and [57].

Recall that the one-dimensional residue of a meromorphic function is defined as a contour integral around an isolated singularity of the function. When extending the notion of residue to higher dimensions several difficulties arise. In particular the zero set of a holomorphic mapping is no longer a discrete set of points, but an analytic variety that may have singularities. There are basically two different approaches to multidimensional residue theory. The so-called classical approach deals with integration of closed differental forms over cycles and includes the Grothendieck residue, see for example [33]. In this thesis we focus on the current approach which started out in the fifties, after the concept of currents was introduced and the importance of the $\bar\partial$-operator was pointed out by Dolbeault.

If $f = (f_1, \dots, f_r)$ is a holomorphic mapping defined in some domain in $\mathbb{C}^n$ that is a complete intersection, which means that $f^{-1}(0)$ is of codimension r, then there is a "canonical" residue current associated with f, namely the Coleff-Herrera product $\bar\partial[1/f_r]\wedge\ldots\wedge\bar\partial[1/f_1]$, introduced in [25]. The Duality Theorem, due to Passare [42] and Dickenstein-Sessa [26], which asserts that the ideal (f) generated by $f_1, \dots, f_r$ coincides with the annihilator of the Coleff-Herrera product, has turned out to be useful in applications.

If f is not a complete intersection it is not so clear how to find a residue current with the same properties as the Coleff-Herrera product. The key step in the search for a good notion turned out to be the paper [46], by Passare, Tsikh and Yger, in which residue currents of Bochner-Martinelli type were introduced. In [3] these currents were recovered as the coefficients of a residue current constructed from the Koszul complex.

In this thesis the residue currents of Bochner-Martinelli type are further studied and developed. In Paper I products of residue currents of Bochner-Martinelli type are defined, generalizing the Coleff-Herrera product. In Paper II we investigate how far the residue currents of Bochner-Martinelli type are from giving a duality theorem by computing them and their annihilators for monomial ideals. In Paper III, which is a joint work with Mats Andersson, we extend the construction of residue currents from the Koszul complex from [3] to more general complexes; this yields a residue current whose annihilator indeed equals (f). Finally, in Paper IV, we compute the currents from Paper III for monomial ideals.

In this introductory part of the thesis we will start by providing some background on residue currents, including a rather detailed description of Andersson's construction from [3], which is the starting point for our investigations. After that we will give brief overviews of the four papers included in the thesis, and illustrate the results by some examples.

Contents

2. Residue currents

Let us start by observing how a local ideal of holomorphic functions in one variable can be described in terms of vanishing of certain residues. Let f be a holomorphic function defined in some neighborhood Ω of $a \in \mathbb{C}$ and suppose that a is a zero of f of order m. Then a necessary and sufficient condition for a holomorphic function φ to be locally (near a) in the ideal generated by f is that

$$\operatorname{Res}_a \frac{(z-a)^k \varphi}{f} = 0 \tag{2.1}$$

for $k \leq m-1$. Here $\operatorname{Res}_a g$ just denotes the ordinary one-variable residue of the meromorphic function g at a defined by

$$\frac{1}{2\pi i} \int_{\partial\omega} g \, d\zeta,$$

where $\partial\omega$ is the (smooth) oriented boundary of some neighborhood ω of a.

Now let us go on to the higher-dimensional case and introduce residue currents as generalizations of residues. Let X be a domain in $\mathbb{C}^n$ or, more generally, a complex manifold of dimension n and let $f : X \to \mathbb{C}$ be a holomorphic mapping. Schwartz, [51], found that there exists a distribution (or $(0,0)$-current) U such that $fU = 1$. One way of realizing such a U, sometimes denoted by $[1/f]$, is as the *principal value* of $1/f$,

$$\mathcal{D}_{n,n}(X) \ni \xi \mapsto \lim_{\varepsilon \to 0} \int_{|f|>\varepsilon} \frac{\xi}{f}.$$

The existence of this limit was originally proved by Herrera and Lieberman, [34], using Hironaka's theorem on resolution of singularities, [35], to deal with the singularities of $Y = f^{-1}(0)$. Applying the $\bar\partial$-operator to $[1/f]$, we obtain a $(0,1)$-current with support on Y, which we call the *residue current* of f and which we denote by R^f. By Stokes' theorem its action is given by

$$\mathcal{D}_{n,n-1}(X) \ni \xi \mapsto \lim_{\varepsilon \to 0} \int_{|f|=\varepsilon} \frac{\xi}{f}, \tag{2.2}$$

where the limit is taken over the regular values of $|f|$.

Let (f) denote the ideal generated by f, and let φ be a holomorphic function on X. Then we have the following duality:

$$\varphi R^f = 0 \Leftrightarrow \varphi \in (f) \text{ locally.} \tag{2.3}$$

In other words, the principal ideal (f) in a local ring $\mathcal{O}_a$ of holomorphic functions defined in some neighborhood of $a \in X$ can be characterized as the annihilator ideal of the corresponding residue current. To see this, suppose that $\varphi \in (f)$. Then $\varphi = f\psi$ for some $\psi \in \mathcal{O}_a$ and so $\varphi R^f = \psi\bar\partial(f[1/f]) = \psi\bar\partial(1) = 0$. On the other hand, if $\varphi \in \mathcal{O}_a$ satisfies $\varphi R^f = 0$, let $\psi = \varphi[1/f]$. Then $f\psi = \varphi$ in the sense of distributions. By hypothesis $\bar\partial\psi = 0$ and so, by elliptic regularity for the $\bar\partial$-operator, $\psi \in \mathcal{O}_a$ and consequently $\varphi \in (f)$.

Let us consider a simple example.

Example 1. Let $f = z^p : \mathbb{C} \to \mathbb{C}$. Then

$$\bar\partial\left[\frac{1}{z^p}\right].\xi(z)\ dz = \frac{2\pi i}{(p-1)!}\frac{\partial^{p-1}}{\partial z^{p-1}}\xi(0), \tag{2.4}$$

as can be seen from (2.2) by integration by parts. From (2.4) it follows that $\varphi \in \mathcal{O}_0$ annihilates $\bar\partial[1/z^p]$ precisely if $\partial^\ell\varphi/\partial z^\ell(0) = 0$ for $0 \le \ell \le p-1$, that is, if $\varphi \in (z^p)$. It is also clear that this is equivalent to the vanishing of $\mathrm{Res}_0(z^\ell\varphi/z^p)$ for $0 \le \ell \le p-1$; compare with (2.1). □

Finally, let us point out an alternative definition of the principal value and residue currents, which traces back to Bernstein-Gelfand, [21], and Atiyah, [9], and which will be frequently used in this thesis. By resolving singularities one can show that the form $|f|^{2\lambda}/f$ has a meromorphic continuation as a current to the entire plane, with poles on the negative real axis; the value at $\lambda = 0$ yields an extension of $1/f$. Analogously $\bar\partial[1/f]$ can be obtained as the analytic continuation to $\lambda = 0$ of $\bar\partial|f|^{2\lambda}/f$.

2.1. **Coleff-Herrera-Passare products.** Given a holomorphic mapping $f = (f_1, \ldots, f_r) : X \to \mathbb{C}^r$, it is natural to look for an analogue of the current R^f that can be used to characterize the ideal (f) generated by $f_1, \ldots, f_r$. If f is a complete intersection, that is, $\operatorname{codim} f^{-1}(0) = r$, one can give meaning to the expression

$$\bar\partial\left[\frac{1}{f_r}\right]\wedge\ldots\wedge\bar\partial\left[\frac{1}{f_1}\right], \tag{2.5}$$

as was first done by Coleff and Herrera, [25], by proving the existence of certain limits of the so-called residue integral

$$I_f^\xi(\varepsilon) = \frac{1}{(2\pi i)^r}\int_{T_\epsilon^r}\frac{\xi}{f_r\cdots f_1}. \tag{2.6}$$

Here ξ is a test form of bidegree $(n, n-r)$ and $T_\varepsilon^r = \{|f_1| = \varepsilon_1, \ldots, |f_r| = \varepsilon_r\}$ is oriented as the distinguished boundary of the corresponding polyhedron. Coleff and Herrera showed that (2.6) indeed converges when ε tends to zero along certain so-called admissible paths. In general though, the unrestricted limit as ε tends to zero does not exist; a counterexample was first found by Passare and Tsikh in [45]. The convergence of (2.6) has later been the subject of several investigations, including [22], [41], [44], and [59]. Recently, Samuelsson [47], [48], [49], showed that when $r \le 3$ an unrestricted limit can be obtained by smoothing out the integration.

When f is a complete intersection, the *Coleff-Herrera product* (2.5), which we will for short denote by R_{CH}^f, turned out to be a good notion of a residue current. It is a closed current of bidegree $(0, r)$, with support on $Y = f^{-1}(0)$, and it satisfies the Duality Theorem, which generalizes (2.3).

Theorem 2.1 ([26], [42]). *Let X be an n-dimensional complex manifold, $f = (f_1, \dots, f_r) : X \to \mathbb{C}^r$ a holomorphic mapping, and suppose that φ is holomorphic on X. Then*

$$\varphi R^f_{CH} = 0 \text{ if and only if } \varphi \in (f_1, \dots, f_r) \text{ locally} .$$

The "if"-direction of Theorem 2.1 follows from the calculus for residue and principal value currents that was developed in [41]. When f is not a complete intersection, the residue integral (2.6) still converges when ε tends to zero along admissible paths, but in general the limit depends in an essential way on the ordering of the f_i. By taking certain averages of the residue integral (2.6), Passare managed to circumvent this problem and define products

$$\left[\frac{1}{f_r \cdots f_{s+1}}\, \bar\partial \frac{1}{f_s} \wedge \ldots \wedge \bar\partial \frac{1}{f_1}\right], \tag{2.7}$$

which are commuting with respect to the principal value factors $1/f_i$ and anti-commuting with respect to the residue factors $\bar\partial(1/f_i)$, and which satisfy Leibniz' rule. Alternatively, (2.7) can be obtained as the analytic continuation to $\lambda = 0$ of

$$|f_r|^{2\lambda}\frac{1}{f_r} \cdots |f_{s+1}|^{2\lambda}\frac{1}{f_{s+1}}\, \bar\partial |f_s|^{2\lambda}\frac{1}{f_s} \wedge \ldots \wedge \bar\partial |f_1|^{2\lambda}\frac{1}{f_1}, \tag{2.8}$$

as was proved in [40].

Let us denote by R^f_{CHP} the *Coleff-Herrera-Passare* product $[\bar\partial(1/f_1) \wedge \ldots \wedge \bar\partial(1/f_r)]$. If f is a complete intersection, R^f_{CHP} coincides with R^f_{CH}, and moreover (2.7) satisfies the rules

$$f_r \left[\frac{1}{f_r \cdots f_{s+1}}\, \bar\partial \frac{1}{f_s} \wedge \ldots \wedge \bar\partial \frac{1}{f_1}\right] = \left[\frac{1}{f_{r-1} \cdots f_{s+1}}\, \bar\partial \frac{1}{f_s} \wedge \ldots \wedge \bar\partial \frac{1}{f_1}\right] \tag{2.9}$$

and

$$f_s \left[\frac{1}{f_r \cdots f_{s+1}}\, \bar\partial \frac{1}{f_s} \wedge \ldots \wedge \bar\partial \frac{1}{f_1}\right] = 0. \tag{2.10}$$

In particular the "if"-direction of Theorem 2.1 follows. In light of (2.9) and (2.10), note that the Coleff-Herrera product really behaves like an exterior product of the currents $\bar\partial[1/f_r]$ through $\bar\partial[1/f_1]$; this motivates our use of brackets in (2.5).

Let us supply a proof of the other direction in the case when $r = 2$. Suppose that φ is a holomorphic function such that

$$\varphi\, \bar\partial \left[\frac{1}{f_2}\right] \wedge \bar\partial \left[\frac{1}{f_1}\right] = 0, \tag{2.11}$$

and that we wish to find a holomorphic $\psi = (\psi_1, \psi_2)$ such that

$$f_1\psi_1 + f_2\psi_2 = \varphi. \tag{2.12}$$

The basic idea, which will be further developed in Sections 3 and 7, is to start looking for a current solution to (2.12) and then modify it to a holomorphic solution by solving a certain $\bar\partial$-equation. Let $v_1 = \varphi[1/f_1]$. Then $f_1 v_1 = \varphi$ and so $(v_1, 0)$ is a current solution to (2.12). An arbitrary solution can now be written as

$$\psi = (v_1, 0) + \gamma(-f_2, f_1),$$

for some γ. Thus, in order to find a holomorphic solution ψ we need to solve

$$f_2\,\bar\partial\gamma = \bar\partial v_1 = \bar\partial\left[\frac{\varphi}{f_1}\right] \tag{2.13}$$

$$f_1\,\bar\partial\gamma = 0. \tag{2.14}$$

According to the calculus for the Coleff-Herrera-Passare products and the assumption (2.11) we have

$$\bar\partial\left[\frac{1}{f_2}\bar\partial\frac{\varphi}{f_1}\right] = \varphi\,\bar\partial\left[\frac{1}{f_2}\right]\wedge\bar\partial\left[\frac{1}{f_1}\right] = 0,$$

and so locally we can solve $\bar\partial\gamma = [(1/f_2)\bar\partial(1/f_1)]$. Clearly such a solution γ solves (2.13), and moreover $f_1[(1/f_2)\bar\partial(1/f_1)] = 0$ by (2.9) and so γ also satisfies (2.14). Hence we have found a local holomorphic solution to (2.12).

The assumption that f is a complete intersection is crucial; in general Leibniz' rule and (2.10) cannot hold simultaneously, which we illustrate by the following simple example.

Example 2. Let $f = (z^2, zw)$ be defined in some neighborhood of the origin in $\mathbb{C}^2$. Then f is clearly not a complete intersection; indeed $Y = \{z = 0\}$ has codimension 1. By Leibniz' rule

$$R^f_{CHP} = \left[\bar\partial\frac{1}{zw}\wedge\bar\partial\frac{1}{z^2}\right] = \frac{1}{3}\,\bar\partial\left[\frac{1}{z^3}\right]\wedge\bar\partial\left[\frac{1}{w}\right],$$

and so z^2 does not annihilate R^f_{CHP}, whereas w does. More precisely, $\operatorname{Ann} R^f_{CHP} = (z^3, w)$. □

Observe that in this particular example, $\operatorname{Ann} R^f_{CHP} \neq (f)$, which follows immediately from the fact that the zero variety of $\operatorname{Ann} R^f_{CHP}$, $V(\operatorname{Ann} R^f_{CHP})$ is the origin, whereas $Y = \{z = 0\}$.

Remark 1. This idea can in fact be extended to prove that we fail to get duality for the Coleff-Herrera-Passare product R^f_{CHP} as soon as f is not a complete intersection. In [43] it was shown that the support of R^f_{CHP} is contained in a variety of codimension r. Moreover, it is not hard to see that if T is a current with support contained in the analytic variety W, then $V(\operatorname{Ann} T)$ is a subvariety of W. Hence, in the case when f is not a complete intersection the dimensions of Y and $V(\operatorname{Ann} R^f_{CHP})$ are different and so we conclude that $\operatorname{Ann} R^f_{CHP} = (f)$ if and only if f is a complete intersection. $\square$

As we will see below, to capture an ideal that is not a complete intersection will in general take more than one current. In Section 7 we will construct residue currents that extend Theorem 2.1. These currents will have different components corresponding to different primary components of (f). Typically, the current corresponding to a primary component of codimension p will be of bidegree $(0, p)$.

Let us discuss some applications and further properties of the Coleff-Herrera-Passare products.

Residue currents have been used together with weighted integral formulas to obtain explicit formulas for division and interpolation, extending the construction of Berndtsson, [19]. Suppose that $f = (f_1, \ldots, f_r)$ is defined in some strictly pseudoconvex domain Ω in $\mathbb{C}^n$. In [42] Passare constructed a representation formula for $\varphi \in \mathcal{O}(\Omega)$:

$$\varphi(z) = f(z) \cdot \int_\zeta T(\zeta, z)\varphi(\zeta) + \int_\zeta S(\zeta, z)\varphi(\zeta). \tag{2.15}$$

Here T and S are currents that are holomorphic in z and $S(., z)$ is closely related to R^f_{CH}; in particular $S\varphi = 0$ if $\varphi R^f_{CH} = 0$. Note that this yields the "only if"-direction of Theorem 2.1. Division formulas of this kind have been used by several authors for various purposes, such as explicit versions of the Fundamental principle, [20] and [62], sharp approximation by polynomials, [63], estimates for the degree of solutions to the Bezout equation, [16], and residue characterizations of ideals of smooth functions, [2]. See also the monograph [13] and the references quoted there.

Dickenstein and Sessa, [26], showed that if Y is a subvariety of X of pure dimension r, which is locally a complete intersection, then every $\bar\partial$-closed current T on X of bidegree $(0, r)$ with support on Y admits a unique representation: $T = R + \bar\partial S$. Here S is $(0, r-1)$-current with support on Y and R is a local residual current, which means that locally it is of the form hR^f_{CH} for some holomorphic function h and some complete intersection f that vanishes on Y. In other words, the classes in the local cohomology groups $H^r_{[Y]}(X, \mathcal{O})$ have canonical residue current representatives. In [31] Fabre used residue currents to compute Dolbeault cohomology groups.

Let us also relate R^f_{CH} to the current of integration over $Y = f^{-1}(0)$. Coleff and Herrera, [25], showed that if f is a complete intersection, then

$$\bar{\partial}\left[\frac{1}{f_r}\right] \wedge \ldots \wedge \bar{\partial}\left[\frac{1}{f_1}\right] \wedge \frac{df_1 \wedge \ldots \wedge df_r}{(2\pi i)^r} = \sum \alpha_j [Y_j]. \tag{2.16}$$

Here $[V]$ just denotes the current of integration over the variety V, Y_j are the irreducible components of Y, and α_j are the corresponding Hilbert-Samuel multiplicities. This kind of factorization formula has been used, for example, to construct explicit Green currents of analytic cycles, [17], [18]. Observe that the residue current really contains more information than the right hand side of (2.16). Indeed, in contrast to the one-dimensional case, the ideal (f) is not determined by its multiplicities; in particular α_j only depends on $|f|$. Morally, residue currents represent ideals in the same way that currents of integration represent varieties.

The Coleff-Herrera product satisfies the so-called Transformation law, [27]: if $g = \Psi f$ for some biholomorphic mapping $\Psi : \mathbb{C}^r \to \mathbb{C}^r$, then $R^f_{CH} = \det \Psi R^g_{CH}$, so R^f_{CH} really depends on the ideal (f) rather than the particular choice of generators.

If $h \in \mathcal{O}(X)$ vanishes on Y, the Coleff-Herrera product R^f_{CH} is annihilated by $\overline{h}$. Similar statements hold also for the residue currents that will be considered below. Basically this means that on the regular part of $f^{-1}(0)$, R^f_{CH} involves only holomorphic derivatives in the normal direction of $f^{-1}(0)$.

Finally, let us remark that the theory of multidimensional residue currents in general relies heavily on Hironaka's famous theorem on resolution of singularities from 1964, [35]. However, recently Mazzilli, [37], constructed a residue current in the case $r = 1$, by elementary methods using only the Weierstrass preparation theorem, and in [57], Tsikh and Yger used amoebas, in the sense of [32], to prove the convergence of the residue integral (2.6) for complete intersections when $r = n$.

2.2. **Residue currents of Bochner-Martinelli type.** In [46], Passare, Tsikh and Yger introduced an alternative approach to the multidimensional residue current, based on the Bochner-Martinelli kernel. In fact, Bochner-Martinelli type division formulas were already used in [15] to obtain Jacobi formulas and effective versions of the Nullstellensatz. Such formulas also appeared in [13]. For each ordered index set $\mathcal{I} = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$ of cardinality k, let $R^f_{\mathcal{I}}$ be the analytic continuation to $\lambda = 0$ of

$$\bar{\partial}|f|^{2\lambda} \wedge \sum_{\ell=1}^{k} (-1)^{\ell-1} \frac{\overline{f_{i_\ell}} \bigwedge_{q \neq \ell} \overline{df_{i_q}}}{|f|^{2k}}, \tag{2.17}$$

where $|f|^2 = |f_1|^2 + \cdots + |f_r|^2$. Then $R^f_{\mathcal{I}}$ is a well-defined $(0,k)$-current with support on Y, which is annihilated by $h \in \mathcal{O}(X)$ if h is locally in the integral closure of $(f)^k$. Recall that a holomorphic function $h \in \mathcal{O}_a$ belongs to the *integral closure* of (f), denoted by $\overline{(f)}$, if $|h| \leq C|f|$ for some constant C in some neighborhood of a, or equivalently if h satisfies a monic equation $h^s + g_1 h^{s-1} + \cdots + g_s = 0$ with $g_i \in (f)^i$ for $1 \leq i \leq s$. Thus, letting $\operatorname{Ann} R^f$ denote the annihilator ideal, $\{h \text{ holomorphic}, hR^f_{\mathcal{I}} = 0, \forall \mathcal{I}\}$ in $\mathcal{O}_a$, we have that

$$\overline{(f)^\mu} \subseteq \operatorname{Ann} R^f, \tag{2.18}$$

where $\mu = \min(r, n)$.

Moreover, $R^f_{\mathcal{I}}$ vanishes whenever $k < \operatorname{codim} Y$ or $k > \mu$. In particular, if f defines a complete intersection there is only one non-vanishing current, $R^f_{\{1,\ldots,r\}}$. Note that $R^f_{\{1,\ldots,r\}}$ can formally be seen as $\bar\partial f^* B$, where B is the Bochner-Martinelli kernel

$$B(w) = \sum_{\ell=1}^{r} (-1)^{\ell-1} \frac{\bar w_\ell \ \bigwedge_{q\neq\ell} d\bar w_q}{|w|^{2r}}$$

in $\mathbb{C}^r$, whereas R^f_{CH} can be seen as the pullback of the multiple Cauchy kernel $C = \bar\partial(1/w_r) \wedge \ldots \wedge \bar\partial(1/w_1)$. Recall that $\bar\partial B = C = \tau$, where $\tau \wedge dw_1 \wedge \ldots \wedge dw_r/(2\pi i)^r = [0]$, and so $R^w_{\{1,\ldots,r\}} = R^w_{CH}$ for $w = (w_1, \ldots, w_r)$. Thus, provided one can give meaning to f^*B and f^*C, it is reasonable to expect $R^f_{\{1,\ldots,r\}}$ and R^f_{CH} to coincide also for more general f.

Theorem 2.2 ([46]). *Let X be an n-dimensional complex manifold and $f = (f_1, \ldots, f_r) : X \to \mathbb{C}^r$ a holomorphic mapping that is a complete intersection. Then the current $R^f_{\{1,\ldots,r\}}$ coincides with the Coleff-Herrera product R^f_{CH}.*

The residue currents of Bochner-Martinelli type were further studied by Andersson in [3]. From his global construction, which is based on the Koszul complex and which will be described in more detail in Section 3, it follows that $\varphi R^f_{\mathcal{I}} = 0$ for all $\mathcal{I} \subseteq \{1, \ldots, r\}$ implies that the holomorphic function φ belongs to the ideal (f) locally. In other words

$$\operatorname{Ann} R^f \subseteq (f), \tag{2.19}$$

which gives one direction of the Duality Theorem. However, the inclusion is strict in general; in Paper II we show that in the case of Artinian monomial ideals it is always strict unless (f) is a complete intersection. Still, $\operatorname{Ann} R^f$ in some sense captures the "size" of (f). In particular, combined with (2.18) it yields a new proof of the classical Briançon-Skoda theorem [53]:

$\overline{(f)^\mu} \subseteq (f)$. A proof of the Briançon-Skoda Theorem based on residue calculus and division formulas was in fact already obtained in [13].

The residue currents of Bochner-Martinelli type have been used for different purposes; in particular for investigations in the non-complete intersection case, [18]. Vidras and Yger, [60], used residue currents of Bochner-Martinelli type to prove some generalizations of Jacobi's theorem on vanishing of residues, which were further developed in [14]. Andersson, [4], obtained factorization formulas in terms of $R^f_{\mathcal{I}}$ for the current of integration of the component of Y of highest dimension, generalizing (2.16). In [52] a local residue was defined following the ideas from [46]. The global construction from [3] was used in [7] to obtain solutions of membership problems with control of the polynomial degrees, and in [3] a new geometric and simpler proof of the Jacobi-type theorem from [60] was given; this idea was further developed in [50].

3. Residue currents constructed from the Koszul complex

For future reference, we will give a description of Andersson's construction of residue currents from the Koszul complex. Suppose that, given the holomorphic mapping $f = (f_1, \dots, f_r) : X \to \mathbb{C}^r$, we are looking for a local holomorphic solution $\psi = (\psi_1, \dots, \psi_r)$ to

$$f_1\psi_1 + \cdots + f_r\psi_r = \varphi. \tag{3.1}$$

We will discuss how this division problem can be solved (when possible) in terms of the Koszul complex. As in our proof of Theorem 2.1 we will start by looking for a smooth or current solution to (3.1) and then modify this to a holomorphic solution by solving certain $\bar\partial$-equations. We adopt an invariant point of view and assume that f is a holomorphic section of the dual bundle E^* of a holomorphic r-bundle $E \to X$. If $e_1, \dots, e_r$ is a local holomorphic frame for E and $e_1^*, \dots, e_r^*$ is the dual frame, we can write f as $\sum f_j e_j^*$. Now the *Koszul complex* of f is the complex

$$0 \xrightarrow{\delta_f} \Lambda^r E \xrightarrow{\delta_f} \Lambda^{r-1} E \xrightarrow{\delta_f} \dots \xrightarrow{\delta_f} \Lambda^2 E \xrightarrow{\delta_f} E \xrightarrow{\delta_f} \mathbb{C} \times X \longrightarrow 0, \tag{3.2}$$

where δ_f is contraction with f, that is, locally

$$\delta_f : e_{i_1} \wedge \ldots \wedge e_{i_k} \mapsto \sum_\ell (-1)^{\ell-1} f_{i_\ell}\, e_{i_1} \wedge \ldots \wedge e_{i_{\ell-1}} \wedge e_{i_{\ell-1}} \wedge \ldots \wedge e_{i_k}.$$

Note in particular that δ_f acts on a section $\psi = \sum \psi_j e_j$ of E as $\delta_f \psi = \sum f_j \psi_j$, and thus (3.1) can expressed as

$$\delta_f \psi = \varphi. \tag{3.3}$$

The Koszul complex is pointwise exact outside $Y = f^{-1}(0)$ and thus outside Y one can find a smooth section u_1 of E which satisfies (3.3). In fact we can choose u_1 as $\sum \bar{f}_j e_j^*/|f|^2$. In general u_1 will not be holomorphic, and we need to compensate for that. Let us introduce the spaces $\mathcal{E}_{0,k}(X, \Lambda^\ell E)$ of smooth sections of $\Lambda(T^*_{0,1} \oplus E)$ (where $d\bar{z}_i \wedge e_j = -e_j \wedge d\bar{z}_i$), that are $(0,k)$-forms taking values in $\Lambda^\ell E$, and the corresponding spaces $\mathcal{D}'_{0,k}(X, \Lambda^\ell E)$ of currents. Note that δ_f and $\bar\partial$ extend to $\Lambda(T^*_{0,1} \oplus E)$, where they anti-commute. Thus, $\delta_f \bar\partial u_1 = -\bar\partial \delta_f u_1 = 0$, and due to the exactness at $\Lambda^2 E$ we can find a $u_2 \in \mathcal{E}_{0,1}(X, \Lambda^2 E)$ such that $\delta_f u_2 = \bar\partial u_1$. We proceed by successively solving

$$\delta_f u_1 = \varphi, \quad \delta_f u_k = \bar\partial u_{k-1}, \quad k \geq 1 \tag{3.4}$$

where $u_k \in \mathcal{E}_{0,k-1}(X, \Lambda^k E)$. Now suppose that all u_k have current extension over Y such that (3.4) still holds. Then in particular $\bar\partial u_r = 0$, so by successively solving equations

$$\bar\partial v_{k-1} = u_{k-1} + \delta_f v_k \tag{3.5}$$

for $k \leq r$, we finally arrive at the desired local holomorphic solution

$$\psi = u_1 + \delta_f v_2$$

to (3.3). Now, if we introduce the notation $\nabla_f = \delta_f - \bar\partial$, the system of equations (3.3) and (3.4) can be expressed as

$$\nabla_f u^f = \varphi,$$

where $u^f = u_1 + \cdots + u_r$. Note that ∇_f is an anti-derivation on $\bigoplus \mathcal{E}_{0,k}(X, \Lambda^\ell E)$.

Notice that if $\nabla_f u^f = 1$, it then follows that $\nabla_f(\varphi u^f) = \varphi$ if φ is holomorphic. To find a solution to $\nabla_f u^f = 1$ in $X \setminus Y$, let us assume that E is equipped with some Hermitian metric and let σ be the section of E with pointwise minimal norm such that $\delta_f \sigma = 1$. Outside Y, the full Cauchy-Fantappiè-Leray form, introduced in [1],

$$u^f = \frac{\sigma}{\nabla_f \sigma} = \frac{\sigma}{\delta_f \sigma - \bar\partial \sigma} = \frac{\sigma}{1 - \bar\partial \sigma} = \sum_\ell \sigma \wedge (\bar\partial \sigma)^{\ell-1},$$

is well-defined (observe that $\bar\partial \sigma$ is of even degree), and $\nabla_f u^f = 1$. Now, the form $|f|^{2\lambda} u^f$ can be extended as a current to $\operatorname{Re} \lambda > -\epsilon$, and the value at $\lambda = 0$, which we denote by U^f, yields an extension of u^f over Y. By analogy with the one-dimensional case, we will sometimes refer to U^f as a principal value current. Moreover, $\nabla_f U^f = 1 - R^f$, where $R^f = \bar\partial |f|^{2\lambda} \wedge u^f|_{\lambda=0}$ now defines the residue current of f. Clearly R^f will have support on Y and $R^f = R_p + \cdots + R_\mu$, where $R_k \in \mathcal{D}'_{0,k}(X, \Lambda^k E)$, $p = \operatorname{codim} Y$ and $\mu = \min(r, n)$. In particular, if f is a complete intersection, then $R = R_r$.

Furthermore, suppose that φ is a holomorphic function on X such that $\varphi R^f = 0$. Then $\nabla_f(\varphi U^f) = \varphi \nabla_f U^f = \varphi(1 - R^f) = \varphi$. Hence $\nabla_f w = \varphi$ has a current solution, and therefore by solving $\bar\partial$-equations (3.5), a local holomorphic solution. Thus we have proved (2.19).

Note that if $E = \mathbb{C}^r \times X$ with the trivial metric, then the coefficients of R^f are just the residue currents of Bochner-Martinelli type, $R^f_{\mathcal{I}}$. Indeed, if $f = \sum f_j e_j^*$, then $\sigma = \sum_j \bar f_j e_j/|f|^2$ and

$$u^f = \sum_\ell \frac{\sum \bar f_j e_j \wedge (\sum \bar\partial \bar f_j e_j)^{\ell-1}}{|f|^{2\ell}}.$$

Hence the coefficient of $e_{i_k} \wedge \ldots \wedge e_{i_1}$ will just be $R^f_{\{i_1,\ldots,i_k\}}$. In this case we will say that R^f is of Bochner-Martinelli type; in the more general case we will say that it is of Cauchy-Fantappiè-Leray type.

4. Paper I

Given a tuple f of holomorphic functions $f_1, \ldots, f_r$ defined on X we saw above that the value at $\lambda = 0$ of (2.8) defines a product of the principal value currents $[1/f_i]$ and residue currents $\bar\partial[1/f_i]$ and moreover that this product provides a natural notion of a residue current of f if f is a complete intersections. In Paper I we extend this construction to allow also for the more general currents of Cauchy-Fantappiè-Leray type U^{f_i} and R^{f_i} from [3], where each f_i is itself a tuple of functions or, more generally, a section of a Hermitian vector bundle.

Regarding each function f_i as a section of the dual bundle E_i^* of a trivial line bundle $E_i \to X$ over X with frame e_i^* and e_i, respectively, the Cauchy-Fantappiè-Leray form from Section 3 is just $u^{f_i} = 1/f_i e_i$ and so (2.8) times $\pm e_1 \wedge \ldots \wedge e_r$ can be expressed as

$$|f_r|^{2\lambda} \wedge u^{f_r} \wedge \ldots \wedge |f_{s+1}|^{2\lambda} \wedge u^{f_{s+1}} \wedge \bar\partial |f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1}. \quad (4.1)$$

This formulation suggests how to extend the definition of products. Let the f_i be holomorphic sections of the dual bundles E_i^* of Hermitian m_i-bundles $E_i \to X$ over X and let u^{f_i} be the Cauchy-Fantappiè-Leray form with respect to some Hermitian metric, see Section 3. Then, for $\operatorname{Re}\lambda$ large enough, (4.1) can be seen as a form taking values in the exterior algebra over $E = E_1 \oplus \ldots \oplus E_r$. The analytic continuation to $\lambda = 0$, which we denote by

$$T = U^{f_r} \wedge \ldots \wedge U^{f_{s+1}} \wedge R^{f_s} \wedge \ldots \wedge R^{f_1},$$

defines a (globally defined) product of the currents of Cauchy-Fantappiè-Leray type U^{f_i} and R^{f_i}. It is commuting with respect to the principal value factors U^{f_i} and anti-commuting with respect to the residue current factors R^{f_i}, and its support is contained in $\bigcap_{i=1}^s Y_i$, where $Y_i = f_i^{-1}(0)$.

Moreover $T = T_p + \cdots + T_\mu$, where $T_\ell \in \mathcal{D}'_{0,\ell}(\Lambda^\bullet E)$, $p = \text{codim} \bigcap_{i=1}^s Y_i$, and $\mu = \min(m, n)$, where $m = m_1 + \cdots + m_r$. In particular, if f is a complete intersection, $R^{f_r} \wedge \ldots \wedge R^{f_1}$ consists of only one term of top degree m. If E is a trivial bundle over X endowed with the trivial metric, the coefficient of this current defines a product of the residue currents of Bochner-Martinelli type $R^{f_i}_{\{1,\ldots,m_i\}}$; compare with the discussion in the previous section.

Hence, given a tuple of functions f we can define different residue currents by dividing f into subtuples and taking the product of the residue currents of Bochner-Martinelli type of each of these. In particular, letting the subtuples consist of single functions one recovers the Coleff-Herrera-Passare product. Our main result, which generalizes Theorem 2.2, asserts that when f is a complete intersection these currents all coincide.

Theorem 4.1. *Let f_i be a holomorphic section of the Hermitian m_i-bundle E_i^* and let f denote the section $f_1 \oplus \ldots \oplus f_r$ of $E^* = E_1^* \oplus \ldots \oplus E_r^*$. Suppose that f is a complete intersection, that is, codim $f^{-1}(0) = m_1 + \cdots + m_r$. Then*

$$R^{f_r} \wedge \ldots \wedge R^{f_1} = R^f.$$

The theorem is proved by finding currents V and $V \wedge U^f$ such that $\nabla_f V = 1 - R^{f_r} \wedge \ldots \wedge R^{f_1}$ and $\nabla_f(V \wedge U^f) = V - U^f$. Then (recall that ∇_f is an anti-derivation)

$$0 = \nabla_f^2(V \wedge U^f) = \nabla_f(V - U^f) = R^{f_r} \wedge \ldots \wedge R^{f_1} - R^f,$$

and the result follows. The idea comes from Proposition 4.2 in [3] where similar potentials were constructed to prove Theorem 2.2. The technical core of Paper I consists of verifying the formal computations that allow us to find such potentials.

If f is not a complete intersection, Theorem 4.1 in general fails to hold, as is discussed and illustrated by examples in the last section of Paper I.

5. Monomial ideals

Parts of this thesis concern monomial ideals. The theory of monomial ideals is one of the strong links between commutative algebra, algebraic geometry and combinatorics and it has been extensively developed in recent years, see for example [29], [38] and [55]. Because of their simplicity and nice combinatorial description monomial ideals serve as a good toy model for illustrating general ideas and results in commutative algebra and algebraic geometry, such as resolution of singularities and Briançon-Skoda type theorems. On the other hand many results for general ideals can be proved by specializing to monomial ideals, for example by the use of Gröbner bases. In fact, the existence of the analytic continuations of the forms (2.8), (2.17) et cetera is indeed proved by reducing to a monomial situation via resolution of singularities.

Monomial ideals are therefore a natural starting point for explicit computations of residue currents. A first result in this direction was obtained in [46] where $R^f_{\{1,\dots,n\}}$ was computed explicitly for monomial ideals generated by exactly n monomials. Also, in [3] and Paper I some residue currents of monomial ideals were computed.

We will consider monomial ideals in the local ring $\mathcal{O}_0^n$ of holomorphic functions defined in some neighborhood of $0 \in \mathbb{C}^n_z$ and in the polynomial ring $S = \mathbb{C}[z_1, \dots, z_n]$. An ideal in either of these rings is said to be *monomial* if it can be generated by monomials, $z^a = z_1^{a_1} \cdots z_n^{a_n}$ for $a = (a_1, \dots, a_n) \in \mathbb{N}^n$. One can prove that a monomial ideal has a unique minimal set of monomial generators. Another "minimal" description is given by its irredundant irreducible decomposition. A monomial ideal is said to be *irreducible* if it is generated by powers of variables. For $b = (b_1, \dots, b_n) \in \mathbb{N}^n$ we will use the notation $\mathfrak{m}^b$ for the irreducible ideal $(z_1^{b_1}, \dots, z_n^{b_n})$. An *irreducible decomposition* of a monomial ideal M is an expression $M = \cap_{i=1}^q \mathfrak{m}^{b^i}$, for some $b^i \in \mathbb{N}^n$. If no intersectand can be omitted the decomposition is said to be *irredundant* and the ideals $\mathfrak{m}^{b^i}$ are then called the *irreducible components* of M. One can prove that each monomial ideal indeed has a unique irredundant irreducible decomposition.

For $A \subseteq \mathbb{N}^n$ let z^A denote the tuple of monomials $\{z^a\}_{a \in A}$. Observe that the monomial ideal (z^A) is precisely the set of functions that have support,

$$\operatorname{supp} \sum_{a \in \mathbb{Z}^n} c_a z^a = \{a \in \mathbb{Z}^n | c_a \neq 0\},$$

in $\bigcup_{a \in A}(a + \mathbb{R}^n_+)$ and thus the ideal can be represented by this set, see Figures 1 and 2 and also the figures in Paper II and Paper IV. Here the box constellations in the figures should be thought of as the area "below" $\bigcup_{a \in A}(a + \mathbb{R}^n_+)$. Because of their, at least in two dimensions, staircase-like appearance, these pictures of monomial ideals are usually referred to as *staircase diagrams.* Note that the coordinates of the "inner corners" are precisely the set of minimal generators, whereas the "outer corners" correspond to the irreducible components.

6. Paper II

In Paper II we compute residue currents of Bochner-Martinelli type associated with monomial ideals. The main motivation is to investigate by how much we fail to get duality for these currents. In light of (2.19) it is natural to ask how big the annihilator of R^f is. Could it happen that $\operatorname{Ann} R^f = (f)$

without f being a complete intersection? We are also curious about the inclusion (2.18). The Briançon-Skoda Theorem is sharp, but still (2.18) might always be strict.

Our main results concern Artinian monomial ideals (z^A), that is, $\{z^A = 0\} = \{0\}$, in $\mathcal{O}_0^n$. Then, a priori, R^{z^A} consists of one component R_n, which could be seen as a vector valued current with one entry R_B for each subset $B \subseteq A$ of cardinality n. Note that from general distribution theory we know that each R_B will be a sum of Dirac measures. The *Newton polyhedron* of A is then defined as the convex hull of $\cup_{a\in A}(a + \mathbb{R}^n_+) \subseteq \mathbb{R}^n$, and the *Newton diagram* of A is the union of all compact faces of the Newton polyhedron. We will say that a subset $B = \{a_1, \ldots, a_n\} \subseteq A$ is *essential* if there exists a facet (face of maximal dimension) F of the Newton diagram of A such that B lies in F and if in addition B spans $\mathbb{R}^n$, that is, $\det(a_1, \ldots, a_n) \neq 0$. Our main theorem states that R_B is non-vanishing precisely when B is essential.

Theorem 6.1. *Let (z^A) be an Artinian monomial ideal in $\mathcal{O}_0^n$, and let R^{z^A} be the corresponding residue current of Bochner-Martinelli type. Then*

$$R^{z^A} = (R_B)_{B \text{ essential}},$$

where

$$R_B = C_B\ \bar\partial\left[\frac{1}{z_1^{\alpha_1^B}}\right]\wedge\ldots\wedge\bar\partial\left[\frac{1}{z_n^{\alpha_n^B}}\right]. \tag{6.1}$$

Here C_B is a non-vanishing constant and $(\alpha_1^B, \ldots, \alpha_n^B) = \alpha_B = \sum_{a\in B} a$.

An immediate consequence is that

$$\operatorname{Ann} R_B = (z_1^{\alpha_1^B}, \ldots, z_n^{\alpha_n^B}) = \mathfrak{m}^{\alpha_B}$$

if B is essential, and since to annihilate R^{z^A} one has to annihilate all entries, we find that

$$\operatorname{Ann} R^{z^A} = \bigcap_{B \text{ essential}} \mathfrak{m}^{\alpha_B}.$$

Thus we get an explicit description of $\operatorname{Ann} R^{z^A}$ in terms of the Newton diagram of A; its irreducible components correspond precisely to the essential sets. In particular, it turns out that the inclusion (2.19) is always strict unless (z^A) is a complete intersection, which should be compared with Remark 1.

Theorem 6.2. *Let (z^A) be an Artinian monomial ideal in $\mathcal{O}_0^n$, and let R^{z^A} be the corresponding residue current of Bochner-Martinelli type. Then*

$$Ann\, R^{z^A} = (z^A)$$

if and only if (z^A) can be generated by a complete intersection.

From Theorem 6.1 one can, for example, also see that not all monomial ideals are annihilator ideals of residue currents of Bochner-Martinelli type (of monomial ideals), as shown in Example 3 in Paper II. It also follows that different ideals can have the same annihilator ideal, as illustrated in Example 4 in Paper II. Moreover, we show that the inclusion (2.18) is always strict for $n \geq 2$; this is Corollary 3.5 in Paper II.

The proof of Theorem 6.1 is an explicit computation based on the proof of the existence of residue currents of Bochner-Martinelli type in [46] and [3]. It relies on an idea originally due to Khovanskii [36] and Varčhenko [58]; from the Newton polyhedron of A one can construct a certain toric manifold in which it locally holds that the pullback of one of the monomials $z^a, a \in A$, divides that of the others. The current R^{z^A} will be computed as the push forward of certain currents on this toric manifold.

We also provide partial results in the non-Artinian case. When the variety of (z^A) is of positive dimension the computations get more involved. Now $R^{z^A} = R_p + \cdots + R_\mu$ for $p = \operatorname{codim} Y$ and $\mu = \min(r, n)$ if r is the number of elements in A; here $R_k \in \mathcal{D}'_{0,k}(\mathbb{C}^n, \Lambda^k E)$. Parts of the top degree term R_n can be computed by the techniques from the Artinian case. Our method for dealing with the terms of lower degree is to perform the computations outside certain varieties, wherein some of the coordinates are zero. This amounts to projecting A and brings us back to the top degree case in a lower dimension. In this way we manage to determine precisely which entries of R_k are non-vanishing, but we fail in general to get an explicit formula like (6.1) for all of them, and thereby to fully describe the annihilator of R^{z^A}. See Theorem 5.2 in Paper II for a precise statement. Still our result allows us to extend Theorem 6.2 to a much larger class of ideals; in particular it holds for all monomial ideals when $n = 2$.

In view of the discussion in Remark 4 and Example 6 in Paper II, it is reasonable to believe that Theorem 6.2 extends to all monomial ideals. Yet it is not clear what one should hope for in the general case. Of course, monomial ideals are very special ideals and very far from being complete intersections (when they are not). On the other hand Theorem 6.2 suggests that we rather seldom get equality in (2.19), which motivates the search for a new notion of residue currents in the non-complete intersection case.

We should also point out that in the Artinian case the annihilator does not depend on the particular choice of generators of the monomial ideal, whereas in the non-Artinian case it does. This latter phenomenon is somewhat unsatisfactory and further motivates the search for an alternative notion of residue currents.

Finally, let us illustrate Theorem 6.1 by an example.

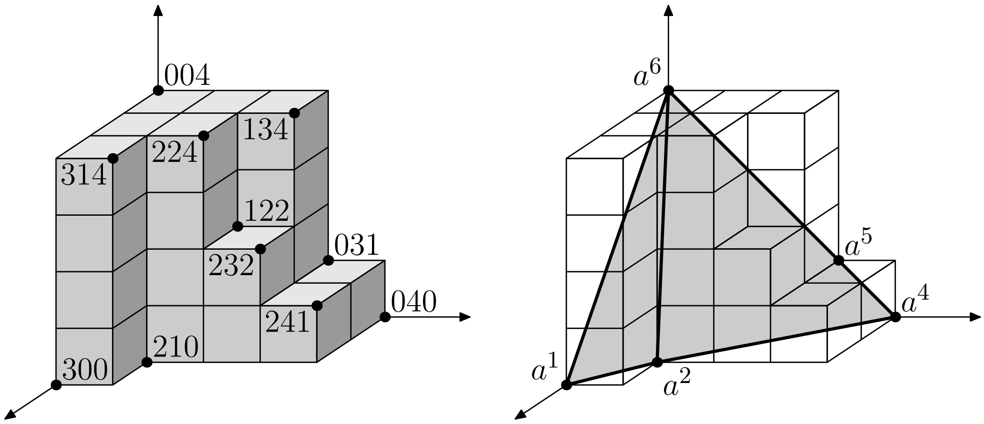

FIGURE 1. The ideal (z^A) and the Newton diagram in Example 3.

Example 3. Let

$$A = \{a^1 = (3,0,0), a^2 = (2,1,0), a^3 = (1,2,2),$$
$$a^4 = (0,4,0), a^5 = (0,3,1), a^6 = (0,0,4)\} \subset \mathbb{N}^3,$$

and let us use the notation x, y, z for the variables in $\mathbb{C}^3$. The corresponding Artinian ideal $(z^A) = (x^3, x^2y, xy^2z^2, y^4, y^3z, z^4)$ is depicted in Figure 1. Note that A is precisely the set of coordinates of the inner corners of the staircase. To the left we have also drawn the the Newton diagram of A; it should be regarded as lying just below the transparent staircase, touching the inner corners. Observe that the Newton diagram has two compact facets with vertices $\{a^1, a^2, a^6\}$ and $\{a^2, a^4, a^6\}$ respectively and moreover that a^5 lies on the second facet. Hence we have the following essential sets:

$$\{a^1, a^2, a^6\},\ \{a^2, a^4, a^5\},\ \{a^2, a^4, a^6\},\ \text{and}\ \{a^2, a^5, a^6\},$$

with

$$\alpha_{126} = (5,1,4),\ \alpha_{245} = (2,8,1),\ \alpha_{246} = (2,5,4),\ \text{and}\ \alpha_{256} = (2,4,5),$$

respectively. Note that the set $B = \{a^4, a^5, a^6\}$ is indeed a set of cardinality 3 that is contained in a facet of the Newton diagram. However, B does not span $\mathbb{R}^3$ since the generators lie on a line and consequently B is not essential.

Now, according to Theorem 6.1 the corresponding residue of Bochner-Martinelli type R^{z^A} has one entry for each essential set. For example

$$R_{126} = C\ \bar\partial\left[\frac{1}{x^5}\right] \wedge \bar\partial\left[\frac{1}{y}\right] \wedge \bar\partial\left[\frac{1}{z^4}\right],$$

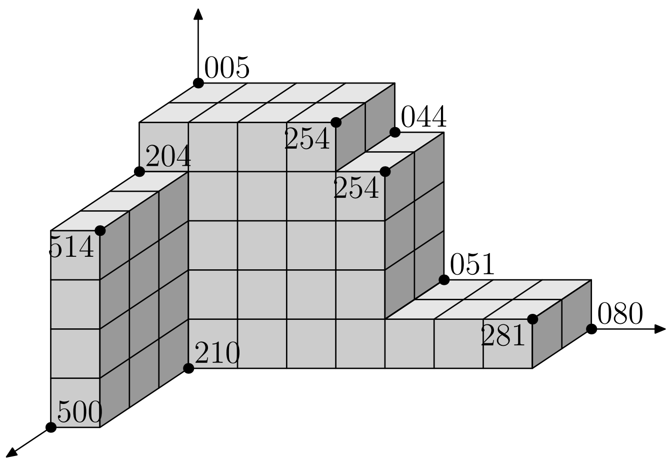

FIGURE 2. The ideal $\operatorname{Ann} R^{z^A}$ in Example 3.

for some constant $C \neq 0$, and so $\operatorname{Ann} R_{126} = (x^5, y, z^4)$. It follows that

$$\operatorname{Ann} R^{z^A} = (x^5, y, z^4) \cap (x^2, y^8, z) \cap (x^2, y^5, z^4) \cap (x^2, y^4, z^5),$$

which is equal to the ideal $(x^5, x^2y, x^2z^4, y^8, y^5z, y^4z^4, z^5)$, depicted in Figure 2. Observe that the essential sets correspond to the outer corners of the staircase diagram of $\operatorname{Ann} R^{z^A}$. Note also that $\operatorname{Ann} R^{z^A}$ does not depend on a^3 which lies in the interior of the Newton polyhedron. □

7. PAPER III

The construction of residue currents from the Koszul complex in [3] was further developed in [8] and [5] to produce residue currents from the Buchsbaum-Rim and Eagon-Northcott complexes, respectively; for a description of these complexes we refer to [29]. The currents were used to obtain, for example, effective results for polynomial mappings related to classical results by Macaulay and Max Nöther and explicit versions of the Briançon-Skoda theorem. In our third paper we extend these ideas further and construct residue currents from arbitrary complexes of vector bundles. The aim is, given an ideal or, more generally, an ideal sheaf, $\mathcal{J}$, to construct a residue current whose annihilator is precisely $\mathcal{J}$. It turns out that our construction gives such "good" currents if the vector bundle complex comes from a locally free resolution of $\mathcal{J}$. The basic philosophy is that to get all necessary information about $\mathcal{J}$ one needs to know not only the generators of $\mathcal{J}$ but also of its higher syzygies.

Before presenting the general construction let us return to Example 2, and see how we can in this case obtain a current with the "right" annihilator.

Example 4. Let $f = (z^2, zw)$ be a section of the dual bundle E^* of a trivial 2-bundle $E \to \Omega$ over some neighborhood Ω of the origin in $\mathbb{C}^2$. Then the Koszul complex (3.2) of f is given by

$$0 \to \Lambda^2 E \xrightarrow{\begin{bmatrix} -zw \\ z^2 \end{bmatrix}} E \xrightarrow{\begin{bmatrix} z^2 & zw \end{bmatrix}} \mathbb{C} \times \Omega \longrightarrow 0. \tag{7.1}$$

It is easy to see that the corresponding complex of germs of holomorphic sections at the origin is not exact at E. Indeed, $(w, -z)^T$ is in the kernel of the right-hand map but not in the image of the left-hand one. Suppose that E is equipped with the trivial metric. Then the residue current of Bochner-Martinelli type R^f has two components

$$R_1 = \bar\partial |f|^{2\lambda} \sigma|_{\lambda=0} = \left[\frac{1}{w}\right] \wedge \bar\partial \left[\frac{1}{z}\right]$$

and

$$R_2 = \bar\partial |f|^{2\lambda} \sigma \wedge \bar\partial\sigma|_{\lambda=0} = \frac{1}{2}\, \bar\partial \left[\frac{1}{z^3}\right] \wedge \bar\partial \left[\frac{1}{w}\right].$$

For the computation, see Example 3 in Paper III. Hence $\operatorname{Ann} R^f = (z) \cap (z^3, w) = (z^3, zw)$, which is strictly included in (f).

In this particular case it is easy to see how to make an exact complex out of (7.1); just divide both entries in the left-hand map by the greatest common divisor z, that is, replace the left-hand map by $F_2 = \begin{bmatrix} -w \\ z \end{bmatrix}$. Moreover, from this exact complex we can now construct a residue current whose annihilator is precisely (f). Let σ_2 be the minimal inverse of F_2 with respect to the trivial metric. (See below for a definition of the minimal inverse.) In fact $\sigma_2 = (f_2^* f_2)^{-1} f_2^*$, where f_2^* is the adjoint of f_2. Furthermore, let

$$\tilde R_2 = |f|^{2\lambda} \sigma_2 \wedge \bar\partial\sigma|_{\lambda=0} = \bar\partial \left[\frac{1}{z^2}\right] \wedge \bar\partial \left[\frac{1}{w}\right].$$

Now, $\operatorname{Ann} \tilde R_2 = (z^2, w)$ and so, if we let $\tilde R = R_1 + \tilde R_2$, we have indeed constructed a residue current from the modified Koszul complex which satisfies the Duality Theorem. □

In Paper III the idea of Example 4 is carried through more systematically. Consider an arbitrary complex of Hermitian holomorphic vector bundles over the complex n-dimensional manifold X,

$$0 \longrightarrow E_N \xrightarrow{F_N} \dots \xrightarrow{F_2} E_1 \xrightarrow{F_1} E_0 \longrightarrow 0, \tag{7.2}$$

that is exact outside an analytic variety Z of positive codimension, and that is equipped with some Hermitian metrics. Outside Z, let σ_k be the minimal inverse of F_k, that is, $F_k\sigma_k$ is the identity on $\operatorname{Im} F_k$, σ_k vanishes on $(\operatorname{Im} F_k)^\perp$, and $\operatorname{Im}\sigma_k$ is orthogonal to $\operatorname{Ker} F_k$, and let $u_k^0 = \sigma_k(\bar\partial\sigma_{k-1})\cdots(\bar\partial\sigma_1)$. Then u_k^0 is a $(0, k-1)$-form that takes values in $\operatorname{Hom}(E_0, E_k)$ and moreover

$$F_1 u_1^0 = I_{E_0} \tag{7.3}$$

and

$$F_k u_k^0 = \bar\partial u_{k-1}^0. \tag{7.4}$$

Let E be the bundle $E_1 \oplus \ldots \oplus E_N$. Now, $u^0 = u_1^0 + \cdots + u_N^0$ can be continued as a $\operatorname{Hom}(E_0, E)$-valued current U^0 over Z as the analytic continuation to $\lambda = 0$ of the form $|g|^{2\lambda}u^0$, where g is a holomorphic function (or tuple of functions) that vanishes on Z. However, (7.3) and (7.4) cannot in general hold over Z. If we let $F = F_1 + \cdots + F_r$ and $\nabla = F - \bar\partial$, then (7.3) and (7.4) can be expressed as $\nabla u^0 = I_{E_0}$. Now, $\nabla U^0 = I_{E_0} - R^0$, where $R^0 = \bar\partial|g|^{2\lambda} \wedge u^0|_{\lambda=0}$. Clearly R^0 has support on Z and moreover $R^0 = R_p^0 + \cdots + R_\mu^0$, where R_k^0 is a $(0, k-1)$-current with values in $\operatorname{Hom}(E_0, E_k)$, $p = \operatorname{codim} Z$ and $\mu = \min(N, n)$.

Furthermore, if φ is a holomorphic section of E_0 such that the (E-valued) current $R^0\varphi$ vanishes, then $\nabla(U^0\varphi) = (I_{E_0} - R^0)\varphi = \varphi$ (indeed $F\varphi$ should be interpreted as 0), and by solving $\bar\partial$-equations like (3.5) we can locally find a holomorphic section ψ of E_1 so that $F_1\psi = \varphi$. Let $\mathcal{O}(E_k)$ denote the sheaf of holomorphic sections of E_k, and let $\mathcal{J} = \operatorname{Im}(\mathcal{O}(E_1) \to \mathcal{O}(E_0))$. Note that if $\operatorname{rank} E_0 = 1$ and $F_1 = f$, then $\mathcal{J}$ is just our familiar ideal (f). To sum up, from (7.2) we have constructed a $\operatorname{Hom}(E_0, E)$-valued residue current R^0 whose annihilator (in $\mathcal{O}(E_0)$) is contained in the sheaf $\mathcal{J}$. Note that if (7.2) is the Koszul complex (3.2), then U^0 and R^0 are just the currents of Cauchy-Fantappiè-Leray type U^f and R^f, respectively, from Section 3.

The current R^0 represents one component of the more general construction in Paper III. By extending the algebraic formalism we construct from (7.2) an $\operatorname{End}E$-valued residue current $R = R^0 + \cdots + R^{N-1}$, with R^k taking values in $\operatorname{Hom}(E_k, E)$, which in a sense measures the exactness of the associated complex of locally free sheaves of $\mathcal{O}$-modules

$$0 \longrightarrow \mathcal{O}(E_N) \xrightarrow{F_N} \ldots \xrightarrow{F_2} \mathcal{O}(E_1) \xrightarrow{F_1} \mathcal{O}(E_0) \longrightarrow 0. \tag{7.5}$$

On the one hand, by similar arguments as above, $R^k = 0$ implies that (7.5) is exact at $\mathcal{O}(E_k)$.

The complex (7.5) is said to be a (locally free) resolution of $\mathcal{O}(E_0)/\mathcal{J}$ if it is exact everywhere except in homological degree 0. We will sometimes be sloppy and say that (7.5) is a resolution of $\mathcal{J}$. Now, on the other hand, if (7.5) is a resolution of $\mathcal{O}(E_0)/\mathcal{J}$ it turns out that $R_k = 0$ for $k \geq 1$. To prove this we use the Buchsbaum-Eisenbud theorem (Theorem 20.9 in [29]) which basically says that if (7.5) is exact then the codimension of the set where the rank of F_k is not optimal is greater than or equal to k. Moreover, intuitively speaking, we show that a residue current of bidegree $(0, q)$ cannot have support on a variety of codimension $q + 1$. Together, these two facts force all components except R^0 to vanish when (7.5) is a resolution.

Furthermore, from the construction in Paper III it follows that if $R^1 = 0$ and φ is a section of $\mathcal{J}$ then $R\varphi = 0$. We can now conclude our main result, which extends Theorem 2.1.

Theorem 7.1. *Let* (7.2) *be a complex of Hermitian holomorphic vector bundles that is generically exact and let* R *be the associated residue current. Suppose that the complex* (7.5) *is a resolution of* $\mathcal{O}(E_0)/\mathcal{J}$, *where* $\mathcal{J} = \operatorname{Im}(\mathcal{O}(E_1) \to \mathcal{O}(E_0))$. *Let* φ *be a section of* $\mathcal{O}(E_0)$. *Then the current* $R\varphi$ *vanishes if and only if* φ *is in* $\mathcal{J}$.

In this case we say that $R = R^0$ is a *Noetherian* residue current for $\mathcal{J}$. The notion comes from the analogy with Noetherian operators, introduced in [28] and [39], which are differential operators that can be used to characterize ideals.

Theorem 7.1 asserts that, given an exact complex (7.5), we can construct a Noetherian residue current R whose annihilator is precisely $\mathcal{J}$. We should also mention that, given any subsheaf $\mathcal{J}$ of some locally free sheaf $\mathcal{O}(E_0)$, one can always find such a complex locally. This follows from the syzygy theorem and Oka's lemma, see [33]. If we equip the corresponding vector bundles with Hermitian metrics we get at least locally a Noetherian current for the sheaf $\mathcal{J}$.

To some extent, the Noetherian residue current depends on the choice of resolution and of the Hermitian metrics chosen on the bundles E_k. However, if $\mathcal{O}(E_0)/\mathcal{J}$ is a sheaf of Cohen-Macaulay modules, then it turns out that the associated Noetherian current R is essentially canonical: see Section 6 in Paper III for precise statements. In particular, if $\mathcal{J}$ is an ideal sheaf that is a complete intersection we get back the Coleff-Herrera current (via Theorem 2.2).

The Noetherian residue currents are used to extend several results, previously known for complete intersections. They fit nicely into the framework of integral formulas developed in [6], which gives us explicit division formulas of the type (2.15), realizing the ideal membership. We also provide formulas for polynomial ideals. By means of these we obtain a residue version of the Ehrenpreis-Palamodov Fundamental principle, [28], [39], generalizing the result of Berndtsson-Passare, [20]. Let $f = (f_1, \ldots, f_r) : \mathbb{C}^n \to \mathbb{C}^r$ be a polynomial mapping. Then any smooth solution to $f^T(i\partial/\partial t)\xi = 0$ (where f^T is just the transpose of f) on a smoothly bounded convex set in $\mathbb{R}^n$ can be written

$$\xi(t) = \int_{\mathbb{C}^n} R^T(\zeta)A(\zeta)e^{-i\langle t,\zeta\rangle},$$

for an appropriate explicitly given matrix of smooth functions A; here R^T is the transpose of R, which is a Noetherian residue current for (f). Conversely, any $\xi(t)$ given in this way is a homogeneous solution.

8. Resolutions of monomial ideals

The degree of explicitness of the residue currents computed in Paper III of course depends directly on the degree of explicitness of the resolution (7.5). In some simple cases, such as that of a complete intersection, a resolution of $\mathcal{O}(E_0)/\mathcal{J}$ can be constructed from the generators of the ideal. Indeed, the Koszul complex (3.2) is exact if and only if f is a complete intersection. In general, however, explicit resolutions are harder to find, see for example [10].

The first explicit resolution of an arbitrary monomial ideal was found in 1966 by Diana Taylor [54]. Her construction can be seen as a generalization of our "divided Koszul complex" in Example 4. One nice feature of monomials is that they have well-defined greatest common divisors and least common multiples. Starting with the Koszul complex and dividing out by greatest common divisors in a systematic way actually gives a resolution.

More formally, suppose that $M = (m_1, \dots, m_r)$ is a monomial ideal in $S = \mathbb{C}[z_1, \dots, z_n]$. Let A_k be an S-module of rank $\binom{r}{k}$ with basis $\{e_I\}$, where I runs over all subsets of $\{1, \dots, r\}$ of cardinality k, let $m_I = \operatorname{lcm}\{m_i | i \in I\}$ and let

$$F_k : e_I \mapsto \sum_{\ell=1}^{k} (-1)^{\ell-1} \frac{m_I}{m_{I^\ell}}\, e_{I^\ell};$$

here $I^\ell = \{i_1, \dots, i_{\ell-1}, i_{\ell+1}, \dots, i_k\}$. Now the complex

$$0 \longrightarrow A_r \xrightarrow{F_r} \dots \xrightarrow{F_2} A_1 \xrightarrow{F_1} S \longrightarrow 0 \tag{8.1}$$

is acyclic, that is, it has zero reduced homology, and thus provides a free resolution, the so-called *Taylor resolution*, of S/M.

Note that if M is a compete intersection the monomials m_i do not have any common factors. Hence $m_I/m_{I^\ell} = m_\ell$ and so identifying e_I with $e_{i_1} \wedge \dots \wedge e_{i_k}$ gives back the Koszul complex.

Recall that a graded free resolution $\cdots \to A_k \xrightarrow{f_k} A_{k-1} \to \cdots$ is *minimal* if and only if for each k, f_k maps a basis of A_k to a minimal set of generators of $\operatorname{Im} f_k$ (see Corollary 1.5 in [30]). The Taylor resolution is in general far from being minimal. For example, if the monomial ideal M is of pure dimension, in particular if it is Artinian, then the Taylor resolution is minimal if and only if M is a complete intersection, see [61]. The concept of Taylor resolutions was further developed by Bayer, Peeva and Sturmfels in [11], and later by Bayer and Sturmfels in [12]. Removing superfluous generators of higher order syzygies in a clever way, they constructed smaller acyclic subcomplexes of the Taylor resolution and managed to find a minimal resolution of a certain class of ideals - the so-called generic ideals. More precisely, a monomial $m' \in S$ *strictly divides* another monomial m if m' divides m/z_i for all variables z_i dividing m. We say that a monomial ideal M is *generic* if whenever two distinct minimal generators m_i and m_j have the same positive degree in some variable, then there exists a third generator m_k that strictly divides the least common multiple of m_i and m_j.

The basic idea in [11] is that, by identifying each $I \subseteq \{1, \dots, r\}$ with a face of the $(r-1)$-dimensional simplex Σ, the Taylor resolution can be encoded into Σ if each face I of Σ is equipped with the label m_I. By taking subcomplexes Δ of Σ one gets new algebraic complexes $\mathbb{F}_\Delta$, which can be seen to be acyclic precisely when the underlying labeled simplicial complexes Δ satisfy a certain acyclicity condition (Proposition 4.5 in [38]).

In particular, if M is a generic monomial ideal, then the so-called *Scarf complex* Δ_M of M, which consists of the the collection of subsets $I \subseteq \{1, \dots, r\}$ whose corresponding least common multiple m_I is unique, that is,

$$\Delta_M = \{I \subseteq \{1, \dots, r\} | m_I = m_{I'} \Rightarrow I = I'\},$$

is acyclic. One can prove that Δ_M is a simplicial complex, and that its dimension is at most $n-1$. (In fact, when M is Artinian, Δ_M is a regular triangulation of the $(n-1)$-simplex). Moreover, $\mathbb{F}_{\Delta_M}$ is a minimal resolution of S/M.

In [12] the construction in [11] is extended to more general polyhedral cell complexes X. As above each face F of X is equipped with a label $m_F = \text{lcm}\{m_i | i \text{ vertex of } F\}$. The corresponding algebraic complex $\mathbb{F}_X$ is called a *cellular complex.* If $\mathbb{F}_X$ is acyclic it is said to be a *cellular resolution.* A more detailed description is given in Paper IV.

9. Paper IV

In Paper IV we compute the residue currents R from Paper III in the case when (7.2) comes from cellular resolutions of Artinian monomial ideals in $S = \mathbb{C}[z_1, \dots, z_n]$ and the metrics are trivial. A priori R has one entry R_F for each $(n-1)$-dimensional face F of the underlying polyhedral cell complex X. The main technical result in Paper IV (Proposition 3.1) asserts that each R_F is of the nice form

$$C_F \; \bar\partial \left[\frac{1}{z_1^{\alpha_1}}\right] \wedge \dots \wedge \bar\partial \left[\frac{1}{z_n^{\alpha_n}}\right], \tag{9.1}$$

where $(\alpha_1, \dots, \alpha_n) = \alpha_F$ is the multi-degree of the label m_F associated with the face F, and C_F is a constant; this should be compared with Theorem 6.1. It immediately follows that if C_F is nonzero, then $\operatorname{Ann} R_F$ is the irreducible ideal $\mathfrak{m}^{\alpha_F}$. To annihilate R one has to annihilate each entry R_F, and so

$$\operatorname{Ann} R = \bigcap_{F \text{ face of } X} \operatorname{Ann} R_F \tag{9.2}$$

gives an irreducible decomposition of $\operatorname{Ann} R$, which equals M by Theorem 7.1.

The proof is inspired by Paper II; the residue current R is computed as the push forward of certain currents on a toric manifold. When considering general cellular resolutions the computations get more involved. In particular, we need to compute the minimal inverses of all differentials F_k. Therefore, unfortunately, we do not in general manage to determine whether or not the constant C_F in (9.1) is zero. Nevertheless, if M is generic we do. Recall that a facet of a simplicial complex is a maximal face.

Theorem 9.1. *Let M be an Artinian generic monomial ideal in S and let R be the residue current associated with the cellular resolution $\mathbb{F}_X$. Then*

$$R = (R_F)_{F \text{ facet of } \Delta_M},$$

where Δ_M is the Scarf complex of M, R_F is given by (9.1) *and the constant C_F there is non-vanishing.*

In particular, if we choose X as Δ_M, then all entries are non-vanishing. In fact a generalization of this holds. Theorem 3.5 in Paper IV states that whenever $\mathbb{F}_X$ is a minimal resolution of M (possibly non-generic) then all entries are non-vanishing.

Once we know that R_F is given by (9.1), Theorem 9.1 is an easy consequence of Theorem 3.7 in [11], which states that if M is generic, then

$$M = \bigcap_{F \text{ facet of } \Delta_M} \mathfrak{m}^{\alpha_F}$$

yields the irreducible irredundant decomposition of M.

Thus, the non-vanishing entries of R correspond precisely to irreducible components of the generic ideal M. So in a sense, R contains no superfluous information, which is sound. Exactly those entries that have to be non-vanishing to determine the ideal are non-vanishing. Compare with the fact that the residue currents of Bochner-Martinelli type in the non-Artinian case turned out to in general depend on the generators, as was discussed in Section 6. Also, if M is generic it follows that (9.2) yields the irredundant irreducible decomposition of M.

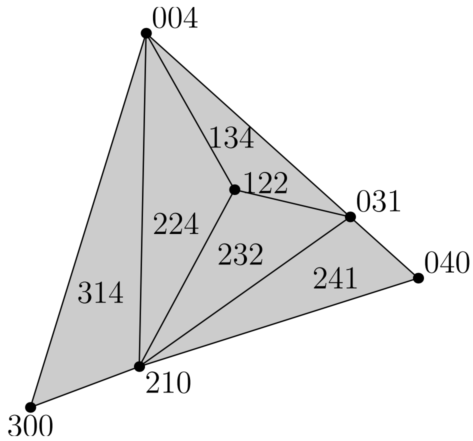

FIGURE 3. The Scarf complex Δ_M of the ideal M in Example 5 (labels on vertices and facets).

Example 5. Let us consider again the ideal $M = (x^3, x^2y, xy^2z^2, y^4, y^3z, z^4) = (m_1, \ldots, m_6)$ from Example 3. Note that M is generic since no generators have the same positive degree in any variable.

The Scarf complex of M, depicted in Figure 3, consists of the facets $\{1,2,6\}, \{2,3,5\}, \{2,3,6\}, \{2,4,5\}$, and $\{3,5,6\}$, with $\alpha_{126} = (3,1,4)$, $\alpha_{235} = (2,3,2)$, $\alpha_{236} = (2,2,4)$, $\alpha_{245} = (2,4,1)$ and $\alpha_{356} = (1,3,4)$, respectively. Compare this to the Newton diagram in Figure 1. Thus according to Theorem 9.1, the Noetherian residue current associated with a cellular resolution of M has one entry for each of these facets, and so "decomposing" M with respect to R gives back the irreducible irredundant decomposition of $M = \operatorname{Ann} R$:

$$M = (x^3, y, z^4) \cap (x^2, y^3, z^2) \cap (x^2, y^2, z^4) \cap (x^2, y^4, z) \cap (x, y^3, z^4).$$

Note that the entries of R correspond to the outer corners of the staircase in Figure 1. $\square$

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Paper I

PRODUCTS OF RESIDUE CURRENTS OF CAUCHY-FANTAPPIÈ-LERAY TYPE

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ABSTRACT. With a given holomorphic section of a Hermitian vector bundle, one can associate a residue current by means of Cauchy-Fantappiè-Leray type formulas. In this paper we define products of such residue currents. We prove that, in the case of a complete intersection, the product of the residue currents of a tuple of sections coincides with the residue current of the direct sum of the sections.

1. INTRODUCTION

Let f be a holomorphic function defined in some domain in $\mathbb{C}^n$ and let $Y = f^{-1}(0)$. Then there exists a distribution U such that $fU = 1$, as shown by Schwartz [16]. For example, one can let U be the principal value distribution $[1/f]$, defined as

$$\mathcal{D}_{n,n} \ni \phi \mapsto \lim_{\varepsilon \to 0} \int_{|f|>\varepsilon} \frac{\phi}{f}.$$

The existence of this limit was proven by Herrera and Lieberman, [9], using Hironaka's desingularization theorem. By the Mellin transform, see for example [13], one can show that the limit is equal to the analytic continuation to $\lambda = 0$ of

$$\lambda \mapsto \int |f|^{2\lambda} \frac{\phi}{f}. \tag{1.1}$$

The *residue current* associated with f is defined as $\bar\partial[1/f]$; it has support on Y and its action on a test form $\phi \in \mathcal{D}_{n,n-1}$ is given by the analytic continuation to $\lambda = 0$ of

$$\lambda \mapsto \int \bar\partial |f|^{2\lambda} \wedge \frac{\phi}{f}.$$

This paper concerns products of residue currents. Recall that it is in general not possible to multiply currents (or distributions). However, given

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a tuple of holomorphic functions $f=(f_1,\dots,f_m)$, by certain limiting processes one can give meaning to the expression

$$\bar\partial\Big[\frac{1}{f_1}\Big]\wedge\ldots\wedge\bar\partial\Big[\frac{1}{f_m}\Big], \tag{1.2}$$

as was first done by Coleff and Herrera, [7]. By the Mellin transform, this so called Coleff-Herrera current, denoted by R^f_{CH}, can be realized as the analytic continuation to $\lambda=0$ of

$$\bar\partial|f_1|^{2\lambda}\frac{1}{f_1}\wedge\ldots\wedge\bar\partial|f_m|^{2\lambda}\frac{1}{f_m}. \tag{1.3}$$

In case f defines a complete intersection, that is, the codimension of $Y=f^{-1}(0)$ is m, then R^f_{CH} has especially nice calculus properties. For example $f_iR^f_{CH}=0$ for all i, see [12], which yields one direction of the duality theorem, due to Passare, [11], and Dickenstein-Sessa, [8], that asserts that if f is a complete intersection, then a holomorphic function φ belongs to the ideal (f) if and only if $\varphi R^f_{CH}=0$.

In [14] Passare, Tsikh and Yger introduced an alternative approach to multidimensional residue currents by constructing currents based on the Bochner-Martinelli kernel. For each ordered index set $\mathcal{I}\subseteq\{1,\dots,m\}$ of cardinality k, let $R^f_{\mathcal{I}}$ be the analytic continuation to $\lambda=0$ of

$$\bar\partial|f|^{2\lambda}\wedge\sum_{\ell=1}^{k}(-1)^{\ell-1}\frac{\overline{f_{i_\ell}}\bigwedge_{q\neq\ell}\overline{df_{i_q}}}{|f|^{2k}},$$

where $|f|^2=|f_1|^2+\ldots+|f_m|^2$. Then $R^f_{\mathcal{I}}$ is a well-defined $(0,k)$-current with support on Y, that vanishes whenever $k<\operatorname{codim}Y$ or $k>\min(m,n)$. If f is a complete intersection, there is only one nonvanishing current, namely $R^f_{\{1,\dots,m\}}$, which corresponds to the classical Bochner-Martinelli kernel and which we denote by R^f_{BM}. Then we have the following result.

Theorem 1.1 (Passare, Tsikh, Yger [14]). *Assume that f is a complete intersection. Then*

$$R^f_{BM}=R^f_{CH}.$$

The Bochner-Martinelli residue currents $R^f_{\mathcal{I}}$ have been used for investigations in the non-complete intersection case; for example, in [6], Berenstein and Yger used them to construct Green currents.

Based on the work in [14] Andersson, [1], introduced more general globally defined residue currents by means of Cauchy-Fantappiè-Leray type formulas. Let us briefly recall his construction. Assume that f is a holomorphic section of the dual bundle E^* of a holomorphic m-bundle $E\to X$ over a complex manifold X. On the exterior algebra over E we have mappings $\delta_f:\Lambda^{\ell+1}E\to$

$\Lambda^\ell E$ of interior multiplication by f, and $\delta_f^2 = 0$. Let $\mathcal{E}_{0,k}(X, \Lambda^\ell E)$ be the space of smooth sections of the exterior algebra of $E^* \oplus T^*_{0,1}$ which are $(0,k)$-forms with values in $\Lambda^\ell E$, and let $\mathcal{D}'_{0,k}(X, \Lambda^\ell E)$ be the corresponding space of currents. The mappings δ_f extend to these spaces, where it anti-commutes with $\bar\partial$. Thus $\mathcal{D}'_{0,k}(X, \Lambda^\ell E)$ is a double complex and the corresponding total complex is

$$\cdots \xrightarrow{\nabla_f} \mathcal{L}^{r-1}(X,E) \xrightarrow{\nabla_f} \mathcal{L}^{r}(X,E) \xrightarrow{\nabla_f} \cdots,$$

where $\mathcal{L}^r(X,E) = \bigoplus_{k+\ell=r} \mathcal{D}'_{0,k}(X, \Lambda^{-\ell} E)$ and $\nabla_f = \delta_f - \bar\partial$. The exterior product, $\wedge$, induces a mapping

$$\wedge : \mathcal{L}^r(X,E) \times \mathcal{L}^s(X,E) \to \mathcal{L}^{r+s}(X,E)$$

when possible, and ∇_f is an antiderivation with respect to $\wedge$.

If φ is a holomorphic function such that is $\varphi = \nabla_f v$ for some $v \in \mathcal{L}^{-1}(X,E)$, one can prove, provided X is Stein, that there is a holomorphic solution ψ to the division problem $\sum \psi_j f_j = \varphi$. Andersson's idea to find such a v was to start looking for a solution to $\nabla_f u = 1$. Assume that E is equipped with some Hermitian metric and let s be the section of E with pointwise minimal norm such that $\delta_f s = |f|^2$ and let

$$u^f = \frac{s}{\nabla_f s} = \frac{s}{\delta_f s - \bar\partial s} = \sum_\ell \frac{s \wedge (\bar\partial s)^{\ell-1}}{(\delta_f s)^\ell} = \sum_\ell \frac{s \wedge (\bar\partial s)^{\ell-1}}{|f|^{2\ell}}$$

be the Cauchy-Fantappiè-Leray form, introduced in [2] in order to construct integral formulas in a convenient way. Clearly $u^f \in \mathcal{L}^{-1}$ is well-defined outside Y and since $\nabla_f s$ is of even degree the expression $s/\nabla_f s$ makes sense, and it follows that $\nabla_f u^f = 1$ outside Y. In [1] it is proved that the form $|f|^{2\lambda} u^f$ has an analytic continuation as a current to $\operatorname{Re}\lambda > -\epsilon$. The value at $\lambda = 0$, denoted by U^f, yields an extension of u^f over Y. In analogy with the one function case, we will sometimes refer to U^f as the principal value current. Clearly, if $Y \neq \emptyset$, U^f can not fulfill $\nabla_f U^f$. In fact, $\nabla_f U^f = 1 - R^f$, where $R^f = \bar\partial |f|^{2\lambda} \wedge u^f|_{\lambda=0}$ now defines the residue current of f. It holds that $R^f = R_p + \ldots + R_\mu$, where $R_j \in \mathcal{D}'_{0,j}(X, \Lambda^j)$, $p = \operatorname{codim} Y$ and $\mu = \min(m,n)$. Moreover, if $\varphi R^f = 0$, then $v = Uf$ yields the desired solution to $\nabla_f v = \varphi$ and thus φ belongs to the ideal generated by f locally.

If E is a trivial bundle endowed with the trivial metric, the coefficients of R^f will actually be the Bochner-Martinelli currents $R^f_{\mathcal{I}}$. If f is a complete intersection, the only nonvanishing coefficient will be R^f_{BM}.

Our first goal is to define products of currents of the type U^f and R^f. Let us consider (1.3). If we assume that each f_i is a section of the dual bundle E_i^* of a line bundle E_i with frame e_i and dual frame e_i^*, the Cauchy-Fantappiè-Leray form u^{f_i} is just e_i/f_i, so in fact (1.3) times the element

$e_1 \wedge \ldots \wedge e_r$ can be expressed as

$$\bar\partial|f_1|^{2\lambda} \wedge u^{f_1} \wedge \ldots \wedge \bar\partial|f_r|^{2\lambda} \wedge u^{f_r}. \tag{1.4}$$

In light of this, it is most tempting to extend this product to include not only sections of line bundles but sections f_i of bundles of arbitrary rank. To be more accurate, we assume that f_i is a section of the dual bundle of a holomorphic m_i-bundle $E_i \to X$. Further, we assume that each E_i is equipped with a Hermitian metric, we let s_i be the section of E_i of minimal norm such that $\delta_{f_i} s_i = |f_i|^2$, and we let u^{f_i} be the corresponding Cauchy-Fantappiè-Leray form. Then (1.4) has meaning as a form taking values in the exterior algebra over $E = E_1 \oplus \cdots \oplus E_r$. Thus, in accordance with the line bundle case, we can take the value at $\lambda = 0$ of (1.4) as a definition of $R^{f_1} \wedge \ldots \wedge R^{f_r}$, provided that the analytic continuation exists. However, this is assured by Theorem 1.2, where products are defined also of principal value currents.

Theorem 1.2. *Let f_i be holomorphic sections of the Hermitian m_i-bundles $E_i^* \to X$. Let u^{f_i} be the corresponding Cauchy-Fantappiè-Leray forms and let $Y_i = f_i^{-1}(0)$. Then*

$$\lambda \mapsto |f_r|^{2\lambda} u^{f_r} \wedge \ldots \wedge |f_{s+1}|^{2\lambda} u^{f_{s+1}} \wedge \bar\partial|f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \bar\partial|f_1|^{2\lambda} \wedge u^{f_1} \tag{1.5}$$

has an analytic continuation as a current to $\operatorname{Re}\lambda > -\epsilon$.

We define $T = U^{f_r} \wedge \ldots \wedge U^{f_{s+1}} \wedge R^{f_s} \wedge \ldots \wedge R^{f_1}$ as the value at $\lambda = 0$. Then T has support on $\bigcap_{i=1}^s Y_i$ and it is alternating with respect to the principal value factors U^{f_i} and symmetric with respect to the residue factors R^{f_i}.

Of course there is nothing special about the ordering that we have chosen; we can just as well mix U's and R's.

If the bundle E is trivial, endowed with the trivial metric, and moreover if $f_1 \oplus \cdots \oplus f_r$ is a complete intersection, then $R^{f_r} \wedge \ldots \wedge R^{f_1}$ will consist of only one term, which can be interpreted as a product of the corresponding Bochner-Martinelli currents $R^{f_i}_{BM}$. In general, however, there will also occur terms of lower degree.

Theorem 1.3. *Let*

$$T = U^{f_r} \wedge \ldots \wedge U^{f_{s+1}} \wedge R^{f_s} \wedge \ldots \wedge R^{f_1}$$

be defined as above. Let $m = m_1 + \ldots + m_r$. Then $T = T_p + \ldots + T_q$, where $T_\ell \in \mathcal{D}'_{0,\ell}(\Lambda^\bullet E)$, $p = \operatorname{codim} Y_1 \cap \ldots \cap Y_s$ and $q = \min(m, n)$. In particular, if $f = f_1 \oplus \cdots \oplus f_r$ is a complete intersection, then $R^{f_r} \wedge \ldots \wedge R^{f_1}$ consists of only one term of top degree m.

Observe that Theorems 1.2 and 1.3 extend Theorem 1.1 in [1].

Our next aim is to prove a generalized version of Theorem 1.1. Since, in the particular case when the bundles E_i are all line bundles, the current $R^{f_1} \wedge \ldots \wedge R^{f_r}$ is just the Coleff-Herrera current of f times $e_1 \wedge \ldots \wedge e_r$, we can formulate the equivalence in the theorem as

$$R^{f_1 \oplus \ldots \oplus f_r} = R^{f_1} \wedge \ldots \wedge R^{f_r}. \tag{1.6}$$

Now, the obvious question is, does this equality extend to hold for sections of vector bundles of arbitrary rank. Our main result states that this is indeed the case.

Theorem 1.4. *Let f_i be holomorphic sections of the Hermitian m_i-bundles E_i^* and let f be the section $f_1 \oplus \cdots \oplus f_r$ of $E^* = E_1^* \oplus \cdots \oplus E_r^*$. If f is a complete intersection, that is,* $\operatorname{codim} f^{-1}(0) = m_1 + \ldots + m_r$*, then*

$$R^f = R^{f_1} \wedge \ldots \wedge R^{f_r}.$$

That is, in a local perspective, given a tuple of functions split into subtuples, the product of the Bochner-Martinelli currents of each subtuple is equal to the Bochner-Martinelli current of the whole tuple of functions. We give an explicit proof of Theorem 1.4 based on the existence of two ∇_f-potentials.

Theorem 1.5. *Let $f = f_1 \oplus \cdots \oplus f_r$ be a section of $E^* = E_1^* \oplus \cdots \oplus E_r^*$. Assume that f is a complete intersection. Then there exists a current V such that*

$$\nabla_f V = 1 - R^{f_1} \wedge \ldots \wedge R^{f_r}, \tag{1.7}$$

and furthermore a current $U^f \wedge V$ such that

$$\nabla_f (U^f \wedge V) = V - U^f.$$

At first it might seem a bit peculiar to denote the second potential by $U^f \wedge V$. However, notice that on a formal level, if we were allowed to multiply currents so that ∇_f acted as an antiderivation on the products, then

$$\nabla_f (U^f \wedge V) = (1 - R^f) \wedge V - U^f \wedge (1 - R^{f_1} \wedge \ldots \wedge R^{f_r}),$$

since U^f is of odd degree. From Theorem 1.3 we know that R^f and $R^{f_1} \wedge \ldots \wedge R^{f_r}$ take values in $\Lambda^m E$, since f is a complete intersection. But since V and U^f have positive degree in e_j it is reasonable to expect the products $V \wedge R^f$ and $U^f \wedge R^{f_1} \wedge \ldots \wedge R^{f_r}$ to vanish. Thus we are left with $V - U^f$, and the notation is motivated.

Proof of Theorem 1.4. Recall that $\nabla_f U^f = 1 - R^f$. Hence, applying ∇_f twice to $U^f \wedge V$ yields

$$0 = \nabla_f^2 (U^f \wedge V) = \nabla_f (U^f - V) = R^{f_1} \wedge \ldots \wedge R^{f_r} - R^f,$$

and thus we are done. □

The disposition of this paper is as follows. In Section 2 we give proofs of Theorem 1.2 and Theorem 1.3. In Section 3 we prove Theorem 1.5. Finally, in Section 4 we give an example of products of Cauchy-Fantappiè-Leray currents and also discuss a possible generalization of Theorem 1.4.

2. Products of residue currents of Cauchy-Fantappiè-Leray type

We start with the proof of Theorem 1.2. For further use a slightly more general formulation is appropriate. Indeed, the proof of Theorem 1.5 requires a broader definition of products of currents. We need to allow also products of currents of sections of the bundle E, that are not necessarily orthogonal, at least in certain cases. Thus we give a new, somewhat unwieldy, version of Theorem 1.2 that however covers all the currents that we will be concerned with.

By the notion that a form (or current) is of degree k in $d\bar{z}_j$, we will just mean that it is a $(\bullet, k)$-form. In the same manner, we will say that a form is of degree ℓ in e_j when it takes values in $\Lambda^\ell E$.

Proposition 2.1. *Let $f = f_1 \oplus \ldots \oplus f_r$ be a holomorphic section of the bundle $E^* = E_1^* \oplus \ldots \oplus E_r^*$, where E_i^* is a Hermitian m_i-bundle. For a subset $I = \{I_1, \ldots, I_p\}$ of $\{1, \ldots, r\}$, let f_I denote the section $f_{I_1} \oplus \ldots \oplus f_{I_p}$ of $E_I^* = E_{I_1}^* \oplus \ldots \oplus E_{I_p}^*$, let u^{f_I} be the corresponding Cauchy-Fantappiè-Leray form, let $Y_I = f_I^{-1}(0)$, and let $m_I = m_{I_1} + \ldots + m_{I_p}$. If $I^1, \ldots, I^t$ are subsets of $\{1, \ldots, r\}$, then*

(2.1)
$$\lambda \mapsto |f_{I^t}|^{2\lambda} u^{f_{I^t}} \wedge \ldots \wedge |f_{I^{s+1}}|^{2\lambda} u^{f_{I^{s+1}}} \wedge \bar{\partial}|f_{I^s}|^{2\lambda} \wedge u^{f_{I^s}} \wedge \ldots \wedge \bar{\partial}|f_{I^1}|^{2\lambda} \wedge u^{f_{I^1}}$$

has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$.

We define $T = U^{f_{I^t}} \wedge \ldots \wedge U^{f_{I^{s+1}}} \wedge R^{f_{I^s}} \wedge \ldots \wedge R^{f_{I^1}}$ as the value at $\lambda = 0$. Then T has support on $\bigcap_{i=1}^s Y_{I^k}$ and it is alternating with respect to the principal value factors U and commutative with respect to the residue factors R.

Note that Theorem 1.2 corresponds to the particular case when each I^j is just a singleton. The proof of Proposition 2.1 is very much inspired by the proof of Lemma 2.2 in [14] and Theorem 1.1 in [1]. It is based on the possibility of resolving singularities by Hironaka's theorem, see [3], and the following lemma, which is proven essentially by integration by parts.

Lemma 2.2. *Let v be a strictly positive smooth function in $\mathbb{C}$, φ a test function in $\mathbb{C}$, and p a positive integer. Then*

$$\lambda \mapsto \int v^\lambda |s|^{2\lambda} \varphi(s) \frac{ds \wedge d\bar{s}}{s^p}$$

and

$$\lambda \mapsto \int \bar{\partial}(v^\lambda |s|^{2\lambda}) \wedge \varphi(s) \frac{ds}{s^p}$$

both have meromorphic continuations to the entire plane with poles at rational points on the negative real axis. At $\lambda = 0$ they are both independent of v, and the second one only depends on the germ of φ at the origin. Moreover, if $\varphi(s) = \bar{s}\psi(s)$ or $\varphi = d\bar{s} \wedge \psi$, then the value of the second integral at $\lambda = 0$ is zero.

Proof of Proposition 2.1. We may assume that the bundle $E = E_1 \oplus \cdots \oplus E_r$ is trivial since the statement is clearly local. Note that $f_i = \sum f_{i,j} e^*_{i,j}$, where $e^*_{i,j}$ is the trivial frame. The proof is based on the possibility to resolve singularities locally using Hironaka's theorem. Given a small enough neighborhood $\mathcal{U}$ of a given point in X there exist a n-dimensional manifold $\widetilde{\mathcal{U}}$ and a proper analytic map $\Pi_h : \widetilde{\mathcal{U}} \to \mathcal{U}$ such that if $Z = \{\prod_{i,j} f_{i,j} = 0\}$ and $\widetilde{Z} = \Pi_h^{-1}(Z)$, then $\Pi : \widetilde{\mathcal{U}} \setminus \widetilde{Z} \to \mathcal{U} \setminus Z$ is biholomorphic and such that moreover $\widetilde{Z}$ has normal crossings in $\widetilde{\mathcal{U}}$. This implies that locally in $\widetilde{\mathcal{U}}$ we have that $\Pi_h^* f_{i,j} = a_{i,j}\mu_{i,j}$, where $a_{i,j}$ are non-vanishing and $\mu_{i,j}$ are monomials in some local coordinates τ_k. Further, given a finite number of monomials $\mu_1 \ldots, \mu_m$ in some coordinates τ_k defined in an n-dimensional manifold $\mathcal{U}_t$, there exists a toric variety $\widetilde{\mathcal{U}}_t$ and a proper analytic map $\Pi_t \colon \widetilde{\mathcal{U}}_t \to \mathcal{U}_t$ such that Π_t is biholomorphic outside the coordinate axes and moreover, locally it holds that, for some i, $\Pi_t^* \mu_i$ divides all $\Pi_t^* \mu_j$, see [5] and [10]. Clearly, if μ_i divides μ_j in $\mathcal{U}_t$ then $\Pi_t^* \mu_i$ divides $\Pi_t^* \mu_j$ in $\widetilde{\mathcal{U}}_t$. Thus after a number, say q, of such toric resolutions Π_{t_i} we can locally consider each section f_{I^i} as a monomial times a non-vanishing section. More precisely we have that $\Pi^* f_{I^i} = \mu_i f'_{I^i}$, where $\Pi = \Pi_{t_q} \circ \cdots \circ \Pi_{t_1} \circ \Pi_h$, μ_i is a monomial and f'_{I^i} is a non-vanishing section of $E^*_{I^i}$.

Let ϕ be a test form with compact support. After a partition of unity we may assume that it has support in a neighborhood $\mathcal{U}$ as above. Then, since Π_h is proper, the support of $\Pi_h^* \phi$ can be covered by a finite number of neighborhoods in which it holds that $\Pi_h^* \phi = a_{i,j}\mu_{i,j}$. If ψ is a test form with support in such a neighborhood, then the support of $\Pi_{t_1}^* \psi$ can be covered by finitely many neighborhoods in which we have the desired property that the pull-back of one monomial divides some of the other ones, and so on. Thus, for $\operatorname{Re}\lambda > 2\max_i m_{I^i}$, (2.1) is in L^1_{loc}, and since Π is biholomorphic outside a set of measure zero we have that

$$\int |f_{I^t}|^{2\lambda} u^{f_{I^t}} \wedge \ldots \wedge |f_{I^{s+1}}|^{2\lambda} u^{f_{I^{s+1}}} \wedge \bar\partial |f_{I^s}|^{2\lambda} \wedge u^{f_{I^s}} \wedge \ldots \wedge \bar\partial |f_{I^1}|^{2\lambda} \wedge u^{f_{I^1}} \wedge \phi$$

is equal to a finite number of integrals of the form

(2.2)

$$\int \Pi^*(|f_{I^t}|^{2\lambda} u^{f_{I^t}} \wedge \ldots \wedge |f_{I^{s+1}}|^{2\lambda} u^{f_{I^{s+1}}} \wedge \bar\partial |f_{I^s}|^{2\lambda} \wedge u^{f_{I^s}} \wedge \ldots \wedge \bar\partial |f_{I^1}|^{2\lambda} \wedge u^{f_{I^1}}) \wedge \tilde\phi.$$

Here

$$\tilde\phi = \rho_{t_q} \Pi^*_{t_q}(\ldots \rho_{t_1} \Pi^*_{t_1}(\rho_h \Pi^*_h(\phi))),$$

where the $\rho_\bullet$'s are functions from some partitions of unity, so that the test form $\tilde\phi$ has support in a neighborhood where it holds that $\Pi^* f_{I^i} = \mu_i f'_{I^i}$. In such a coordinate neighborhood the pullback of s_{I^i} is $\bar\mu_i$ times a smooth form, so that $\Pi^*(s_{I^i} \wedge (\bar\partial s_{I^i})^{\ell-1})$ is $\bar\mu_i^\ell$ times a smooth form. Moreover $\Pi^*|f_{I^i}|^2 = |\mu_i|^2 a_i$, where a_i is a strictly positive smooth function. Thus

$$\Pi^* u^{f_{I^i}} = \sum_\ell \frac{\bar\mu_i^\ell \alpha_{i,\ell}}{|\mu_i|^{2\ell}} = \sum_\ell \frac{\alpha_{i,\ell}}{\mu_i^\ell},$$

where $\alpha_{i,\ell}$ are smooth forms taking values in $\Lambda^\ell E$, and so (2.2) is equal to a finite sum of integrals

$$\int |\mu_t|^{2\lambda} a_t^\lambda \frac{\alpha_{t,\ell_t}}{\mu_t^{\ell_t}} \wedge \ldots \wedge |\mu_{s+1}|^{2\lambda} a_{s+1}^\lambda \frac{\alpha_{s+1,\ell_{s+1}}}{\mu_{s+1}^{\ell_{s+1}}} \wedge \bar\partial(|\mu_s|^{2\lambda} a_s^\lambda) \wedge \frac{\alpha_{s,\ell_s}}{\mu_s^{\ell_s}} \wedge \ldots \wedge \bar\partial(|\mu_1|^{2\lambda} a_1^\lambda) \wedge \frac{\alpha_{1,\ell_1}}{\mu_1^{\ell_1}} \wedge \tilde\phi. \tag{2.3}$$

Expanding each factor $\bar\partial(|\mu_j|^{2\lambda} a_j^\lambda)$ by Leibniz' rule results in a finite sum of terms. Letting $\bar\partial$ fall only on the monomials μ_i yields integrals of the form

$$\int a^\lambda |\mu'|^{2\lambda} \frac{\alpha_L}{\mu_L} \wedge \bar\partial |\sigma_s^{q_s}|^{2\lambda} \wedge \ldots \wedge \bar\partial |\sigma_1^{q_1}|^{2\lambda} \wedge \tilde\phi, \tag{2.4}$$

where σ_i is one of the coordinate functions τ_j that divide μ_i, $a = a_t \cdots a_1$ is a strictly positive smooth function, $\mu_L = \mu_t^{\ell_t} \cdots \mu_1^{\ell_1}$ is a monomial in τ_j, μ' is a monomial in τ_j not divisible by any σ_i and $\alpha_L = C\alpha_{t,\ell_t} \wedge \ldots \wedge \alpha_{1,\ell_1}$ is a smooth form, where C is just a constant that depends on the relation between q_i and the number of σ_i's in μ_i. The remaining integrals, that arise when $\bar\partial$ falls on any of the a_i, vanish in accordance with Lemma 2.2. Indeed, consider one of the integrals obtained when $\bar\partial$ falls on a_1,

$$\lambda \int a^\lambda |\mu'|^{2\lambda} \frac{\alpha_L}{\mu_L} \wedge \bar\partial |\sigma_s^{q_s}|^{2\lambda} \wedge \ldots \wedge \bar\partial |\sigma_2^{q_2}|^{2\lambda} \wedge \bar\partial a_1 \wedge \tilde\phi.$$

This is just λ times an integral of the form (2.4), so provided that we can prove the existence of an analytic continuation of (2.4), it must clearly vanish at $\lambda = 0$.

Now an application of Lemma 2.2 for each τ_k that divides any of the μ_j's gives the desired analytic continuation of (2.4) to $\operatorname{Re}\lambda > -\epsilon$. Note that for $\sigma_1, \ldots, \sigma_s$ we get integrals of the second type, for the remaining τ_i integrals of the first type, so that the value at $\lambda = 0$ is a current with support on $\{\sigma_s = 0\} \cap \ldots \cap \{\sigma_1 = 0\}$. Thus the value of (2.3) at $\lambda = 0$ has support on

$$\{\mu_s = 0\} \cap \ldots \cap \{\mu_1 = 0\} = \widetilde{Y}_{I^s} \cap \ldots \cap \widetilde{Y}_{I^1},$$

where $\widetilde{Y}_\bullet = \Pi^{-1} Y_\bullet$, and accordingly $U^{f_{I^t}} \wedge \ldots \wedge U^{f_{I^{s+1}}} \wedge R^{f_{I^s}} \wedge \ldots \wedge R^{f_{I^1}}$ is a current with support on $Y_{I^s} \cap \ldots \cap Y_{I^1}$.

Since the form (2.1) is alternating with respect to the factors $|f_{I^i}|^{2\lambda} u^{f_{I^i}}$ and symmetric with respect to the factors $\bar\partial |f_{I^i}|^{2\lambda} \wedge u^{f_{I^i}}$, it follows that $U^{f_{I^t}} \wedge \ldots \wedge U^{f_{I^{s+1}}} \wedge R^{f_{I^s}} \wedge \ldots \wedge R^{f_{I^1}}$ is alternating with respect to the principal value factors and symmetric with respect to the residue factors. $\square$

We continue with the proof of Theorem 1.3.

Proof of Theorem 1.3. Notice that T_ℓ is the analytic continuation to $\lambda = 0$ of the terms

$$|f_r|^{2\lambda} u^{f_r}_{\ell_r} \wedge \ldots \wedge |f_{s+1}|^{2\lambda} u^{f_{s+1}}_{\ell_{s+1}} \wedge \bar\partial |f_s|^{2\lambda} \wedge u^{f_s}_{\ell_s} \wedge \ldots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1}_{\ell_1}, \tag{2.5}$$

where

$$u^{f_i}_{\ell_i} = \frac{s_i \wedge (\bar\partial s_i)^{\ell_i - 1}}{|f_i|^{2\ell_i}}$$

and the total degree in $d\bar z_j$ (that is $\ell_1 + \ldots + \ell_r - r + s$) is ℓ.

Following the proof of Proposition 2.1, a term of the form (2.5), integrated against a test form ϕ, is equal to a sum of terms like

$$\int |\mu_r|^{2\lambda} a_r^\lambda \frac{\alpha_{r,\ell_r}}{\mu_r^{\ell_r}} \wedge \ldots \wedge |\mu_{s+1}|^{2\lambda} a_{s+1}^\lambda \frac{\alpha_{s+1,\ell_{s+1}}}{\mu_{s+1}^{\ell_{s+1}}} \wedge \\ \bar\partial(|\mu_s|^{2\lambda} a_s^\lambda) \wedge \frac{\alpha_{s,\ell_s}}{\mu_s^{\ell_s}} \wedge \ldots \wedge \bar\partial(|\mu_1|^{2\lambda} a_1^\lambda) \wedge \frac{\alpha_{1,\ell_1}}{\mu_1^{\ell_1}} \wedge \tilde\phi, \tag{2.6}$$

where the α_{i,ℓ_i}'s are smooth forms of degree ℓ_i in e_j, the a_i's are non-vanishing functions, the μ_i's are monomials in some local coordinates τ_j and $\tilde\phi$ is as in the previous proof. We can find a toric resolution such that locally one of $\mu_1, \ldots, \mu_s$ divides the other ones, so without loss of generality we may assume that μ_1 divides $\mu_2, \ldots, \mu_s$.

We expand $\bar\partial(|\mu_1|^{2\lambda} a_1^\lambda)$ by Leibniz' rule. Observe that when $\bar\partial$ falls on a_1^λ the integral vanishes as in the proof of Proposition 2.1, and thus it suffices to consider the case when $\bar\partial$ falls on one of the τ_j that divide μ_1, say on $|\sigma|^{2\lambda}$.

If $\ell < p$, we claim that this part of (2.6) vanishes when integrating with respect to σ. In fact, we may assume that $\phi = \phi_I \wedge d\bar{z}_I$, where ϕ_I is an $(n,0)$-form and $d\bar{z}_I = d\bar{z}_{I_1} \wedge \ldots \wedge d\bar{z}_{I_{n-\ell}}$. Now $d\bar{z}_I$ vanishes on the variety $Y_1 \cap \ldots \cap Y_s$ of codimension p for degree reasons. Consequently $\Pi^*(d\bar{z}_I)$ vanishes on $\widetilde{Y}_1 \cap \ldots \cap \widetilde{Y}_s$, and in particular on $\{\sigma = 0\}$. However, this is a form in $d\bar{\tau}_k$ with antiholomorphic coefficients since Π is holomorphic, and therefore each of its terms contains a factor $d\bar{\sigma}$ or a factor $\bar{\sigma}$. Indeed, if $\Psi(\tau)$ is a form in $d\bar{\tau}_k$ with antiholomorphic coefficients we can write

$$\Psi(\tau) = \Psi'(\tau) \wedge d\bar{\sigma} + \Psi''(\tau),$$

where $\Psi''(\tau)$ does not contain $d\bar{\sigma}$. The first term clearly vanishes on $\{\sigma = 0\}$ since $d\bar{\sigma}$ does. If $\Psi(\tau)$ vanishes on $\{\sigma = 0\}$, then $\Psi''(\tau)$ does, and hence it contains a factor $\bar{\sigma}$ due to antiholomorphicity. In both cases the σ-integral, and thereby (2.6), vanishes according to Lemma 2.2. $\square$

3. The complete intersection case

Our way of proving Theorem 1.4, that is, via Theorem 1.5, is inspired by Proposition 4.2 in [1], in which potentials were used to prove Theorem 1.1. The proof is self-contained and we hope that this construction of potentials will be of use for further investigations in the case of a non-complete intersection.

Proof of Theorem 1.5. We let

$$V = U^{f_1} + U^{f_2} \wedge R^{f_1} + U^{f_3} \wedge R^{f_2} \wedge R^{f_1} + \ldots + U^{f_r} \wedge R^{f_{r-1}} \wedge \ldots \wedge R^{f_1}.$$

To motivate this choice of V, note that on a formal level

$$(3.1) \quad \nabla_f (U^{f_i} \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1}) = R^{f_{i-1}} \wedge \ldots \wedge R^{f_1} - R^{f_i} \wedge \ldots \wedge R^{f_1},$$

so that

$$\nabla_f V = 1 - R^{f_r} \wedge \ldots \wedge R^{f_1}.$$

Indeed, observe that ∇_f acts on U^{f_i} just as ∇_{f_i}, so that $\nabla_f U^{f_i} = 1 - R^{f_i}$. Thus, to prove the first claim of the theorem we have to make this computation legitimate.

First, notice that if a form $A(\lambda)$, depending on a parameter λ, has an analytic continuation as a current to $\lambda = 0$, then clearly $\nabla_f A(\lambda)$ has one. The action on a test form ϕ is given by

$$\pm \int A(\lambda) \wedge \nabla_f \phi.$$

However, by integration by parts with respect to ∇_f and due to the uniqueness of analytic continuations, this is equal to

$$\int \nabla_f A(\lambda) \wedge \phi.$$

To be able to perform the integration by parts in a stringent way we have to regard the currents $T \in \mathcal{D}'_{0,k}(\Lambda^\ell E)$ as functionals on $\mathcal{D}_{n,n-k}(\Lambda^{n-\ell} E \wedge \Lambda^n E^*)$. So far we have been a little sloppy about this.

Thus, to compute $\nabla_f V$ we consider the form

$$v^\lambda = |f_1|^{2\lambda} u^{f_1} + |f_2|^{2\lambda} u^{f_2} \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1} + \dots \\ \dots + |f_r|^{2\lambda} u^{f_r} \wedge \bar\partial |f_{r-1}|^{2\lambda} \wedge u^{f_{r-1}} \wedge \dots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1},$$

since, by definition, $v^\lambda|_{\lambda=0} = V$, and accordingly $\nabla_f V = (\nabla_f v^\lambda)|_{\lambda=0}$. More precisely, to verify (3.1), let us consider (recall that $\nabla_f u^{f_i} = 1$)

$$\nabla_f (|f_i|^{2\lambda} u^{f_i} \wedge \bar\partial |f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \dots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1}) = \\ - \bar\partial |f_i|^{2\lambda} \wedge u^{f_i} \wedge \bar\partial |f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \dots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1} + \\ |f_i|^{2\lambda} \bar\partial |f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \dots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1} + \mathcal{R},$$

where $\mathcal{R}$ is a sum of terms of the form

$$|f_i|^{2\lambda} u^{f_i} \wedge \bar\partial |f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \dots \bar\partial |f_j|^{2\lambda} \wedge \bar\partial |f_{j-1}|^{2\lambda} \wedge \dots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1},$$

that arise when ∇_f falls on any u^{f_j}, $j < i$. The value at $\lambda = 0$ of the first term is just $-R^{f_i} \wedge R^{f_{i-1}} \wedge \dots \wedge R^{f_1}$, and it follows from Lemma 3.1 that the second term has an analytic continuation to $\lambda = 0$ equal to $R^{f_{i-1}} \wedge \dots \wedge R^{f_1}$. The remaining terms, $\mathcal{R}$, vanish according to Lemma 3.3. Thus (1.7) is proved, and thereby the first part of the theorem.

Furthermore, let

$$U^f \wedge V = U^f \wedge U^{f_1} + U^f \wedge U^{f_2} \wedge R^{f_1} + \\ U^f \wedge U^{f_3} \wedge R^{f_2} \wedge R^{f_1} + \dots + U^f \wedge U^{f_r} \wedge R^{f_{r-1}} \wedge \dots \wedge R^{f_1}.$$

We compute ∇_f of each term. To do this we use a form as above whose analytic continuation to $\lambda = 0$ is equal to this particular current. Now, we actually need the extended version of Theorem 1.2, that is Proposition 2.1.

Indeed, consider

$$\begin{aligned}
&\nabla_f(|f|^{2\lambda}u^f \wedge |f_i|^{2\lambda}u^{f_i} \wedge \bar\partial|f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar\partial|f_1|^{2\lambda} \wedge u^{f_1}) = \\
&\quad - \bar\partial|f|^{2\lambda} \wedge u^f \wedge |f_i|^{2\lambda}u^{f_i} \wedge \bar\partial|f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar\partial|f_1|^{2\lambda} \wedge u^{f_1} + \\
&\qquad |f|^{2\lambda}|f_i|^{2\lambda}u^{f_i} \wedge \bar\partial|f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar\partial|f_1|^{2\lambda} \wedge u^{f_1} + \\
&\quad |f|^{2\lambda}u^f \wedge \bar\partial|f_i|^{2\lambda} \wedge u^{f_i} \wedge \bar\partial|f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar\partial|f_1|^{2\lambda} \wedge u^{f_1} \\
&\quad - |f|^{2\lambda}u^f \wedge |f_i|^{2\lambda} \wedge \bar\partial|f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar\partial|f_1|^{2\lambda} \wedge u^{f_1} + \\
&\qquad\qquad |f|^{2\lambda}u^f \wedge \mathcal{R}.
\end{aligned}$$

The first term corresponds to $-R^f \wedge U^{f_i} \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1}$. Since f is a complete intersection and R^f therefore is of top degree in $d\bar z_j$ according to Theorem 1.3, it is most reasonable to expect also this product to be of top degree in $d\bar z_j$, but because of the factor $U^{f_i} \in \mathcal{L}^{-1}(E_i)$ that is apparently not possible unless the product vanishes. This is indeed the case, as follows from Lemma 3.2. The second, third and fourth terms have analytic continuations as $U^{f_i} \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1}$, $U^f \wedge R^{f_i} \wedge \ldots \wedge R^{f_1}$ and $-U^f \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1}$, respectively, by Lemma 3.1. The remaining terms vanish according to Lemma 3.3. Hence

$$\begin{aligned}
\nabla_f(U^f \wedge V) &= \sum_{i=1}^{r} U^{f_i} \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1} \\
&\quad - \sum_{i=1}^{r} (U^f \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1} - U^f \wedge R^{f_i} \wedge \ldots \wedge R^{f_1}) = \\
&\qquad V - U^f + U^f \wedge R^{f_r} \wedge \ldots \wedge R^{f_1}.
\end{aligned}$$

Finally, the term $U^f \wedge R^{f_r} \wedge \ldots \wedge R^{f_1}$ vanishes by Lemma 3.4, and thus taking the lemmas 3.1 to 3.4 for granted, the theorem is proved. $\square$

What remains is the technical part, to prove the lemmas. We have tried to put them as simply as possible. Still the formulations may seem a bit strained. Hopefully, the remarks will shed some light on what matters. We will use the word codegree for the difference between the dimension n of X and the degree.

Lemma 3.1. *Let $f = f_1 \oplus \cdots \oplus f_r$ be a section of $E^* = E_1^* \oplus \cdots \oplus E_r^*$. Assume that f is a complete intersection. Let $s < r$ and $s \leq r' \leq r$. If $h = f$, or if $h = f_i$ for some $i > s$, then*

$$|h|^{2\lambda}|f_{r'}|^{2\lambda}u^{f_{r'}} \wedge \ldots \wedge |f_{s+1}|^{2\lambda}u^{f_{s+1}} \wedge \bar\partial|f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \bar\partial|f_1|^{2\lambda} \wedge u^{f_1} \tag{3.2}$$

has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$, *which for* $\lambda = 0$ *is equal to the current* $U^{f_{r'}} \wedge \ldots \wedge U^{f_{s+1}} \wedge R^{f_s} \wedge \ldots \wedge R^{f_1}$.

Moreover,
(3.3)
$$|h|^{2\lambda}|f|^{2\lambda}u^f\wedge|f_{r'}|^{2\lambda}u^{f_{r'}}\wedge\ldots\wedge|f_{s+1}|^{2\lambda}u^{f_{s+1}}\wedge\bar\partial|f_s|^{2\lambda}\wedge u^{f_s}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}$$
has an analytic continuation to $\mathrm{Re}\,\lambda>-\epsilon$*, which for* $\lambda=0$ *is equal to the current* $U^f\wedge U^{f_{r'}}\wedge\ldots\wedge U^{f_{s+1}}\wedge R^{f_s}\wedge\ldots\wedge R^{f_1}$*.*

Remark 1. The crucial point is that inserting a factor $|h|^{2\lambda}$, where h is any tuple of holomorphic functions and $|\cdot|$ is any Hermitian metric, has no effect on the value at $\lambda=0$, as long as

$$\mathrm{codim}\,\{h=0\}\cap Y_s\cap\ldots\cap Y_1>\mathrm{codim}\,Y_s\cap\ldots\cap Y_1,$$

since then all possibly "dangerous" contributions to the current will vanish for degree reasons as in the proof of Theorem 1.3. That the currents are unaffected by the factor $|h|^{2\lambda}$ is closely related to them being their own standard extensions in the sense of Barlet [4]. □

Proof. We give a proof of the first claim of the lemma. The second one, concerning (3.3), can be proved along the same lines.

For a compactly supported test form ϕ, we consider

$$\int|h|^{2\lambda}|f_{r'}|^{2\lambda}u^{f_{r'}}\wedge\ldots\wedge|f_{s+1}|^{2\lambda}u^{f_{s+1}}\wedge\bar\partial|f_s|^{2\lambda}\wedge u^{f_s}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}\wedge\phi.$$

After a resolution of singularities as described in the proof of Proposition 2.1, for $\mathrm{Re}\,\lambda$ large enough, this integral is equal to a sum of

$$\int|\mu_h|^{2\lambda}|\mu_{r'}|^{2\lambda}a_{r'}^{\lambda}\frac{\alpha_{r',\ell_{r'}}}{\mu_{r'}^{\ell_{r'}}}\wedge\ldots\wedge|\mu_{s+1}|^{2\lambda}a_{s+1}^{\lambda}\frac{\alpha_{s+1,\ell_{s+1}}}{\mu_{s+1}^{\ell_{s+1}}}\wedge \\ \bar\partial(|\mu_s|^{2\lambda}a_s^{\lambda})\wedge\frac{\alpha_{s,\ell_s}}{\mu_s^{\ell_s}}\wedge\ldots\wedge\bar\partial(|\mu_1|^{2\lambda}a_1^{\lambda})\wedge\frac{\alpha_{1,\ell_1}}{\mu_1^{\ell_1}}\wedge\tilde\phi, \tag{3.4}$$

where the a_j's are strictly positive functions, the μ_j's are polynomials in some local coordinates τ_j, the α_{j,ℓ_j}'s are smooth forms and $\tilde\phi$ is as in the proof of Proposition 2.1. The existence of the analytic continuation to $\mathrm{Re}\,\lambda>-\epsilon$ follows from Lemma 2.2 as before.

Our aim is to prove that the factor $|h|^{2\lambda}$ does not affect the value at $\lambda=0$. Let σ be one of the coordinate functions τ_k that divides μ_h. When expanding each factor $\bar\partial(|\mu_j|^{2\lambda}a_j^{\lambda})$ by Leibniz' rule we get two different types of terms, integrals with an occurrence of a factor $\bar\partial|\sigma^\alpha|^{2\lambda}$ for some α, and integrals with no such factors. In the second case the extra factor $|\sigma|^{2\lambda}$ does no harm, since, in fact, the value at $\lambda=0$ is independent of the number of $|\sigma|^{2\lambda}$'s in the numerator as long as there is no $\bar\sigma$ in the denominator. Furthermore, we claim that each integral of the first kind actually vanishes at $\lambda=0$. The argument is analogous to the one in the proof of Theorem 1.3. Let us first consider the case when $h = f$. Observe that the terms in (3.2) are of degree at most $m_1 + \ldots + m_{r'} - r' + s \leq m - 1$ in $d\bar{z}_j$, where $m = m_1 + \ldots + m_r$. The crucial term -1 appears because of the (at least for the proof) necessary condition that $r > s$, that is that we have at least one factor U. Thus, it is enough to consider test forms of codegree in $d\bar{z}$ at most $m - 1$. We assume that $\phi = \phi_I \wedge d\bar{z}_I$, where ϕ_I is a smooth $(n, 0)$-form and $d\bar{z}_I = d\bar{z}_{I_1} \wedge \ldots \wedge d\bar{z}_{I_p}$ where $p \geq n - (m_1 + \ldots + m_r) + 1$. Now, $d\bar{z}_I$ vanishes on the variety $Y = f^{-1}(0)$, since it has codimension m, and accordingly $\Pi^*(d\bar{z}_I)$ vanishes on $\widetilde{Y} = \Pi^{-1}Y$, and in particular on $\{\sigma = 0\}$. Since it is a form in $d\bar{\tau}_j$ with antiholomorphic coefficients, each of its terms contains a factor $\bar{\sigma}$ or $d\bar{\sigma}$, see the proof of Theorem 1.3, and so in both cases the σ-integrals vanish according to Lemma 2.2.

In the second case, when $h = f_i$, the proof becomes slightly more complicated. We want to prove that the σ-integral vanishes due to the occurrence of a factor $\bar{\sigma}$ or $d\bar{\sigma}$ as above, but now the desired factors $\bar{\sigma}$ and $d\bar{\sigma}$ do not necessarily divide the test form $\tilde{\phi}$. We need to look at a "larger" form than ϕ, in fact at the "largest" possible "σ-free" form. Without loss of generality we may assume that, for some numbers s' and r'', $1 \leq s' \leq s \leq r'' \leq r'$, σ divides $\mu_{s'+1}, \ldots, \mu_s$ and $\mu_{r''+1}, \ldots, \mu_{r'}$ but neither $\mu_1, \ldots, \mu_{s'}$ nor $\mu_{s+1}, \ldots, \mu_{r''}$. Recall that $u^{f_i} = \sum_\ell v^{f_i}_{\ell_i}/|f_i|^{2\ell}$, where $v^{f_i}_{\ell_i} = s_i \wedge (\bar{\partial} s_i)^{\ell_i - 1}$. Let the smooth form

$$v^{f_{r''}}_{\ell_{r''}} \wedge \ldots \wedge v^{f_{s+1}}_{\ell_{s+1}} \wedge \bar{\partial}|f_{s'}|^2 \wedge v^{f_{s'}}_{\ell_{s'}} \wedge \ldots \wedge \bar{\partial}|f_1|^2 \wedge v^{f_1}_{\ell_1}$$

be denoted by F_ℓ, and let

$$Y' = \{f_{s'+1} = \ldots = f_s = f_{r''+1} = \ldots = f_{r'} = h = 0\}.$$

As above we may assume that ϕ consists of only one term $\phi_I \wedge d\bar{z}_I$. Then, by inspection, the form $F_\ell \wedge d\bar{z}_I$ is of codegree at most

$$m_{s'+1} + \ldots + m_s + m_{r''+1} + \ldots + m_{r'} - r' + r''$$

in $d\bar{z}_j$, which is strictly less than

$$\operatorname{codim} Y' = m_{s'+1} + \ldots + m_s + m_{r''+1} + \ldots + m_{r'} + m_h,$$

because of the assumptions of complete intersection. Consequently $F_\ell \wedge d\bar{z}_I$ vanishes on Y', and thus $\Pi^*(F_\ell \wedge d\bar{z}_I)$ vanishes on $\Pi^{-1}Y'$, and in particular on $\{\sigma = 0\}$. Since it is a form in $d\bar{\tau}_j$ with antiholomorphic coefficients, each of its terms contains a factor $\bar{\sigma}$ or a factor $d\bar{\sigma}$. Using that $\bar{\partial}|f|^{2\lambda} =$

$\lambda|f|^{2(\lambda-1)}\bar\partial|f|^2$, we can write (3.4) as

$$\pm\int |\mu_h|^{2\lambda}|\mu_{r'}|^{2\lambda}a_{r'}^{\lambda}\frac{\alpha_{r',\ell_{r'}}}{\mu_{r'}^{\ell_{r'}}}\wedge\ldots\wedge|\mu_{r''+1}|^{2\lambda}a_{r''+1}^{\lambda}\frac{\alpha_{r''+1,\ell_{r''+1}}}{\mu_{r''+1}^{\ell_{r''+1}}}\wedge$$
$$\bar\partial(|\mu_s|^{2\lambda}a_s^{\lambda})\wedge\frac{\alpha_{s,\ell_s}}{\mu_s^{\ell_s}}\wedge\ldots\wedge\bar\partial(|\mu_{s'+1}|^{2\lambda}a_1^{\lambda})\wedge\frac{\alpha_{s'+1,\ell_{s'+1}}}{\mu_{s'+1}^{\ell_{s'+1}}}\wedge$$
$$\frac{|\mu_{r''}|^{2\lambda}a_{r''}^{\lambda}\cdots|\mu_1|^{2\lambda}a_1^{\lambda}}{|\mu_{r''}|^{2\ell_{r''}}\cdots|\mu_{s+1}|^{2\ell_{s+1}}}\lambda^{s'}\frac{|\mu_{s'}|^{2\lambda}a_{s'}^{(\lambda-1)}\cdots|\mu_1|^{2\lambda}a_1^{(\lambda-1)}}{|\mu_{s'}|^{2(\ell_{s'}+1)}\cdots|\mu_1|^{2(\ell_1+1)}}\,\Pi^*(F_\ell)\wedge\tilde\phi,$$

where the sign depends on the relation between r', r'', s and s'. Now the only way a factor $\bar\sigma$ in the numerator (more precisely in $\Pi^*(F_\ell)\wedge\tilde\phi$) could be cancelled out when λ is small is by the occurrence of a factor $\bar\sigma$ in one of $\mu_1,\ldots,\mu_{s'}$, but that would obviously contradict the assumption made above. Hence each term in the integral must contain a factor $\bar\sigma$ or $d\bar\sigma$ independently of the value of λ and thus the σ-integral vanishes according to Lemma 2.2. □

Lemma 3.2. *Let $f = f_1\oplus\cdots\oplus f_r$ be a section of $E^* = E_1^*\oplus\cdots\oplus E_r^*$ of rank m and let $h = f\oplus f'$, where f' is a section of the dual bundle of a holomorphic m'-bundle E'. Assume that h is a complete intersection. If $r > s$, then*

(3.5)
$$\bar\partial|h|^{2\lambda}\wedge u^h\wedge|f_r|^{2\lambda}u^{f_r}\wedge\ldots\wedge|f_{s+1}|^{2\lambda}u^{f_{s+1}}\wedge\bar\partial|f_s|^{2\lambda}\wedge u^{f_s}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}$$

has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ *that vanishes at* $\lambda = 0$.

Remark 2. Notice that the value at $\lambda = 0$ corresponds to the current $R^h\wedge U^{f_r}\wedge\ldots\wedge U^{f_{s+1}}\wedge R^{f_s}\wedge\ldots\wedge R^{f_1}$. Since h is a complete intersection, R^h is of degree $m+m'$ in $d\bar z_j$ according to Theorem 1.3, and therefore it is reasonable to expect also the product to be of degree $m+m'$ in $d\bar z_j$. However, since the product contains at least one principal value factor, the degree in e_j must be strictly larger than the degree in $d\bar z_j$, and so, the product must vanish. We will see that the assumption that $r > s$ is crucial also for the proof. □

Proof. After a resolution of singularities as described in the proof of Proposition 2.1, we can write (3.5) integrated against a test form ϕ as a sum of terms of the type

$$\int\bar\partial(|\mu_h|^{2\lambda}a_h^{\lambda})\frac{\alpha_{h,\ell_h}}{\mu_h^{\ell_h}}\wedge|\mu_r|^{2\lambda}a_r^{\lambda}\frac{\alpha_{r,\ell_r}}{\mu_r^{\ell_r}}\wedge\ldots\wedge|\mu_{s+1}|^{2\lambda}a_{s+1}^{\lambda}\frac{\alpha_{s+1,\ell_{s+1}}}{\mu_{s+1}^{\ell_{s+1}}}\wedge$$
$$\bar\partial(|\mu_s|^{2\lambda}a_s^{\lambda})\wedge\frac{\alpha_{s,\ell_s}}{\mu_s^{\ell_s}}\wedge\ldots\wedge\bar\partial(|\mu_1|^{2\lambda}a_1^{\lambda})\wedge\frac{\alpha_{1,\ell_1}}{\mu_1^{\ell_1}}\wedge\tilde\phi,$$

where the α_{i,ℓ_i}'s are smooth forms of degree ℓ_i in e_j, the a_i's are non-vanishing functions and the μ_i's are monomials in some local coordinates τ_k and $\tilde{\phi}$ is as in the previous proofs.

We expand the factor $\bar\partial(|\mu_h|^{2\lambda}a_h^\lambda)$ by Leibniz' rule and consider the term obtained when $\bar\partial$ falls on $|\sigma|^{2\lambda}$, where σ is one of the τ_k's that divide μ_h. We prove that this term vanishes when integrating with respect to σ. The term that arises when $\bar\partial$ falls on a_h^λ clearly vanishes as before, see the proof of Proposition 2.1. Since the rank of $E\oplus E'$ is $m+m'$, the terms in (3.5) are of degree at most $m+m'-1$ in $d\bar z$, since we have at least one U-factor. Thus it is enough to consider test forms of codegree in $d\bar z$ at most $m+m'-1$. As in the previous proofs we may assume that $\phi=\phi_I\wedge d\bar z_I$. It follows that $d\bar z_I$ vanishes on $Y=h^{-1}(0)$ for degree reasons, and thus $\Pi^*(d\bar z_I)$ vanishes on $\Pi^{-1}Y$. Since this is a form in $d\bar\tau_j$ with antiholomorphic coefficients, each of its terms contains a factor $\bar\sigma$ or $d\bar\sigma$ and consequently the σ-integral vanishes according to Lemma 2.2. □

Lemma 3.3. *Let $f=f_1\oplus\cdots\oplus f_r$ be a section of $E^*=E_1^*\oplus\cdots\oplus E_r^*$. Assume that f is a complete intersection and let $s<r$. Then*

$$|f_r|^{2\lambda}u^{f_r}\wedge\ldots\wedge|f_{s+1}|^{2\lambda}u^{f_{s+1}}\wedge\bar\partial|f_s|^{2\lambda}\wedge u^{f_s}\wedge\ldots\wedge\bar\partial|f_t|^{2\lambda}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1} \tag{3.6}$$

and

$$|f|^{2\lambda}u^f\wedge|f_r|^{2\lambda}u^{f_r}\wedge\ldots\wedge|f_{s+1}|^{2\lambda}u^{f_{s+1}}\wedge\bar\partial|f_s|^{2\lambda}\wedge u^{f_s}\wedge \ldots\wedge\bar\partial|f_t|^{2\lambda}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1} \tag{3.7}$$

have analytic continuations to $\operatorname{Re}\lambda>-\epsilon$ *that vanish at* $\lambda=0$.

Remark 3. Morally, what this lemma says is that when applying Leibniz' rule to ∇_f acting on a product of principal value and residue currents, there will be no contributions from ∇_f falling on a residue factor. Of course this is expected, since the residue currents are ∇_f-closed. □

Proof. For (3.6) the result follows from Lemma 3.1 after an integration by parts with respect to ∇_f. (Recall that ϕ is a form taking values in $\Lambda^{n-\ell}E\wedge\Lambda^nE^*$.) Note that $\bar\partial|f_t|^{2\lambda}=-\nabla_f|f_t|^{2\lambda}$. By Stokes' theorem,

$$\int|f_r|^{2\lambda}u^{f_r}\wedge\ldots\wedge|f_{s+1}|^{2\lambda}u^{f_{s+1}}\wedge\bar\partial|f_s|^{2\lambda}\wedge u^{f_s}\wedge\ldots$$
$$\ldots\wedge\nabla_f|f_t|^{2\lambda}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}\wedge\phi=$$
$$\pm\int(|f_t|^{2\lambda}-1)\nabla_f(|f_r|^{2\lambda}u^{f_r}\wedge\ldots\wedge|f_{s+1}|^{2\lambda}u^{f_{s+1}}\wedge$$
$$\bar\partial|f_s|^{2\lambda}\wedge u^{f_s}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}\wedge\phi),$$

so it is enough to prove that this expression vanishes at $\lambda = 0$. Now, applying Leibniz' rule to

$$\nabla_f(|f_r|^{2\lambda}u^{f_r} \wedge \ldots \wedge |f_{s+1}|^{2\lambda}u^{f_{s+1}} \wedge \bar\partial|f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \bar\partial|f_1|^{2\lambda} \wedge u^{f_1} \wedge \phi)$$

gives a sum of terms, of which the ones arising when ∇_f falls on a factor u^{f_t} for $1 \leq t \leq r$ will vanish for degree reasons, whereas the others will be precisely as in the hypothesis of Lemma 3.1. Moreover f_t is an h of the second kind, so according to Lemma 3.1 the factor $|f_t|^{2\lambda}$ does not have any effect on the value at $\lambda = 0$. Thus we are done.

In the case of (3.7), after an integration by parts, we have to prove that

$$\int (|f_t|^{2\lambda} - 1)\nabla_f(|f|^{2\lambda}u^f \wedge |f_r|^{2\lambda}u^{f_r} \wedge \ldots \wedge |f_{s+1}|^{2\lambda}u^{f_{s+1}} \wedge \\ \bar\partial|f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \bar\partial|f_1|^{2\lambda} \wedge u^{f_1} \wedge \phi)$$

vanishes at $\lambda = 0$. The term when ∇_f falls on the factor $|f|^{2\lambda}u^f$ is of the type in Lemma 3.2. It is easy to see from the proof that the factor $|f_t|^{2\lambda}$ does not affect the value at $\lambda = 0$ and so this term vanishes. The remaining part is as in the hypothesis of the latter statement of Lemma 3.1, thus the result follows as above. □

Lemma 3.4. *Let $f = f_1 \oplus \cdots \oplus f_r$ be a section of $E^* = E_1^* \oplus \cdots \oplus E_r^*$. Assume that f is a complete intersection. Let $h = f_{I_1} \oplus \cdots \oplus f_{I_p}$, where $I = \{I_1, \ldots, I_p\} \subset \{1, \ldots, r\}$. Then*

$$|h|^{2\lambda}u^h \wedge \bar\partial|f_r|^{2\lambda} \wedge u^{f_r} \wedge \ldots \wedge \bar\partial|f_1|^{2\lambda} \wedge u^{f_1} \tag{3.8}$$

has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ *that vanishes at* $\lambda = 0$.

Remark 4. The value at $\lambda = 0$ corresponds to the current $U^h \wedge R^{f_1} \wedge \ldots \wedge R^{f_r}$. Since the R-part is of top degree according to Theorem 1.3 this product should formally vanish by arguments similar to those in Remark 2. □

Proof. As in the proofs of the previous lemmas we start by a resolution of singularities. Thus, the form (3.8) integrated against a test form ϕ is equal to a sum of terms of the type

$$\int |\mu_h|^{2\lambda}a_h^\lambda \frac{\alpha_{h,\ell_h}}{\mu_h^{\ell_h}} \wedge \bar\partial(|\mu_r|^{2\lambda}a_r^\lambda) \wedge \frac{\alpha_{r,\ell_r}}{\mu_r^{\ell_r}} \wedge \ldots \wedge \bar\partial(|\mu_1|^{2\lambda}a_1^\lambda) \wedge \frac{\alpha_{1,\ell_1}}{\mu_1^{\ell_1}} \wedge \tilde\phi,$$

where α_{i,ℓ_i}, a_i, μ_i and $\tilde\phi$ are as above. Further, we can find a resolution to a certain toric variety so that locally one of the monomials $\mu_1, \ldots, \mu_r$ divides the other ones. Without loss of generality we may assume that μ_1 divides all μ_j's. We expand $\bar\partial(|\mu_1|^{2\lambda}a_1^\lambda)$ by Leibniz' rule. The term obtained when $\bar\partial$ falls on a_1^λ vanishes as in the proof of Proposition 2.1, so it is enough to consider the terms that arise when $\bar\partial$ falls on $|\sigma|^{2\lambda}$, where σ is one of the coordinates in μ_1.

We claim that the σ-integral vanishes at $\lambda = 0$. As usual, we observe that the terms of (3.8) are of degree at most $m-1$ in $d\bar z_j$, where the -1 in this case is due to the factor U^h, so it suffices to consider test forms of codegree at most $m-1$. We assume that $\phi = \phi_I \wedge d\bar z_I$, where ϕ_I is an $(n,0)$-form and $d\bar z_I = d\bar z_{I_1} \wedge \ldots \wedge d\bar z_{I_p}$, where $p \leq n-m+1$. Then $d\bar z_I$ vanishes on the variety $Y = f^{-1}(0)$ for degree reasons, and accordingly $\Pi^*(d\bar z)$ vanishes on $\Pi^{-1}Y$, and in particular on $\{\sigma_1 = 0\}$. By arguments as in the proof of Theorem 1.3 it follows that $\Pi^*(d\bar z)$ must contain a factor $\bar\sigma$ or $d\bar\sigma$ since it is a form in $d\bar\tau_k$ with antiholomorphic coefficients, and hence the σ-integral vanishes as before. □

Remark 5. If $f = f_1, f_2$ defines a complete intersection, then

$$\lambda = (\lambda_1, \lambda_2) \mapsto \int \bar\partial |f_1|^{2\lambda_1} \frac{1}{f_1} \wedge \bar\partial |f_2|^{2\lambda_2} \frac{1}{f_2} \wedge \phi$$

is holomorphic at $\lambda = 0$, see [13], as was first proven by Berenstein and Yger. The result has been claimed to extend to any finite number of functions f_i, but we have found no proofs in the literature. It was recently verified to be true in the case of three functions by Samuelsson [15]. His proof shows that, in the three- (or more-) dimensional case, the question of analyticity becomes a global problem in the resolutions, which makes it much more involved.

Provided the Mellin transform of the residue integral is shown to be analytic in $\lambda = (\lambda_1, \ldots, \lambda_r)$, most likely, similar arguments could be used to prove that

$$t(\lambda) := \bar\partial |f_r|^{2\lambda_r} \wedge u^{f_r} \wedge \ldots \wedge \bar\partial |f_1|^{2\lambda_1} \wedge u^{f_1}$$

is analytic in λ. Note that $t((\lambda, \ldots, \lambda))|_{\lambda=0}$ by definition is our current $R^{f_1} \wedge \ldots \wedge R^{f_r}$. Presuming $t(\lambda)$ to be analytic, we can give a soft proof of Theorem 1.4, based on Theorem 1.1. Indeed, let

$$t^f_{CFL}(\lambda) = \bar\partial |f|^{2\lambda} \wedge u^f,$$

and

$$t^f_{CH}(\lambda) = \bar\partial |f_1|^{2\lambda} \frac{1}{f_1} \wedge \ldots \wedge \bar\partial |f_m|^{2\lambda} \frac{1}{f_m},$$

where CFL and CH of course stand for Cauchy-Fantappiè-Leray and Coleff-Herrera, respectively. With this notation the equality in Theorem 1.1 can be expressed as

$$t^f_{CFL}(\lambda)|_{\lambda=0} = t^f_{CH}(\lambda)|_{\lambda=0}. \tag{3.9}$$

Now let f and g be sections of the bundles E_1^* and E_2^*, respectively, and assume that $f \oplus g$ is a complete intersection. By definition,

$$R^f \wedge R^g = t^f_{CFL}(\lambda) \wedge t^g_{CFL}(\lambda)|_{\lambda=0},$$

and

$$R^{f\oplus g} = t^{f\oplus g}_{CFL}(\lambda)|_{\lambda=0},$$

so we need to prove that

$$t^f_{CFL}(\lambda) \wedge t^g_{CFL}(\lambda)|_{\lambda=0} = t^{f\oplus g}_{CFL}(\lambda)|_{\lambda=0}.$$

If $\operatorname{Re}\lambda_2$ is large enough, $t^g_{CFL}(\lambda_2)$ is in L^1_{loc}, and so by (3.9)

$$t^f_{CFL}(\lambda_1) \wedge t^g_{CFL}(\lambda_2)|_{\lambda_1=0} = t^f_{CH}(\lambda_1) \wedge t^g_{CFL}(\lambda_2)|_{\lambda_1=0},$$

and analogously, if $\operatorname{Re}\lambda_1$ is large enough

$$t^f_{CH}(\lambda_1) \wedge t^g_{CFL}(\lambda_2)|_{\lambda_2=0} = t^f_{CH}(\lambda_1) \wedge t^g_{CH}(\lambda_2)|_{\lambda_2=0}.$$

Now, by assumption

$$(\lambda_1, \lambda_2) \mapsto t^f_\bullet(\lambda_1) \wedge t^g_\bullet(\lambda_2),$$

where $\bullet$ stands for either CFL or CH, is holomorphic at the origin, and thus it follows that

$$t^f_{CFL}(\lambda) \wedge t^g_{CFL}(\lambda)|_{\lambda=0} = t^f_{CH}(\lambda) \wedge t^g_{CH}(\lambda)|_{\lambda=0},$$

but the right hand side is, by (3.9), equal to $t^{f\oplus g}_{CFL}(\lambda)|_{\lambda=0}$, and so we obtain Theorem 1.4 for $r = 2$. However, the argument easily extends to arbitrary r, since

$$t^{f_1}_\bullet(\lambda_1) \wedge \ldots \wedge t^{f_i}_{CFL}(\lambda_i) \wedge \ldots \wedge t^{f_r}_\bullet(\lambda_r)|_{\lambda_i=0} = \\ t^{f_1}_\bullet(\lambda_1) \wedge \ldots \wedge t^{f_i}_{CH}(\lambda_i) \wedge \ldots \wedge t^{f_r}_\bullet(\lambda_r)|_{\lambda_i=0}$$

by (3.9) if $\lambda_j, j \neq i$ are large enough.

We should mention that the above method actually gives a proof of Theorem 1.4 in the special case when f is of rank 2 and g is of rank 1. It follows from Samuelsson's result, [15], and the fact that $t(\lambda)$ indeed is holomorphic if $r = 2$. The latter statement is not hard to verify, see for example [17]. □

4. An example

We conclude this paper with an explicit computation, by which we enlighten the possibility of extending Theorem 1.4 to a slightly weaker notion of complete intersection. Indeed, when generalizing Theorem 1.1, or rather its line bundle formulation (1.6), to sections of bundles of arbitrary rank, it is not obvious how one should interpret the assumption of f being a complete intersection. In the formulation of Theorem 1.4 we require the codimension of $f^{-1}(0)$ to be equal to the rank of the bundle E. A less strong hypothesis would be to just demand the f_i's to intersect properly, that is, that $\operatorname{codim} f^{-1}(0) = p_1 + \ldots + p_r$ if $p_i = \operatorname{codim} f_i$. However, the following example shows that Theorem 1.4 does not extend to this case.

Example 1. Let $f_1 = z_1^2$, $f_2 = z_1 z_2$ and $g = z_2 z_3$. Then

$$Y_f = f^{-1}(0) = \{z_1 = 0\}, \quad Y_g = g^{-1}(0) = \{z_2 = 0\} \cup \{z_3 = 0\},$$
$$\text{and } Y = Y_f \cap Y_g = \{z_1 = z_2 = 0\} \cup \{z_1 = z_3 = 0\}.$$

Note that Y_f and Y_g have codimension 1, and that Y has codimension 2. Thus f and g intersect properly, although they do not define a complete intersection.

Let us compute $(R^f \wedge R^g)_2$. Adopting the trivial metric we get

$$s^f = \bar{f}_1 e_1 + \bar{f}_2 e_2 = \bar{z}_1(\bar{z}_1 e_1 + \bar{z}_2 e_2) \text{ and } |f|^2 = |z_1|^2(|z_1|^2 + |z_2|^2),$$

so that

$$u_1^f = \frac{\bar{z}_1 e_1 + \bar{z}_2 e_2}{z_1(|z_1|^2 + |z_2|^2)}.$$

Let ϕ be a test form of bidegree $(3,1)$ with support outside $\{z_2 = 0\}$. Then $R^f \wedge R^g.\phi$ is given by

$$\int \bar\partial(|z_1|^{2\lambda}(|z_1|^2 + |z_2|^2)^\lambda) \wedge \frac{\bar{z}_1 e_1 + \bar{z}_2 e_2}{z_1(|z_1|^2 + |z_2|^2)} \wedge \bar\partial |z_2 z_3|^{2\lambda} \wedge \frac{e_3}{z_2 z_3} \wedge \phi|_{\lambda=0} =$$
$$- \int \bar\partial\Big[\frac{1}{z_1}\Big]\Big[\frac{1}{z_2}\Big] \wedge \bar\partial\Big[\frac{1}{z_3}\Big] \wedge e_2 \wedge e_3 \wedge \phi.$$

Note that the support of the current is on the z_2-axis, as expected since ϕ has support outside the z_3-axis.

To deal with test forms with support intersecting $\{z_2 = 0\}$ we need to resolve the singularity of f at the z_3-axis. Let $\widetilde{\mathcal{U}}$ be the blow-up of $\mathbb{C}^3_z$ along the z_3-axis and let $\Pi : \widetilde{\mathcal{U}} \to \mathbb{C}^3_z$ be the corresponding proper map. We can cover $\widetilde{\mathcal{U}}$ by two coordinate charts,

$$\Omega_1 = \{(\tau_1, \tau_2, z_3);\ (\tau_1, \tau_1\tau_2, z_3) = z \in \mathbb{C}^3_z\}$$
$$\text{and } \Omega_2 = \{(\sigma_1, \sigma_2, z_3);\ (\sigma_1\sigma_2, \sigma_1, z_3) = z \in \mathbb{C}^3_z\}.$$

In Ω_1

$$\Pi^* u_1^f = \frac{e_1 + \bar\tau_2 e_2}{\tau_1^2(1 + |\tau_2|^2)},$$

and thus

$$R^{\Pi^* f} \wedge R^{\Pi^* g} = \bar\partial|\tau_1|^{4\lambda} \wedge \frac{e_1 + \bar\tau_2 e_2}{\tau_1^2(1 + |\tau_2|^2)} \wedge\ \bar\partial|\tau_1\tau_2 z_3|^{2\lambda} \wedge \frac{e_3}{\tau_1\tau_2 z_3}\Big|_{\lambda=0} =$$
$$\frac{2}{3}\ \bar\partial\Big[\frac{1}{\tau_1^3}\Big] \wedge \frac{e_1 + \bar\tau_2 e_2}{\tau_1^3(1 + |\tau_2|^2)} \wedge \bar\partial\Big[\frac{1}{\tau_2 z_3}\Big] \wedge e_3.$$

Let ϕ be a test form of bidegree $(3,1)$; we can write ϕ as $\phi_I \wedge dz$, where $dz = dz_1 \wedge dz_2 \wedge dz_3$ and $\phi_I = \varphi^1(z)d\bar{z}_1 + \varphi^2(z)d\bar{z}_2 + \varphi^3(z)d\bar{z}_3$. Now $\int R^f \wedge R^g \wedge \phi = \int_{\widetilde{\mathcal{U}}} R^{\Pi^* f} \wedge R^{\Pi^* g} \wedge \Pi^*\phi$. To compute the contribution from the chart Ω_1, let $\widetilde{\phi} = \chi \Pi^* \phi$, where χ is a function of some partition of unity with support in Ω_1. We may without loss of generality assume that χ only depends on $|\tau_i|$ and also that $\chi(0,0,z_3) = 1$. Then we get that $R^{\Pi^* f} \wedge R^{\Pi^* g} . \widetilde{\phi}$ is equal to

$$\frac{2}{3}\int \bar{\partial}\Big[\frac{1}{\tau_1^3}\Big] \wedge e_1 \wedge \bar{\partial}\Big[\frac{1}{\tau_2}\Big]\Big[\frac{1}{z_3}\Big] \wedge e_3 \wedge \chi\varphi^3(\tau_1, \tau_1\tau_2, z_3)\, d\bar{z}_3 \wedge d\tau_1 \wedge d(\tau_1\tau_2) \wedge dz_3 =$$
$$(2\pi i)^2 \frac{2}{3} \int \varphi_1^3(0,0,z_3)\frac{1}{z_3} e_1 \wedge e_3 \wedge d\bar{z}_3 \wedge dz_3 =$$
$$-\frac{2}{3}\int \bar{\partial}\Big[\frac{1}{z_1^2}\Big] \wedge \bar{\partial}\Big[\frac{1}{z_2}\Big]\Big[\frac{1}{z_3}\Big] \wedge e_1 \wedge e_3 \wedge \phi,$$

where we have used the well known fact that

$$\int_z \bar{\partial}\Big[\frac{1}{z^p}\Big] \wedge \psi(z)dz = \frac{2\pi i}{(p-1)!}\frac{\partial^{p-1}}{\partial z^{p-1}}\psi(0).$$

Computing $R^{\Pi^* f} \wedge R^{\Pi^* g}$ in Ω_2 gives yet another contribution. Altogether we get

$$(R^f \wedge R^g)_2 = -\bar{\partial}\Big[\frac{1}{z_1}\Big]\Big[\frac{1}{z_2}\Big] \wedge \bar{\partial}\Big[\frac{1}{z_3}\Big] \wedge e_2 \wedge e_3$$
$$-\frac{2}{3}\bar{\partial}\Big[\frac{1}{z_1^2}\Big] \wedge \bar{\partial}\Big[\frac{1}{z_2}\Big]\Big[\frac{1}{z_3}\Big] \wedge e_1 \wedge e_3 + \frac{1}{3}\bar{\partial}\Big[\frac{1}{z_1}\Big] \wedge \bar{\partial}\Big[\frac{1}{z_2^2}\Big]\Big[\frac{1}{z_3}\Big] \wedge e_2 \wedge e_3.$$

Similar computations yield

$$R_2^{f\oplus g} = \bar{\partial}\Big[\frac{1}{z_1^2}\Big] \wedge \bar{\partial}\Big[\frac{1}{z_2}\Big]\Big[\frac{1}{z_3}\Big] \wedge e_1 \wedge e_3 - \bar{\partial}\Big[\frac{1}{z_1}\Big]\Big[\frac{1}{z_2^2}\Big] \wedge \bar{\partial}\Big[\frac{1}{z_3}\Big] \wedge e_2 \wedge e_3.$$

For details, we refer to [17]. See also Theorem 5.2 in [18].

To conclude, $R^f \wedge R^g \neq R^{f\oplus g}$, and hence Theorem 1.4 does not generalize to the case of proper intersections. $\square$

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Paper II

RESIDUE CURRENTS OF MONOMIAL IDEALS

ELIZABETH WULCAN

ABSTRACT. We compute residue currents of Bochner-Martinelli type associated with a monomial ideal I, by methods involving certain toric varieties. In case the variety of I is the origin, we give a complete description of the annihilator of the currents in terms of the associated Newton diagram. In particular, we show that the annihilator is strictly included in I, unless I is defined by a complete intersection. We also provide partial results for general monomial ideals.

1. INTRODUCTION

Let f be a tuple of holomorphic functions $f_1, \dots, f_m$ in $\mathbb{C}^n$ and let $Y = \{f_1 = \dots = f_m = 0\}$. If f is a complete intersection, that is, the codimension of Y is m, the duality theorem, due to Dickenstein-Sessa, [6], and Passare, [10], asserts that a holomorphic function h locally belongs to the ideal $(f) = (f_1, \dots, f_m)$ if and only if $hR^f_{CH} = 0$, where R^f_{CH} is the Coleff-Herrera residue current of f. In [11], Passare, Tsikh and Yger introduced residue currents for arbitrary f by means of the Bochner-Martinelli kernel. For each ordered index set $\mathcal{I} \subseteq \{1, \dots, m\}$ of cardinality k, let $R^f_{\mathcal{I}}$ be the analytic continuation to $\lambda = 0$ of

$$\bar{\partial}|f|^{2\lambda} \wedge \sum_{\ell=1}^{k} (-1)^{\ell-1} \frac{\overline{f_{i_\ell}} \bigwedge_{q \neq \ell} \overline{df_{i_q}}}{|f|^{2k}},$$

where $|f|^2 = |f_1|^2 + \dots + |f_m|^2$. Then $R^f_{\mathcal{I}}$ is a well-defined $(0,k)$-current with support on Y, that vanishes whenever $k < \operatorname{codim} Y$ or $k > \min(m,n)$. In case f defines a complete intersection, the only nonvanishing current, $R^f_{\{1,\dots,m\}}$, is shown to coincide with the Coleff-Herrera current.

The concept of Bochner-Martinelli residue currents was further developed by Andersson in [1]. From his construction, based on the Koszul complex, follows that $hR^f_{\mathcal{I}} = 0$ for all $\mathcal{I}$ implies that the holomorphic function h

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belongs to the ideal (f) locally. Thus, letting $\operatorname{Ann} R^f$ denote the annihilator ideal, $\{h \text{ holomorphic}, hR^f_{\mathcal{I}} = 0, \forall \mathcal{I}\}$, we have that

$$\operatorname{Ann} R^f \subseteq (f). \tag{1.1}$$

The inclusion is strict in general, and thus the currents $R^f_{\mathcal{I}}$ do not fully characterize (f) as in the complete intersection case. Still the ideal $\operatorname{Ann} R^f$ is big enough to catch in some sense the "size" of (f). Recall that a holomorphic function h belongs locally to the *integral closure* of (f), denoted by $\overline{(f)}$, if $|h| \leq C|f|$ for some constant C, or equivalently if h fulfills a monic equation $h^r + g_1 h^{r-1} + \ldots + g_r = 0$ with $g_i \in (f)^i$ for $1 \leq i \leq r$. In [11] it was proved that $hR^f_{\mathcal{I}} = 0$ for any h that is locally in the integral closure of $(f)^k$, where $k = |\mathcal{I}|$, and thus we get

$$\overline{(f)^\mu} \subseteq \operatorname{Ann} R^f, \tag{1.2}$$

where $\mu = \min(m, n)$. Now, combining (1.1) and (1.2) yields a proof of the Briançon-Skoda theorem [5]: $\overline{(f)^\mu} \subseteq (f)$. This motivates us to study the ideal $\operatorname{Ann} R^f$.

In this paper we compute the Bochner-Martinelli currents $R^f_{\mathcal{I}}$ in case the generators f_i are all monomials. Because of their simplicity and nice combinatorial description monomial ideals serve as a good toy model for illustrating general ideas and results in commutative algebra and algebraic geometry, such as resolution of singularities and Briançon-Skoda type theorems, see [13]. On the other hand many results for general ideals can be proved by specializing to monomial ideals. Recall from [11] and [1] that the existence of (Bochner-Martinelli) residue currents indeed is proved by reducing to a monomial situation by resolving singularities. Monomial ideals are therefore a natural starting point for investigating the inclusions (1.1) and (1.2) and for producing examples of $R^f_{\mathcal{I}}$. A first result in this direction was obtained in [11] (Proposition 3.1) where $R^f_{\{1,\ldots,n\}}$ was computed explicitly for monomial ideals generated by exactly n monomials.

Our main result, Theorem 3.1, gives a complete description of $\operatorname{Ann} R^f$ in terms of the Newton diagram associated with the generators, when (f) is a monomial ideal of dimension 0. In particular it turns out that $\operatorname{Ann} R^f$ depends only on (f), not on the particular choice of generators. Also, it follows that we have equality in (1.1) if and only if (f) is a complete intersection and moreover that the inclusion (1.2) is always strict. The proof of Theorem (3.1), given in Section 4, amounts to computing residue currents in a certain toric variety constructed from the generators, using ideas originally from Varchenko, [14], and Khovanskii, [8]. In Section 5 we provide partial results for the case of general monomial ideals.

2. Preliminaries and notation

Let A be a set in $\mathbb{Z}^n_+$ and let z^A denote the tuple of monomials $\{z^a\}_{a\in A}$, where $z^a = z_1^{a_1}\cdots z_n^{a_n}$ if $a=(a_1,\dots,a_n)$. The ideal (z^A) admits a nice geometric interpretation as the set $\cup_{a\in A}(a+\mathbb{R}^n_+)\subset\mathbb{R}^n$. Indeed, a holomorphic function is in the ideal precisely when its support ($\operatorname{supp}\sum\varphi_a z^a = \{a\in\mathbb{Z}^n_+, \varphi_a\neq 0\}$) is in $\cup_{a\in A}(a+\mathbb{R}^n_+)$. The *Newton polyhedron* $\Gamma^+(A)$ of A is defined as the convex hull of $\cup_{a\in A}(a+\mathbb{R}^n_+)\subset\mathbb{R}^n$ and the *Newton diagram* $\Gamma(A)$ of A is the union of all compact faces of the Newton polyhedron. Recall that a face of maximal dimension is called a *facet*. For further reference we remark that the set of vertices of the Newton polyhedron is a subset of A, see for example [15].

We will work in the framework from [1] and use the fact that the currents $R^f_{\mathcal{I}}$ appear as the coefficients of full Bochner-Martinelli current introduced there. We identify z^A with a section of the dual bundle E^* of a trivial vector bundle E over $\mathbb{C}^n$ of rank $m=|A|$ (the number of generators), endowed with the trivial metric. If $\{e_a\}_{a\in A}$ is a global holomorphic frame for E and $\{e^*_a\}_{a\in A}$ is the dual frame, we can write $z^A=\sum_{a\in A}z^a e^*_a$. We let s be the dual section $\sum_{a\in A}\bar{z}^a e_a$ of z^A. Also, we fix an ordering of A.

Next, we let

$$u=\sum_\ell \frac{s\wedge(\bar\partial s)^{\ell-1}}{|z^A|^{2\ell}},$$

where $|z^A|^2=\sum_{a\in A}|z^a|^2$, be the full Bochner-Martinelli form, introduced in [2] in order to construct integral formulas with weight factors in a convenient way. Then u is a smooth section of $\Lambda(E\oplus T^*_{0,1}(\mathbb{C}^n))$ (where $e_a\wedge d\bar{z}_i = -d\bar{z}_i\wedge e_a$), that is clearly well defined outside $Y=f^{-1}(0)$, and moreover

$$\bar\partial|z^A|^{2\lambda}\wedge u$$

has an analytic continuation as a current to $\operatorname{Re}\lambda > -\epsilon$. The *(full) Bochner-Martinelli residue current* R^{z^A} is defined as the value at $\lambda=0$. Then R^{z^A} has support on Y and $R^{z^A}=R_p+\ldots+R_\mu$, where $p=\operatorname{codim}Y$ and $\mu=\min(m,n)$, and where $R_k\in\mathcal{D}'_{0,k}(\mathbb{C}^n,\Lambda^k E)$, by analogy with the fact that the current $R^f_{\mathcal{I}}$ vanishes if $|\mathcal{I}|$ is smaller than p or greater than μ. We should remark that Andersson's construction of residue currents, using kernels of Cauchy-Fantappiè-Leray type, works for sections of any holomorphic vector bundle equipped with some Hermitian metric. Observe that in our case (trivial bundle and trivial metric), though, the coefficients of R^{z^A} are just currents of the type $R^f_{\mathcal{I}}$. Indeed, letting s_B be the section $\sum_{a\in B}\bar{z}^a e_a$, we can write u as a sum, taken over subsets B of A, of terms

$$u_B = \frac{s_B \wedge (\bar\partial s_B)^{k-1}}{|z^A|^{2n}},$$

where k is the cardinality of B. The corresponding current,

$$\bar\partial |z^A|^{2\lambda} \wedge u_B \tag{2.1}$$

evaluated at $\lambda = 0$, denoted by $R_B^{z^A}$ or R_B for short, is then merely the current $R^f_{\mathcal{I}}$ with $\mathcal{I}$ corresponding to the subset B, times the basis element $e_B = \bigwedge_{a\in B} e_a$, where the wedge product is taken with respect to the ordering. Henceforth we will deal with the Bochner-Martinelli currents rather then currents $R^f_{\mathcal{I}}$.

Let us make an observation that will be of further use. If the section s can be written as $\mu s'$ for some smooth function μ we have the following homogeneity:

$$s \wedge (\bar\partial s)^{k-1} = \mu^k s' \wedge (\bar\partial s')^{k-1}, \tag{2.2}$$

that holds since s is of odd degree.

We will use the notation $\bar\partial[1/f]$ for the value at $\lambda = 0$ of $\bar\partial |f|^{2\lambda}/f$ and analogously by $[1/f]$ we will mean $|f|^{2\lambda}/f|_{\lambda=0}$, that is just the principal value of $1/f$. By iterated integration by parts we have that

$$\int_z \bar\partial\Big[\frac{1}{z^p}\Big] \wedge \varphi dz = \frac{2\pi i}{(p-1)!}\frac{\partial^{p-1}}{\partial z^{p-1}}\varphi(0). \tag{2.3}$$

In particular, the annihilator of $\bar\partial[1/z^p]$ is (z^p). The currents $R_B^{z^A}$ will typically be tensor products of currents of this type.

3. Main results

Our main result is an explicit computation of the Bochner-Martinelli residue current R^{z^A} in case Y is the origin. Before stating it let us introduce some notation. We say that a subset $B = \{a_1, \ldots, a_n\} \subseteq A$ is *essential* if there exists a facet F of $\Gamma^+(A)$ such that B lies in F and if in addition B spans $\mathbb{R}^n$, that is $\det(a_1, \ldots, a_n) \neq 0$. It follows, when $Y = \{0\}$, that the essential sets are contained in the Newton diagram $\Gamma(A)$. Indeed, $Y = \{0\}$ precisely when A intersects all axes in $\mathbb{Z}^n$ and thus the only non-compact faces of Γ^+ are contained in the coordinate planes in $\mathbb{Z}^n$. But if B is contained in a coordinate plane, B can not span $\mathbb{R}^n$. Also, when $Y = \{0\}$, all points in $A \cap \Gamma(A)$ are in fact contained in some essential set. Next, if B is a subset of A, let $\alpha^B = \sum_{a\in B} a$. Notice that if B is essential, then α^B lies on $n\Gamma$. In fact, α^B/n is the barycenter of the simplex spanned by B. We are now ready to formulate our main theorem.

Theorem 3.1. *Let $z^A, A \subseteq \mathbb{Z}_+^n$ be a tuple of monomials in $\mathbb{C}^n$ such that $\{z^A = 0\} = \{0\}$, and let R^{z^A} be the corresponding Bochner-Martinelli residue current . Then*

$$R^{z^A} = \sum_{B \subseteq A} R_B,$$

where

$$R_B = C_B\ \bar\partial\Big[\frac{1}{z_1^{\alpha_1^B}}\Big] \wedge \ldots \wedge \bar\partial\Big[\frac{1}{z_n^{\alpha_n^B}}\Big] \wedge e_B, \tag{3.1}$$

and where C_B is a constant that is nonzero if B is an essential set and zero otherwise.

An immediate consequence is that if B is essential then

$$\operatorname{Ann} R_B = (z_1^{\alpha_1^B}, \ldots, z_n^{\alpha_n^B}),$$

where $\operatorname{Ann} R_B$ just denotes the ideal of holomorphic functions annihilating R_B. Note in particular that $\operatorname{Ann} R_B$ depends only on the set B and not on the remaining A. Furthermore, since the basis elements e_B are all different it follows that

$$\operatorname{Ann} R^{z^A} = \bigcap_{B \text{ essential}} \operatorname{Ann} R_B.$$

Thus, $\operatorname{Ann} R^{z^A}$ is fully determined by the Newton diagram $\Gamma(A)$ and the points in A lying on it. In particular $\operatorname{Ann} R^{z^A}$ depends only on the ideal, not on the particular choice of generators. We also see that different monomial ideals (z^A) and $(z^{A'})$ give rise to the same annihilator ideal if and only if $A \cap \Gamma(A) = A' \cap \Gamma(A')$.

Furthermore, Theorem 3.1 implies that the inclusion (1.1) is strict unless we have a complete intersection.

Theorem 3.2. *Let $z^A, A \subseteq \mathbb{Z}_+^n$, be a tuple of monomials such that $\{z^A = 0\} = \{0\}$, and let R^{z^A} be the corresponding Bochner-Martinelli residue current. Then*

$$Ann\, R^{z^A} = (z^A)$$

if and only if (z^A) can be generated by a complete intersection.

For the proof we need a simple lemma.

Lemma 3.3. *Let B be an essential subset of A such that $(z^B) \subseteq Ann\, R_B$. Then (z^B) is a complete intersection.*

Proof. Denote the elements in B by $a_i, i = 1, \ldots, n$ and let $\geq$ be the natural partial order on $\mathbb{Z}^n$. Suppose that $(z^B) \subseteq \operatorname{Ann} R_B$. We have that $z^{a_i} \in$

$\operatorname{Ann} R_B$ precisely when one of the generators of $\operatorname{Ann} R_B$ divides z^{a_i}, that is, when

$$\begin{aligned}(a_{1i}, \dots, a_{ni}) &\geq (\sum_j a_{1j}, 0, \dots, 0) \text{ or}\\ (a_{1i}, \dots, a_{ni}) &\geq (0, \sum_j a_{2j}, 0, \dots, 0) \text{ or}\\ &\vdots\\ (a_{1i}, \dots, a_{ni}) &\geq (0, \dots, 0, \sum_j a_{nj}).\end{aligned}$$

This set of inequalities holds for all $1 \leq i \leq n$, and it is easy to see that this implies first that $a_{k\ell} \neq 0$ for at most one k, which means that a_ℓ lies in one of the coordinate axes, and second that there is at least one a_ℓ intersecting each coordinate axis. Thus, B intersects all coordinate axes in $\mathbb{Z}^n$, which in turn implies that (z^B) is a complete intersection. $\square$

Proof of Theorem 3.2. We need to show the "only if" direction. Suppose that $(z^A) = \operatorname{Ann} R^{z^A}$ and let B be an essential subset. Clearly essential subsets always exist, since otherwise $R^{z^A} = 0$ and $\operatorname{Ann} R^{z^A} = (z^A)$ is the whole ring of holomorphic functions, which contradicts that $Y = \{0\}$. Now, in particular $(z^B) \subseteq \operatorname{Ann} R_B$, and by Lemma 3.3, (z^B) is a complete intersection. Thus

$$(z^B) = \operatorname{Ann} R^{z^B} = \operatorname{Ann} R_B^{z^A} \supseteq \operatorname{Ann} R^{z^A} = (z^A) \supseteq (z^B),$$

where the second equality follows since $\operatorname{Ann} R_B$ only depends on B and not on A. Hence $(z^A) = (z^B)$ and the result follows. $\square$

We give some examples to illustrate Theorems 3.1 and 3.2.

Example 1. Let

$$A = \{a^1 = (8,0), a^2 = (6,1), a^3 = (2,3), a^4 = (1,5), a^5 = (0,6)\} \subseteq \mathbb{Z}^2.$$

We identify the ideal (z^A) with the set $\bigcup_{a \in A}(a + \mathbb{R}^n_+)$ as in Figure 1, where we have also depicted the Newton diagram Γ. Such pictures of monomial ideals are usually referred to as *staircase diagrams*, see [9]. The points in A should be recognized as the "inner corners" of the staircase. The Newton diagram $\Gamma(A)$ consists of two facets, one with vertices a^1 and a^3 and the other one with vertices a^3 and a^5, and thus we have the essential sets

$$\{a^1, a^2\}, \{a^1, a^3\}, \{a^2, a^3\}, \{a^3, a^5\},$$

with

$$\alpha^{12} = (14,1), \alpha^{13} = (10,3), \alpha^{23} = (8,4), \alpha^{35} = (2,9),$$

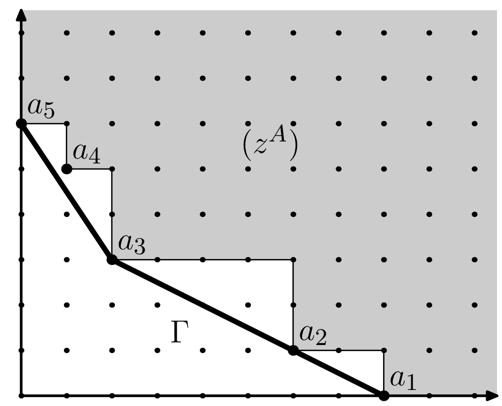

FIGURE 1. The ideal (z^A) and the Newton diagram $\Gamma(A)$ in Example 1

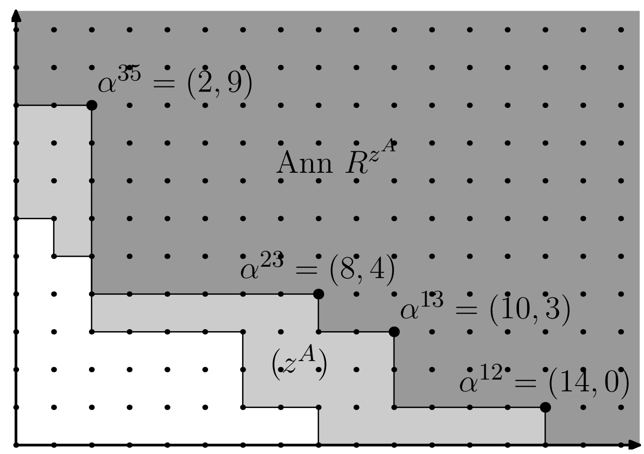

FIGURE 2. The ideals $\operatorname{Ann} R^{z^A}$ (dark gray) and (z^A) (light gray) in Example 1

respectively. It follows from Theorem 3.1 that

$$\operatorname{Ann} R^{z^A} = (z_1^{14}, z_2) \cap (z_1^{10}, z_2^3) \cap (z_1^8, z_2^4) \cap (z_1^2, z_2^9),$$

which is equal to the ideal $(z_1^{14}, z_1^{10}z_2, z_1^8z_2^3, z_1^2z_2^4, z_2^9)$, see Figure 2. Observe that $\operatorname{Ann} R^{z^A}$ is given by the staircase diagram with α^{ij} as "outer corners". Note also that $\operatorname{Ann} R^{z^A}$ does not depend on a^4, which lies in the interior of $\Gamma^+(A)$. $\square$

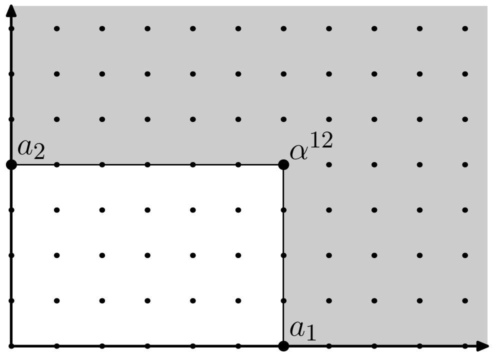

FIGURE 3. A complete intersection

Example 2. Consider the complete intersection $\{z_1^{a^1}, \ldots, z_n^{a^n}\}$. The associated Newton diagram is the n-simplex spanned by

$$A = \{(a^1, 0, \ldots, 0), \ldots, (0, \ldots, 0, a^n)\}$$

and there exists only one essential set, namely A itself, with $\alpha^A = (a^1, \ldots, a^n)$. Thus according to Theorem 3.1,

$$\operatorname{Ann} R^{z^A} = (z_1^{a^1}, \ldots, z_n^{a^n}),$$

so the annihilator ideal is equal to (z^A), which we already knew. Figure 3 illustrates the two ways of thinking of the ideal when $n = 2$; either as a staircase with $(a^1, 0)$ and $(0, a^2)$ as inner corners or as a staircase with $\alpha^A = (a^1, a^2)$ as the (only) outer corner. □

Example 3. We should remark that not all monomial ideals arise as annihilator ideals associated with monomial ideals. The idea is that the outer corners of the staircase of an annihilator ideal must lie on a hypothetical Newton diagram. Indeed, from the discussion just before Theorem 3.1 we know that each α^B corresponding to an essential set B lies on $n\Gamma$. In other words, the lines joining adjacent outer corners must lie on the boundary of a convex domain above the staircase, and thus a necessary condition is that the "slope" of the staircase decreases while we are descending it.

For example, consider the ideal

$$I = (z_1^5, z_1^4 z_2^2, z_1 z_2^4, z_2^5)$$

with staircase diagram as in Figure 4, where we have also marked the slope. Clearly, the outer corners cannot lie on the boundary of a convex Newton polyhedron, and thus I is not an annihilator ideal. □

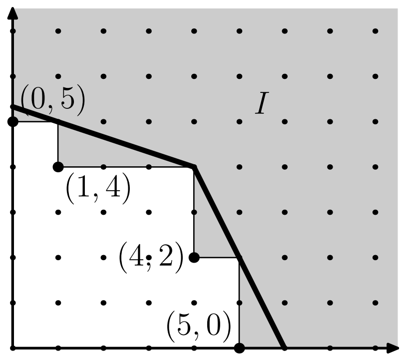

FIGURE 4. The ideal in Example 3. The thick lines illustrate the "slope" of the staircase.

Remark 1. Observe that adding an extra generator to an ideal (z^A) does not necessarily make the corresponding annihilator ideal smaller or larger. However, with a fixed Newton diagram an extra generator can only make the annihilator ideal smaller. In fact, given Γ, $\operatorname{Ann} R^{z^A}$ is maximal if A is chosen as the vertex set of Γ and minimal if A is all integer points on Γ, as we will see in Example 4. □

Let us now consider the inclusion (1.2). We start by interpreting the left hand side in case f is monomial.

Lemma 3.4. *The integral closure of the monomial ideal (z^A) is the monomial ideal generated by $z^a, a \in \Gamma^+(A)$.*

The result is well known from algebraic contexts, see for example [12].

Next, we claim that the ideal $\overline{(z^A)^r}$ is generated by $z^a, a \in r\Gamma^+(A)$. The ideal $(z^A)^r$ is generated by $z^a, a \in A + \ldots + A$ (r times), so we need to show that the Newton polytope of $A + \ldots + A$ is equal to $r\Gamma^+(A)$. But $A + \ldots + A \supseteq rA$ and thus $\Gamma^+(A + \ldots + A) \supseteq \Gamma^+(rA) = r\Gamma^+(A)$. On the other hand $A + \ldots + A \subseteq \Gamma^+(A) + \ldots + \Gamma^+(A) = r\Gamma^+(A)$, where the equality holds since $\Gamma^+(A)$ is a convex set, and so it follows that $\Gamma^+(A + \ldots + A) \subseteq r\Gamma^+(A)$.

Corollary 3.5. *Suppose $n \geq 2$. Let z^A be as in Theorem 3.1. Then the integral closure of the ideal $(z^A)^n$ is strictly included in $\operatorname{Ann} R^{z^A}$.*

Observe that Corollary 3.5 fails when $n = 1$. Then, in fact, $(z^A) = \operatorname{Ann} R^{z^A} = \overline{(z^A)}$.

Proof. Let $(b_1, 0, \ldots, 0)$ be the intersection between $\Gamma(A)$ and the x_1-axis and let $f = z_1^{nb_1 - 1}$. Then $(nb_1 - 1, 0, \ldots, 0) \notin n\Gamma^+(A)$ and thus $f \notin \overline{(z^A)^n}$. However, $f \in \operatorname{Ann} R_B$ for all essential B. To see this, observe that the

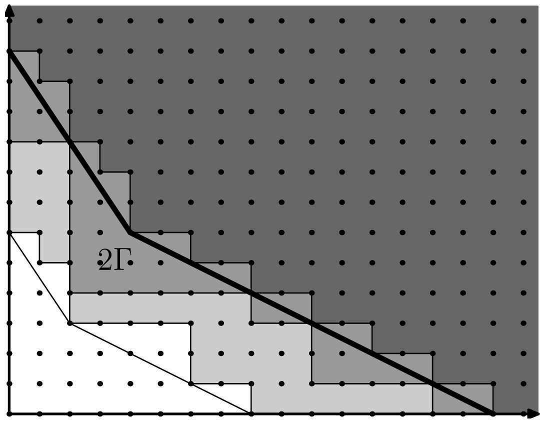

FIGURE 5. The ideals (z^A) (light gray) $\operatorname{Ann} R^{z^A}$ (medium gray) and $\overline{(z^A)^2}$ (dark gray) in Example 1

simplex spanned by the intersection points between Γ and the axes separates Γ from $\{x_1 = b_1\}$, and so Γ intersects the hyperplane $\{x_1 = b_1\}$ only at the point $(b_1, 0, \dots, 0)$. This implies in particular that $\alpha_1^B \leq nb_1 - (n-1)$ for all essential B and thus $f \in (z_1^{\alpha_1^B}) \subseteq \operatorname{Ann} R_B$. Hence we have found a function f in $\operatorname{Ann} R^{z^A} \setminus \overline{(z^A)^n}$. $\square$

Another, probably more illuminating, way of thinking of the ideals is in terms of staircase diagrams as in the examples above. The fact that the ideal $\overline{(z^A)^n}$ is generated by $\{z^a\}, a \in n\Gamma^+$ means that its staircase lies just above $n\Gamma$. On the other hand we know that the outer corners of the staircase of $\operatorname{Ann} R^{z^A}$, the α^B, lie on $n\Gamma$ and therefore the staircase must lie under $n\Gamma$. Thus the staircase of $\operatorname{Ann} R^{z^A}$ is "strictly lower" than the staircase of $\overline{(z^A)^n}$ and so the corresponding inclusion of ideals is strict. For an illustration, see Figure 5, where we have drawn the staircases of the three ideals (z^A), $\operatorname{Ann} R^{z^A}$ and $\overline{(z^A)^n}$ for A from Example 1.

Example 4. Let Γ be the simplex with vertices $(3,0)$ and $(0,3)$. In Figure 6 we have drawn the staircases of the ideals (z^A) (light gray), $\operatorname{Ann} R^{z^A}$ (medium gray) and $\overline{(z^A)^2}$ (dark gray) for different $A = A_i$ with Γ as Newton diagram; more precisely for $A_1 = \{(3,0),(0,3)\}$, $A_2 = \{(3,0),(2,1),(0,3)\}$, and finally for $A_3 = \{(3,0),(2,1),(1,2),(0,3)\}$. We see that $\operatorname{Ann} R^{z^A}$ decreases when we add points to A. In particular integrally closed ideals, that is ideals I such that $\overline{I} = I$, have the smallest annihilator ideals. $\square$

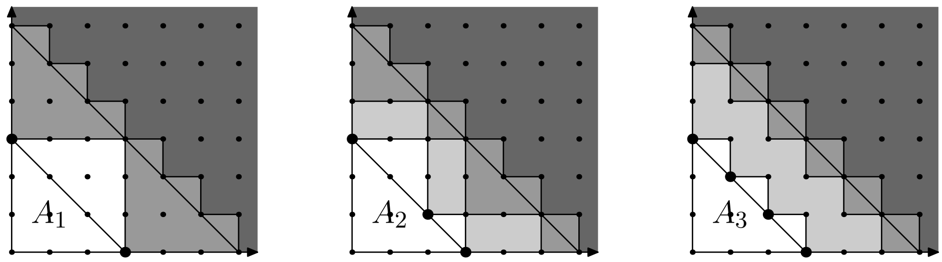

FIGURE 6. The various ideals in Example 4

4. PROOF OF THEOREM 3.1

The proof of Theorem 3.1 is very much inspired by the proof of Lemma 2.2 in [11] and the proof of Theorem 1.1 in [1]. We will compute R^{z^A} as the push-forward of a corresponding current on a certain toric variety $\mathcal{X}$ constructed from the Newton polyhedron $\Gamma^+(A)$. To do this we will have use for the following simple lemma which is proved essentially by integration by parts.

Lemma 4.1. *Let v be a strictly positive smooth function in $\mathbb{C}$, φ a test function in $\mathbb{C}$, and p a positive integer. Then*

$$\lambda \mapsto \int v^\lambda |s|^{2\lambda} \varphi(s) \frac{ds \wedge d\bar{s}}{s^p}$$

and

$$\lambda \mapsto \int \bar{\partial}(v^\lambda |s|^{2\lambda}) \wedge \varphi(s) \frac{ds}{s^p}$$

both have meromorphic continuations to the entire plane with poles at rational points on the negative real axis. At $\lambda = 0$ they are both independent of v and the second one only depends on the germ of φ at the origin. Moreover, if $\varphi(s) = \bar{s}\psi(s)$ or $\varphi = d\bar{s} \wedge \psi$, then the value of the second integral at $\lambda = 0$ is zero.

Throughout this section we will write $\mathbf{1}$ for the unit vector $(1, 1, \ldots, 1)$. We will regard the elements in A as column vectors and denote by B the matrix with the vectors in the set B as columns. Also we will use the notation $\widehat{\alpha_i}$ for $\alpha_1 \wedge \ldots \wedge \alpha_{i-1} \wedge \alpha_{i+1} \wedge \ldots \wedge \alpha_n$.

Let us start by describing $\mathcal{X}$, following [4]. Let $\mathcal{S}$ be the set of normal directions to the facets of Γ^+ represented by vectors ρ with minimal integer non-negative coefficients. Then $\mathcal{S}$ provides a partition of the first orthant of $\mathbb{R}^n$ into a finite number of distinct n-dimensional cones. Such a system of cones with the same apex together with their faces is called a *fan*. We say that the fan is generated by $\mathcal{S}$ and we denote it by $\Delta(\mathcal{S})$. By techniques due to Mumford et al., [7], $\mathcal{S}$ can be completed into a system $\widetilde{\mathcal{S}}$ of vectors ρ

such that if $\rho_1, \dots, \rho_n$ generate one of the n-dimensional cones of $\Delta(\widetilde{\mathcal{S}})$, then $\det(\rho_1, \dots, \rho_n) = \pm 1$. Such a fan is called *regular.* We will construct $\mathcal{X}$ by gluing together different copies of $\mathbb{C}^n$, one for each n-dimensional cone of $\Delta(\widetilde{\mathcal{S}})$. Let τ be such a cone and denote its generators by $\rho_1, \dots, \rho_n$. Let $\mathcal{U}$ be the corresponding copy of $\mathbb{C}^n$ with local coordinates $t = (t_1, \dots, t_n)$. Let P be the matrix with $\rho_i = (\rho_{1i}, \dots, \rho_{ni})$ as rows and let Π be the mapping

$$\begin{aligned} \Pi : \mathcal{U} &\to \mathbb{C}^n \\ t &\mapsto t^P, \end{aligned}$$

where t^P is a shorthand notation for $(t_1^{\rho_{11}} \cdots t_n^{\rho_{n1}}, \dots, t_1^{\rho_{1n}} \cdots t_n^{\rho_{nn}})$.

Two points $t \in \mathcal{U}$ and $t' \in \mathcal{U}'$ are identified if the monoidal map $\Pi'^{-1} \circ \Pi : \mathcal{U} \to \mathcal{U}'$ is defined at t and maps t to t'. Gluing the charts $\mathcal{U}$ together induces a proper map $\widetilde{\Pi} : \mathcal{X} \to \mathbb{C}^n$ that is biholomorphic from $\mathcal{X} \setminus \widetilde{\Pi}^{-1}(\{z_1 \cdots z_n = 0\})$ to $\mathbb{C}^n \setminus \{z_1 \cdots z_n = 0\}$, that is, outside the coordinate planes. It holds that $\widetilde{\Pi}^{-1}(\{z_1 \cdots z_n = 0\})$ is a set of measure zero in $\mathcal{X}$, and moreover $\widetilde{\Pi}^{-1}(0)$ consists of a system of various $\mathbb{CP}^{n-i}$, corresponding to i-dimensional cones of the fan $\Delta(\widetilde{\mathcal{S}})$. In particular, each vector ρ, that generates a 1-dimensional cone, corresponds to a $\mathbb{CP}^{n-1}$, denoted by S_ρ and obtained by gluing together parts of the charts from the cones determined by n-dimensional cones that have ρ as one of its generators. In fact, if the vector ρ determines the coordinate t_1 in $\mathcal{U}$, then S_ρ is covered by the $\{t_1 = 0\}$-part of $\mathcal{U}$.

Observe that $R^{z^A} = R_n$ since $Y = \{0\}$. Therefore, we only need to compute the currents R_B when B is a subset of cardinality n. For $\operatorname{Re}\lambda$ large enough, (2.1) is integrable and since $\widetilde{\Pi}$ is biholomorphic outside a set of measure zero it holds that

$$\int_{\mathbb{C}^n} \bar\partial |z^A|^{2\lambda} \wedge u_B \wedge \phi = \int_{\mathcal{X}} \widetilde{\Pi}^*(\bar\partial |z^A|^{2\lambda} \wedge u_B) \wedge \widetilde{\Pi}^* \phi,$$

if ϕ is a test form of bidegree $(n,0)$. It is easy to see that the analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ of $\widetilde{\Pi}^*(\bar\partial |z^A|^{2\lambda} \wedge u_B)$ exists in each chart $\mathcal{U}_\tau$; we will actually compute it below. Thus, because of the uniqueness of analytic continuations,

$$\widetilde{R}_B := \widetilde{\Pi}^*(\bar\partial |z^A|^{2\lambda} \wedge u_B)|_{\lambda=0}$$

defines a (globally defined) current on $\mathcal{X}$ such that $\widetilde{\Pi}_* \widetilde{R}_B = R_B$. We will start by computing $\widetilde{R}_B$ in a fixed chart $\mathcal{U}_0$ parametrized by Π corresponding to the cone τ_0.

Claim 1. *The current $\widetilde{R}_B$ vanishes in $\mathcal{U}_0$ whenever B is not contained in a facet whose normal direction is one of the generators of τ_0. Moreover $\widetilde{R}_B$ vanishes if* $\det B = 0$.

In particular, a necessary condition for $\widetilde{R}_B$ not to vanish is that B is essential.

Proof. First, note that the pullback Π^* transforms the exponents of monomials by the linear mapping P;

$$\Pi^* z^a = \Pi^* z_1^{a_1} \cdots z_n^{a_n} = t_1^{\rho_{11}a_1+\ldots+\rho_{1n}a_n} \cdots t_n^{\rho_{n1}a_1+\ldots+\rho_{nn}a_n} = t^{Pa}. \tag{4.1}$$

It is well known that for some $a_0 \in A, \Pi^* z^{a_0}$ divides $\Pi^* z^a$ for all $a \in A$, and moreover, in view of (4.1) one easily checks that a_0 has to be a vertex of $\Gamma^+(A)$. Using this we can write

$$\Pi^* s = \bar{t}^{Pa_0} s',$$

where s' is the nonvanishing section

$$s' = \sum_{a\in A} \bar{t}^{P(a-a_0)} e_a,$$

and furthermore

$$\Pi^* |z^A|^2 = |t|^{2Pa_0} \nu(t),$$

where

$$\nu(t) = \sum_{a\in A} |t|^{2P(a-a_0)n}$$

is nonvanishing. By homogeneity, see (2.2),

$$\Pi^*(s \wedge (\bar\partial s)^{n-1}) = \bar{t}^{nPa_0} s' \wedge (\bar\partial s')^{n-1},$$

and thus

$$\widetilde{R}_B = \bar\partial(|t|^{2\lambda Pa_0} \nu^\lambda) \left. \frac{s'_B \wedge (\bar\partial s'_B)^{n-1}}{t^{nPa_0} \nu(t)^n} \right|_{\lambda=0}. \tag{4.2}$$

By Leibniz' rule and Lemma 4.1, (4.2) is equal to a sum of currents

$$\bar\partial \left[\frac{1}{t_i^{n\rho_i \cdot a_0}} \right] \otimes \left[\frac{1}{\prod_{j\neq i} t_j^{n\rho_j \cdot a_0}} \right] \wedge \frac{s'_B \wedge (\bar\partial s'_B)^{n-1}}{\nu(t)^n}. \tag{4.3}$$

We need to compute $s'_B \wedge (\bar\partial s'_B)^{n-1}$. Denote the elements in B by $b_1, \ldots, b_n$ in such a way that $e_B = e_{b_n} \wedge \ldots \wedge e_{b_1}$. Furthermore, let C be the matrix with columns $Pb_i - Pa_0$ so that

$$s'_B = \sum_i \bar{t}_1^{c_{1i}} \cdots \bar{t}_n^{c_{ni}} e_{b_i},$$

and let D_i be the determinant of C with row i replaced with the unit vector $\mathbf{1}$. Then we have the following lemma.

Lemma 4.2. *We have that*

$$s'_B \wedge (\bar\partial s'_B)^{n-1} = (n-1)! \bar t^{C\mathbf{1}} \sum_i (-1)^{i-1}\ D_i \widehat{\frac{d\bar t_i}{\bar t_i}} \wedge e_B, \tag{4.4}$$

where

$$\widehat{\frac{d\bar t_i}{\bar t_i}} = \frac{d\bar t_1}{\bar t_1} \wedge \ldots \wedge \frac{d\bar t_{i-1}}{\bar t_{i-1}} \wedge \frac{d\bar t_{i+1}}{\bar t_{i+1}} \wedge \ldots \wedge \frac{d\bar t_n}{\bar t_n}.$$

Observe that all $\bar t_i$ in the denominator are cancelled since (4.4) is in fact smooth.

Proof. Let $\alpha_j = \bar t_1^{c_{1i}} \cdots \bar t_n^{c_{ni}} e_{b_j}$ and $\beta_i = \frac{d\bar t_i}{\bar t_i}$. Then $s'_B = \sum_{j=1}^n \alpha_j$ and

$$\bar\partial s'_B = \sum_{j=1}^n \sum_{i=1}^n c_{ij} \frac{d\bar t_i}{\bar t_i} \wedge \bar t_1^{c_{1i}} \cdots \bar t_n^{c_{ni}} e_{b_j} = \sum_{j=1}^n \sum_{i=1}^n c_{ij} \beta_i \wedge \alpha_j.$$

Thus we get

$$s'_B \wedge (\bar\partial s'_B)^{n-1} = \sum_{j=1}^n \alpha_j \wedge (\sum_{j=1}^n \sum_{i=1}^n c_{ij}\beta_i \wedge \alpha_j)^{n-1} =$$

$$\sum_{\sigma \in S^n} \sum_{\tau \in S^n} c_{\sigma(2)\tau(2)} \cdots c_{\sigma(n)\tau(n)} \alpha_{\tau(1)} \wedge \beta_{\sigma(2)} \wedge \alpha_{\tau(2)} \wedge \ldots \wedge \beta_{\sigma(n)} \wedge \alpha_{\tau(n)} =$$

$$\sum_{\sigma \in S^n} \sum_{\tau \in S^n} c_{\sigma(2)\tau(2)} \cdots c_{\sigma(n)\tau(n)} \beta_{\sigma(2)} \wedge \ldots \wedge \beta_{\sigma(n)} \wedge \alpha_{\tau(n)} \wedge \ldots \wedge \alpha_{\tau(1)} =$$

$$\sum_{\sigma \in S^n} \sum_{\tau \in S^n} (-1)^{\operatorname{sgn}\ \tau} c_{\sigma(2)\tau(2)} \cdots c_{\sigma(n)\tau(n)} \beta_{\sigma(2)} \wedge \ldots \wedge \beta_{\sigma(n)} \wedge \alpha_n \wedge \ldots \wedge \alpha_1 =$$

$$\sum_{i=1}^n \sum_{\sigma \in S^n; \sigma(1)=i} D_i (-1)^{\operatorname{sgn}\ \sigma} \beta_{\sigma(2)} \wedge \ldots \wedge \beta_{\sigma(n)} \wedge \alpha_n \wedge \ldots \wedge \alpha_1 =$$

$$\sum_{i=1}^n (n-1)! D_i (-1)^{i-1} \beta_1 \wedge \ldots \widehat{\beta_i} \ldots \wedge \beta_n \wedge \alpha_n \wedge \ldots \wedge \alpha_1 =$$

$$(n-1)! \bar t^{C\mathbf{1}} \sum_{i=1}^n (-1)^{i-1}\ D_i \widehat{\frac{d\bar t_i}{\bar t_i}} \wedge e_B.$$

Here S^n just denotes the set of permutations of $\{1, \ldots, n\}$. □

Now (4.3) is equal to

$$\bar\partial \left[\frac{1}{t_i^{n\rho_i \cdot a_0}} \right] \otimes \left[\frac{1}{\prod_{j \neq i} t_j^{n\rho_j \cdot a_0}} \right] \wedge \frac{(n-1)!\ D_i\ \bar t^{C\mathbf{1}}}{\nu(t)^n} \widehat{\frac{d\bar t_i}{\bar t_i}} \wedge e_B, \tag{4.5}$$

that can vanish for two reasons. First, by Lemma 4.1, (4.5) vanishes whenever the numerator contains a factor $\bar{t}_i$, that happens if $c_{ij} > 0$ for some j, which means that Pb_j has a greater t_i-coordinate than Pa_0. Thus, a necessary condition for (4.5) not to vanish is that $P(B)$ is contained in the facet of $P(\Gamma^+)$ parallel to the coordinate plane $\{t_i = 0\}$; in other words, since P is invertible, that B is contained in the facet F_i of Γ^+ with normal direction ρ_i. Hence the first part of Claim 1 follows.

Second, (4.5) vanishes if $D_i = 0$. Assume for simplicity that $i = 1$. Then $\rho_1 \cdot a$ is constant and equal to $\rho_1 \cdot a_0$ on F_1, that is, $(PB)_{1j} = (Pa_0)_1$ for all j, and we get

$$D_1 = \begin{vmatrix} 1 & \cdots & 1 \\ c_{21} & \cdots & c_{2n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{vmatrix} = \begin{vmatrix} 1 & \cdots & 1 \\ (PB)_{21} - (Pa_0)_2 & \cdots & (PB)_{2n} - (Pa_0)_2 \\ \vdots & & \vdots \\ (PB)_{n1} - (Pa_0)_n & \cdots & (PB)_{nn} - (Pa_0)_n \end{vmatrix} =$$

$$\begin{vmatrix} 1 & \cdots & 1 \\ (PB)_{21} & \cdots & (PB)_{2n} \\ \vdots & & \vdots \\ (PB)_{n1} & \cdots & (PB)_{nn} \end{vmatrix} = \frac{1}{(Pa_0)_1} \begin{vmatrix} (PB)_{11} & \cdots & (PB)_{1n} \\ (PB)_{21} & \cdots & (PB)_{2n} \\ \vdots & & \vdots \\ (PB)_{n1} & \cdots & (PB)_{nn} \end{vmatrix} = \frac{\det(PB)}{(Pa_0)_1}.$$

But since P is invertible $(Pa_0)_1 \neq 0$ and $\det P \neq 0$. Thus $D_i = 0$ if and only if $\det B = 0$. $\square$

Note that it follows from the proof of Claim 1 that $\widetilde{R}_B$ has support on S_ρ if B is contained in the facet with normal direction ρ. Indeed $\widetilde{R}_B$ survives precisely in the charts corresponding to cones τ with ρ as one of its generators and in each such chart it has support on the part covering S_ρ.

Now let us fix a set B contained in the facet with normal direction ρ_i, so that (4.5) is nonvanishing, and compute the action of $\widetilde{R}_B$ on the pullback of a test form $\phi = \varphi(z)\ dz$ of bidegree $(n, 0)$. Here dz is just a shorthand notation for $dz_1 \wedge \ldots \wedge dz_n$. Let $\{\chi_\tau\}$ be a partition of unity on $\mathcal{X}$ subordinate the cover $\{\mathcal{U}_\tau\}$. It is not hard to see that we can choose the partition in such a way that the χ_τ are circled, that is they only depend on $|t_1|, \ldots, |t_n|$. Now $\widetilde{R}_B = \sum_\tau \chi_\tau \widetilde{R}_B$. We will start by computing the contribution from our fixed chart $\mathcal{U}_0$ where $\widetilde{R}_B$ is realized by (4.5).

Since R has support at the origin it does only depend on finitely many derivatives of φ and therefore to determine R_B it is enough to consider the case when φ is a polynomial. We can write φ as a finite Taylor expansion,

$$\varphi = \sum_{\alpha,\beta} \frac{\varphi_{\alpha,\beta}(0)}{\alpha!\beta!} z^\alpha \bar{z}^\beta,$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\beta=(\beta_1,\ldots,\beta_n)$

$$\varphi_{\alpha,\beta}=\frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}\frac{\partial^{\beta_1}}{\partial \bar z_1^{\beta_1}}\cdots\frac{\partial^{\beta_n}}{\partial \bar z_n^{\beta_n}}\varphi,$$

and $\alpha!=\alpha_1!\cdots\alpha_n!$, $\beta!=\beta_1!\cdots\beta_n!$ with pullback to $\mathcal{U}_0$ given by

$$\Pi^*\varphi=\sum_{\alpha,\beta}\frac{\varphi_{\alpha,\beta}(0)}{\alpha!\beta!}t^{P\alpha}\bar t^{P\beta}.$$

A computation similar to the one in the proof of Lemma 4.2 yields

$$\Pi^* dz=\det P\ t^{(P-I)\mathbf{1}}\ dt.$$

Hence $\chi_\tau\widetilde{R}_B.\Pi^*\phi$ is equal to

$$K\int\bar\partial\left[\frac{1}{t_i^{n\rho_i\cdot a_0}}\right]\otimes\left[\frac{1}{\prod_{j\neq i}t_j^{n\rho_j\cdot a_0}}\right]\wedge\frac{\bar t_i\bar t^{(C-I)\mathbf{1}}\ \widehat{d\bar t_i}\wedge e_B}{\nu(t)^n}\wedge$$
$$\chi_\tau(t)\sum_{\alpha,\beta}\frac{\varphi_{\alpha,\beta}(0)}{\alpha!\beta!}t^{P\alpha}\bar t^{P\beta}t^{(P-I)\mathbf{1}}dt=K\ \sum_{\alpha,\beta}I_{\alpha,\beta}\ \wedge\frac{\varphi_{\alpha,\beta}(0)}{\alpha!\beta!}\ e_B,$$

where $K=(n-1)!D_1\det P$ and

$$I_{\alpha,\beta}=\int\bar\partial\left[\frac{1}{t_i^{\rho_i\cdot(na_0-\alpha-\mathbf{1})+1}}\right]\otimes[\mu_{\alpha,\beta}]\frac{\chi_\tau(t)\ \bar t_i^{\rho_1\cdot\beta}}{\nu(t)^n}\ \wedge\widehat{d\bar t_i}\wedge dt, \tag{4.6}$$

and where $\mu_{\alpha,\beta}$ is the Laurent monomial in t_j and $\bar t_j$ for $j\neq i$:

$$\mu_{\alpha,\beta}=\prod_{j\neq i}t_j^{\rho_j\cdot(\alpha+\mathbf{1}-na_0)-1}\ \bar t_j^{\rho_j\cdot(\beta+B\mathbf{1}-na_0)-1}.$$

Observe that $\rho_j\cdot(\beta+B\mathbf{1}-na_0)-1\geq 0$ so there are no $\bar t_j$ in the denominator. Recalling (2.3), we evaluate the t_i-integral. Since ν and χ_τ depend on $|t_1|,\ldots,|t_n|$ it follows that $\frac{\partial^\ell}{\partial t_i^\ell}\frac{\chi_\tau}{\nu}|_{t_i=0}=0$ for $\ell\geq 1$ and thus (4.6) is equal to

$$2\pi i\int_{\widehat{t_i}}\frac{\chi_\tau(t)|_{t_i=0}[\mu_{\alpha,\beta}]}{\nu(t)^n|_{t_i=0}}\ \widehat{d\bar t_i}\wedge\widehat{dt_i}, \tag{4.7}$$

if

$$\rho_i\cdot(na_0-\alpha-\mathbf{1})+1=1 \tag{4.8}$$

and

$$\rho_i\cdot\beta=0, \tag{4.9}$$

and zero otherwise. Moreover, for symmetry reasons (4.7) vanishes unless

$$\rho_j\cdot(\alpha+\mathbf{1}-na_0)-1=\rho_j\cdot(\beta+B\mathbf{1}-na_0)-1 \tag{4.10}$$

for $j \neq i$. From the discussion just before Theorem 3.1 we know that the facet containing B is compact, which means that its normal vector has nonzero entries. Thus (4.9) implies that $\beta = (0, \dots, 0)$. Using the fact that $\rho_i \cdot a = \rho_i \cdot a_0$ for all $a \in B$ we can rewrite the left hand side of (4.8) as $\rho_i \cdot (B\mathbf{1} - 1 - \alpha - \mathbf{1}) + 1$ and thus summarize the conditions (4.8) and (4.10) on α as

$$P(\alpha + \mathbf{1}) = PB\mathbf{1}. \tag{4.11}$$

But, since P is invertible there exists exactly one α that fulfills (4.11), namely $\alpha = (B - I)\mathbf{1}$, which is precisely $\alpha^B - \mathbf{1}$. With these values of α and β the Laurent monomial $\mu_{\alpha,\beta}$ is nonsingular and so the integrand of (4.7),

$$2\pi i \int_{\widehat{t_i}} \frac{\chi_\tau(t)|_{t_i=0} \prod_{j\neq i} |t_j|^{2(\rho_j \cdot (B\mathbf{1} - na_0) - 1)}}{\nu(t)^n|_{t_i=0}} \widehat{d\bar{t}_i} \wedge \widehat{dt_i} \tag{4.12}$$

becomes integrable.

To compute $\widetilde{R}_B.\widetilde{\Pi}^*\phi$ we want to add contributions from all charts. However, $\mathcal{U}_0$ covers the support of $\widetilde{R}_B$ except for a set of measure zero, since $\widetilde{R}_B$ has support on S_{ρ_i}, and moreover all integrands that appear are of the form (4.12) and therefore integrable. Thus $\widetilde{R}_B.\widetilde{\Pi}^*\phi$ is equal to

$$\int_{\mathcal{X}} \sum_\tau \widetilde{\Pi}^*(\bar{\partial}|z^A|^{2\lambda} \wedge u_B) \wedge \chi_\tau \widetilde{\Pi}^*\phi \Big|_{\lambda=0} =$$
$$\int_{\mathcal{U}_0} \widetilde{\Pi}^*(\bar{\partial}|z^A|^{2\lambda} \wedge u_B) \wedge \widetilde{\Pi}^*\phi \Big|_{\lambda=0} = C_B \, \frac{\varphi_{\alpha^B - \mathbf{1},0}(0)}{\alpha!\beta!} e_B,$$

where

$$C_B = 2\pi i K \int_{\widehat{t_i}} \frac{\prod_{j\neq i} |t_j|^{2(\rho_j \cdot (B\mathbf{1} - na_0) - 1)}}{(\sum_{a\in A} \prod_{j\neq i} |t_j|^{2\rho_j \cdot (a - a_0)})^n} \widehat{d\bar{t}_i} \wedge \widehat{dt_i}.$$

Hence R_B is of the form (3.1) and the result follows.

5. General monomial ideals

If the zero variety of z^A is of positive dimension the computations of R^{z^A} get more involved. Recall that in general $R^{z^A} = R_p + \dots + R_\mu$, where $p = \operatorname{codim} Y$, $\mu = \min(m, n)$ and $R_k \in \mathcal{D}'_{0,k}(\mathbb{C}^n, \Lambda^k E)$. Parts of the top degree term R_n can be computed by the techniques from the proof of Theorem 3.1. Our method for dealing with the terms of lower degree, though, is to perform the computations outside certain varieties, where some of the coordinates are zero. This amounts to projecting A and brings us back to the more familiar top degree case in a lower dimension. The price we have to pay is that we miss parts of $\mathbb{C}^n$. More precisely, we will compute the current R_k outside the $(k+1)$-dimensional variety

$$V_k := \bigcup_{\mathcal{I}, |\mathcal{I}|=k+1} \bigcap_{i\in\mathcal{I}} H_i,$$

where H_i denotes the hyperplane $\{z_i = 0\}$. However, it turns out that R_k will not carry any essential information on such "small" varieties. To be precise, we have the following lemma, which can be proved analogously to the proof of Lemma 2.2 in [3].

Lemma 5.1. *Let $h_1, \ldots, h_s$ be a tuple of holomorphic functions and let $Y_h = \{h_1 = \ldots h_s = 0\}$. Suppose that $\operatorname{codim} Y_h \cap Y > k$. Then the current $|h|^{2\lambda_0} R_k$, where $|h|^2 = |h_1|^2 + \ldots + |h_s|^2$, has an analytic continuation to $\operatorname{Re} \lambda_0 \geq -\epsilon$ and*

$$|h|^{2\lambda_0} R_k|_{\lambda_0=0} = R_k.$$

It follows, in particular, that to annihilate R_k it suffices to do it outside V_k (or any variety of codimension $k+1$). Indeed $hR_k = 0$ outside V_k implies that $hR_k = 0$.

Before stating our result, a word of notation: For $\mathcal{I} = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, let $T_\mathcal{I}$ be the projection

$$\begin{array}{rcl} T_\mathcal{I} : \mathbb{Z}^n & \to & \mathbb{Z}^k \\ (a_1, \ldots, a_n) & \mapsto & (a_{i_1}, \ldots, a_{i_k}). \end{array}$$

We say that $T_\mathcal{I}(B)$ is essential if $T_\mathcal{I}(B)$ is contained in a facet of $\Gamma^+(T_\mathcal{I}(A))$ and if $T_\mathcal{I}(B)$ spans $\mathbb{R}^{|\mathcal{I}|}$.

Theorem 5.2. *Let z^A, $A \subseteq \mathbb{Z}^n_+$, be a tuple of monomials in $\mathbb{C}^n$, and let*

$$R^{z^A} = \sum_{B\subset A} R_B$$

be the corresponding Bochner-Martinelli residue current. Then outside $V_{|B|}$,

$$R_B = \sum_{\mathcal{I}\subset\{1,\ldots,n\}, |\mathcal{I}|=|B|} R_{B,\mathcal{I}},$$

where the current $R_{B,\mathcal{I}}$ vanishes unless $T_\mathcal{I}(B)$ is essential. Moreover if $T_\mathcal{I}(B)$ is essential and contained in a compact facet of $\Gamma^+(T_\mathcal{I}(A))$, then

$$R_{B,\mathcal{I}} = C_{B,\mathcal{I}}(\eta) \otimes \bigwedge_{i\in\mathcal{I}} \bar\partial\left[\frac{1}{z_i^{\alpha_i^B}}\right] \wedge e_B, \tag{5.1}$$

where η denotes the $z_i, i \notin \mathcal{I}$, and $C_{B,\mathcal{I}}(\eta)$ is a smooth function not identically equal to zero.

Several remarks are in order. First, an immediate consequence is that

$$\operatorname{Ann} R_{B,\mathcal{I}} = (z_i^{\alpha_i^B})_{i\in\mathcal{I}}$$

if $R_{B,\mathcal{I}}$ is of the form (5.1), since annihilating such a current clearly is equivalent to annihilating the $\bigwedge_{i\in\mathcal{I}} \bar\partial\left[\frac{1}{z_i^{\alpha_i^B}}\right]$ part. Moreover the support of (5.1) is the set $\cap_{i\in\mathcal{I}}\{z_i = 0\}$. Note that all the computable $R_{B,\mathcal{I}}$ have different supports.

Remark 2. Observe that adding elements to A that lie in any of the non-compact facets of $\Gamma^+(A)$, not contained in any coordinate plane, gives rise to new essential sets. For example we can add redundant generators to (z^A) and thus in general $\operatorname{Ann} R^{z^A}$ is not independent of the choice of generators as in the case of a discrete zero variety. □

Remark 3. Theorem 3.1 is just a special case of Theorem 5.2. Let us say a word about how to see that the currents of lower degree vanish when Y is the origin. This hypothesis means precisely that A intersects all axes, which in turn implies that the image of A under any projection $T_{\mathcal{I}}$, $|\mathcal{I}| < n$, contains the origin. However, if $0 \in A$, the Newton polyhedron $\Gamma^+(A)$ equals the first orthant and there are no essential sets; note that this corresponds to the case when f contains a nonvanishing function. Thus $R^{z^A} = \sum_{|B|=n} R_B$, where $R_B = R_{B,\{1,\dots,n\}}$ and Theorem 3.1 follows. Of course, by slightly refined arguments one can see how the currents R_k, $k < \operatorname{codim} Y$ vanish in general.

We should also mention that Theorem 5.2 includes Proposition 3.1 in [11] except that the smooth contributions $C_{B,\mathcal{I}}$ are not made explicit. □

Remark 4. By Theorem 5.2 we can extend Theorem 3.2 to hold for a much larger class of ideals. Recall that a crucial point of the proof of Theorem 3.2 was the existence of essential sets. If the Newton diagram of A is of dimension $n-1$, though, we can always find essential sets, for example take the vertices of one of the facets, and the proof applies immediately. In fact, one can show that Theorem 3.2 holds unless $\Gamma(A)$ is not parallel to any of the coordinate planes. Yet, there are ideals for which Theorem 5.2 does not give enough information to decide whether the inclusion (1.1) is strict or not, as we will see in Example 6. Still, in this particular case, one can show by explicit computations that the annihilator ideal is strictly included in the ideal and we believe that Theorem 3.2 holds for monomial ideals in general, although we do not know enough to prove it. □

Let us illustrate Theorem 5.2 with some simple examples.

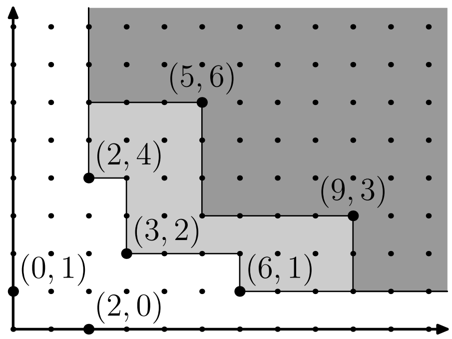

FIGURE 7. The ideals $\operatorname{Ann} R^{z^A}$ (dark gray) and (z^A) (light gray) in Example 5

Example 5. Let $A = \{a^1 = (6,1), a^2 = (3,2), a^3 = (2,4)\}$. There are two essential subsets of A, $\{a^1, a^2\}$ and $\{a^2, a^3\}$, with $\alpha^{12} = (9,3)$ and $\alpha^{23} = (5,6)$, respectively. Moreover, $\Gamma^+(T_{\{1\}}(A))$ is the interval $[2, \infty)$ and consequently $\Gamma(T_{\{1\}}(A)) = \{2\}$. Thus the only set such that its image under $T_{\{1\}}$ is essential is $\{a^3\}$, with $\alpha^3 = a^3$, and according to Theorem 5.2 $\operatorname{Ann} R_{\{a^3\},\{1\}} = (z_1^2)$. Similarly, projecting A on the second axis yields one current, $R_{\{a^1\},\{2\}}$, with annihilator (z_2). Altogether we get

$$\operatorname{Ann} R^{z^A} = (z_1^9, z_2^3) \cap (z_1^5, z_2^6) \cap (z_1^2) \cap (z_2),$$

that is equal to $(z_1^9 z_2, z_1^5 z_2^3, z_1^2 z_2^6)$, see Figure 7. Observe, apropos of Remark 2, that adding a point to A in any of the noncompact facets gives a new essential set and thereby essentially changes R^{z^A}. □

In view of Example 5 it should be clear that Theorem 5.2 actually gives a complete description of $\operatorname{Ann} R^{z^A}$ in case $n = 2$, provided we choose a minimal set of generators (or at least avoid to pick redundant generators from the unbounded facets of $\Gamma^+(A)$).

Example 6. Let I be the ideal (z^A), where $A = \{a^1 = (1,0,1), a^2 = (0,1,1)\} \subset \mathbb{Z}^3$. The codimension of $\{z^a = 0\}$ is 1 and thus I is not a complete intersection (nor can be defined by one). Note that the set A is to small to be essential, whereas the image of A under any projection to $\mathbb{Z}^2$ is, as shown in Figure 8. Still, Theorem 5.2 gives the annihilator ideal only for one of the corresponding currents, namely $\operatorname{Ann} R_{A,\{1,2\}} = (z_1, z_2)$. In both of the other cases the projection of A lies in a noncompact facet of the Newton polyhedron. Furthermore, projecting A to $\mathbb{Z}$ yields the currents $R_{\{a^1\},\{3\}}$ and $R_{\{a^2\},\{3\}}$, both with annihilator (z_3). Observe that the intersection of the computable currents is precisely I. Thus we have found an

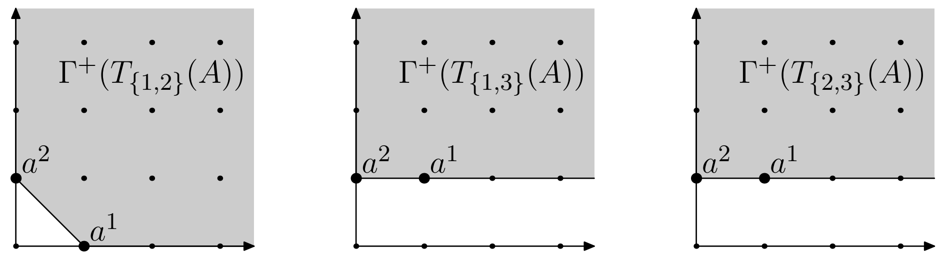

FIGURE 8. The image of A under the various projections to $\mathbb{Z}^2$ in Example 6

example of an non-complete intersection where Theorem 5.2 does not give enough information to decide whether the inclusion (1.1) is strict or not. In this simple example, however, it is easy to compute the remaining parts of R^{z^A} and see that the inclusion is indeed strict. $\square$

Proof of Theorem 5.2. We start by considering the term of top degree,

$$R_n = \sum_{B\subset A, |B|=n} R_B,$$

for which the result follows easily from the proof of Theorem 3.1. To see this, observe first that the proof of Claim 1 does not depend on the codimension of Y. Thus we conclude that $R_B = 0$ unless B is essential.

Next, suppose that B is contained in a compact facet F_B of Γ^+ with normal direction ρ_i. As in the proof of Theorem 3.1 let $\mathcal{U}_0$ be a chart parametrized by Π, determined by the cone τ_0 that has ρ_i as its ith generator. Recall from the proof that the support of $\widetilde{R}_B$ in $\mathcal{U}_0$ is given by $\{t_i = 0\}$. That F_B is compact means precisely that all entries of ρ_i are strictly positive, which implies that $\Pi(\{\rho_i = 0\}) = \{0\}$. Consequently, when computing $\widetilde{R}_B$ in $\mathcal{U}_0$, we only need to consider it acting on test forms $\phi = \varphi\, dz$, where φ is a polynomial. Hence the rest of the proof of Theorem 3.1 applies, and we get that $R_B = R_{B,\{1,\dots,n\}}$ is of the form (3.1) that is equivalent to (5.1) in case $k = n$.

We will compute the terms of lower degree by looking outside certain coordinate planes, which will correspond to projections of A. More precisely, to determine R_k we will look where $n-k$ of the z_i are nonzero. To do this let us fix $\mathcal{I} = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ and let $M_\mathcal{I}$ be the set where z_i is nonvanishing if $i \notin \mathcal{I}$, that is

$$M_\mathcal{I} = (\bigcup_{i\notin\mathcal{I}} H_i)^C.$$

Denote the $z_i, i \in \mathcal{I}$, by ζ and the $z_i, i \notin \mathcal{I}$, by η and write $z^a = \zeta^{a_\zeta}\eta^{a_\eta}$, where a_ζ and a_η are the images of a under $T_\mathcal{I}$ and $T_{\mathcal{I}^C}$, respectively. Let A_ζ and A_η denote the corresponding images of A, and let ϕ be a test form of bidegree (n,k) with (compact) support in $M_\mathcal{I}$. Now R_k acting on ϕ is the analytic continuation to $\lambda = 0$ of

$$\int \bar\partial |z^A|^{2\lambda} \wedge \frac{s \wedge (\bar\partial s)^{k-1}}{|z^A|^{2k}} \wedge \phi(z),$$

that is equal to a sum, taken over B such that $|B| = k$, of terms

$$\int_\eta \int_\zeta \bar\partial |z^A|^{2\lambda} \wedge \frac{s_B \wedge (\bar\partial s_B)^{k-1}}{|z^A|^{2k}} \wedge \varphi(\zeta,\eta)\ d\zeta \wedge d\bar\eta \wedge d\eta. \tag{5.2}$$

It is easily checked that R_k vanishes unless ϕ is of the form $\varphi(\zeta,\eta)\ d\bar\eta \wedge d\eta \wedge d\zeta$. We can now compute the inner integral of (5.2) as in the top degree case (with A^ζ in $\mathbb{C}^k_\zeta$). Indeed, since η is nonvanishing, we can regard z^A as the monomials ζ^{A_ζ} times the parameters η^{A_η}. It follows that, at $\lambda = 0$, (5.2) vanishes unless $T_\mathcal{I}(B)$ is essential, and moreover, if $T_\mathcal{I}(B)$ is contained in a compact facet of $\Gamma(T_\mathcal{I}(A))$, then the inner integral is equal to

$$C_{B,\mathcal{I}}(\eta) \otimes\ \bar\partial\left[\frac{1}{\zeta_1^{\alpha_1^B}}\right] \wedge \ldots \wedge \bar\partial\left[\frac{1}{\zeta_k^{\alpha_k^B}}\right] \wedge e_B \wedge \varphi(\zeta,\eta)\ d\zeta,$$

where $C_{B,\mathcal{I}}$ depends smoothly on η.

In other words, if we let $R_{B,\mathcal{I}}$ be defined by (5.2) (meaning that its action on a test form ϕ is the value of (5.2) at $\lambda = 0$), then $R_{B,\mathcal{I}}$ is of the form (5.1).

When looking in $M_\mathcal{J}$ for each index set $\mathcal{J}$ of cardinality k we miss

$$\Big(\bigcup_{\mathcal{J},|\mathcal{J}|=k} M_\mathcal{J}\Big)^C = \Big(\bigcup_{\mathcal{J},|\mathcal{J}|=k} \big(\bigcup_{i\notin\mathcal{J}} H_i\big)^C\Big)^C =$$

$$\bigcap_{\mathcal{J},|\mathcal{J}|=k}\ \bigcup_{i\notin\mathcal{J}} H_i = \bigcup_{\mathcal{J},|\mathcal{J}|=k+1}\ \bigcap_{i\in\mathcal{J}} H_i,$$

that is precisely V_k. Clearly each current $R_{B,\mathcal{I}}$ extends to $\bigcup_{\mathcal{J},|\mathcal{J}|=k} M_\mathcal{J}$. In fact $R_{B,\mathcal{I}}$ has support only in $M_\mathcal{I}$. Thus outside V_k we have $R_k = \sum R_{B,\mathcal{I}}$, where the $R_{B,\mathcal{I}}$ are of the desired form and we are done. □

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Paper III

NOETHERIAN RESIDUE CURRENTS

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Abstract. Given a coherent ideal sheaf J we construct locally a vector-valued residue current R whose annihilator is precisely the given sheaf. In case J is a complete intersection, R is just the classical Coleff-Herrera product. By means of these currents we can extend various results, previously known for a complete intersection, to general ideal sheaves. We get a residue characterization of the ideal of smooth functions generated by J. If J is a polynomial ideal we get an integral formula that for all polynomials p of a given degree realizes the membership as soon as p belongs to J. By integral formulas we also obtain a residue version of the Ehrenpreis-Palamodov fundamental principle. Analogous results hold true also for a coherent subsheaf of a locally free analytic sheaf.

1. Introduction

Let J be a primary ideal in the local ring $\mathcal{O}_0$ of germs of holomorphic functions at $0 \in \mathbb{C}^n$ and let Z be the associated germ of a variety. There is a finite collection of holomorphic differential operators $\mathcal{L}_1, \dots, \mathcal{L}_\nu$, so-called Noetherian operators, such that a function $\phi \in \mathcal{O}_0$ belongs to J if and only if

$$\mathcal{L}_1\phi = \cdots = \mathcal{L}_\nu\phi = 0 \quad \text{on} \quad Z. \tag{1.1}$$

If J is an arbitrary ideal one obtains a similar description after a primary decomposition $J = \cap_k J_k$. The existence of Noetherian operators (for polynomial ideals) is one of the keystones in the celebrated fundamental principle due to Ehrenpreis and Palamodov, [27] and [39]; for an accessible account of these matters, see [17] and [32]. In one complex variable, a local ideal J is just the set of holomorphic functions that vanish to a given order k at 0, and it is described by the Noetherian operators $\mathcal{L}_j = \partial^j/\partial z^j$, $j = 0, \dots, k-1$. In this case the equalities (1.1) can be collected elegantly in the simple requirement that ϕ annihilates the residue current $R = \bar\partial(1/z^k)$. There is a well-known multivariable generalization of R. Let $h = h_1, \dots, h_m$ be a tuple

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of holomorphic functions at $0 \in \mathbb{C}^n$ such that their common zero set Z has codimension m, and let

$$R^h_{ch} = \bar\partial\frac{1}{h_1}\wedge\ldots\wedge\bar\partial\frac{1}{h_m} \tag{1.2}$$

be the Coleff-Herrera product introduced in [22]. It was proved independently by Dickenstein-Sessa, [25], and Passare, [40], that a holomorphic function ϕ is in the ideal $J(h)$ generated by $h_1, \ldots, h_m$ if and only if the current ϕR^h_{ch} vanishes, i.e., ϕ belongs to the annihilator $\mathrm{ann} R^h_{ch}$. Since R^h_{ch} has support on Z, $\phi \in \mathrm{ann} R^h_{ch}$ means, roughly speaking, that certain derivatives of ϕ vanish on Z. Therefore it seems natural to say that R^h_{ch} is a Noetherian residue current for the ideal $J(h)$. (Our precise definition of Noetherian residue current is given below.)

In the literature this characterization of $J(h)$, i.e., $J(h) = \mathrm{ann} R^h_{ch}$, is often referred to as the duality principle, but we will restrict this term to the (stronger in one direction) fact, also proved in [25] and [40], that $\phi \in J(h)$ if and only if ϕR_{ch} vanishes on all test forms that are $\bar\partial$-closed in a neighborhood of Z.

The Coleff-Herrera product R^h_{ch} has a variety of important applications, and it is therefore desirable to find analogues for more general ideals. Given any ideal J one can find a finite tuple $\gamma = (\gamma_1, \ldots \gamma_\mu)$ of so-called Coleff-Herrera currents such that $I = \mathrm{ann}\gamma = \cap_j \mathrm{ann}\gamma_j$; this is in fact closely related to the existence of Noetherian operators, see [19]. (One should mention that there are constructive ways to find Noetherian operators for a polynomial ideal via a choice of a Gröbner basis, see, [7], [38], [45], and [23].) However, much of the utility of R^h_{ch} depends on the fact that it is quite explicitly constructed from the generators h_j of the ideal $J(h)$ and that it fits into various integral representation formulas, e.g., division-interpolation formulas. Therefore one should look, given a general ideal, for an analogue of the Coleff-Herrera product that also shares these additional properties to some extent. One step in this direction was achieved in [26] where each analytic functional annihilating a Cohen-Macaulay ideal J is represented by a smooth form times a quite explicit Coleff-Herrera product. Explicit (in terms of generators) residue currents of Bochner-Martinelli type related to quite general ideals are also used in [42] and [13]. Inspired by [42], the first author introduced in [1] a vector-valued residue current R^h for an arbitrary tuple h, based on the Koszul complex induced by h, with the property that R^h has support on Z and

$$\overline{J(h)}^{\min(m,n)} \subset \overline{J(h)^{\min(m,n)}} \subset \mathrm{ann} R^h \subset J(h), \tag{1.3}$$

where m is the number of generators and $\bar I$ denote the integral closure of the ideal I. In particular, (1.3) immediately implies the Briançon-Skoda theorem, [21]. In the case of a complete intersection R^h coincides with R^h_{ch} so the rightmost inclusion in (1.3) is an equality, but in general the inclusion is strict; recently the second author has proved, [46], that in case of monomial ideals of dimension zero, the inclusion is always strict unless $J(h)$ is a complete intersection.

Using the Buchsbaum-Rim and the Eagon-Northcott complexes generated by a mapping $h\colon \mathcal{O}_0^{\oplus m} \to \mathcal{O}_0^{\oplus r}$, the construction in [1] was extended, [4] and [5], and in the case when $\operatorname{codim} J(\det h) = m-r+1$, the annihilators of the residue currents so obtained coincide with the module $J(\operatorname{Im} h)$ and the associated determinantal ideal $J(\det h)$, respectively. These constructions are actually global, so if h is globally defined, then the annihilators of the global residue currents coincide with the corresponding coherent sheaves.

The purpose of this paper is to extend these ideas to obtain what we will call a Noetherian residue current for a general coherent ideal sheaf (or coherent subsheaf of $\mathcal{O}^{\oplus r}$), and provide some applications, previously known in the case of a complete intersection. The basic philosophy is that to get all necessary information of an ideal one needs generators not only for the ideal itself but also for all higher syzygies, i.e., a resolution of the ideal, cf., [29]. Our Noetherian residue currents will be constructed from a resolution of the ideal; in simple cases, as a complete intersection, one can easily construct a resolution from a (minimal) set of generators of the ideal.

To begin with we consider an arbitrary complex of Hermitian holomorphic vector bundles over a complex manifold X,

$$0 \to E_N \xrightarrow{f_N} \dots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0, \tag{1.4}$$

that is exact outside an analytic variety Z of positive codimension. To this complex $E_\bullet$ we associate a current $R = R(E_\bullet)$ taking values in $\operatorname{End}(\oplus_k E_k)$ and with support on Z. This current in a certain way measures the lack of exactness of the associated complex of locally free sheaves of $\mathcal{O}$-modules

$$0 \to \mathcal{O}(E_N) \to \cdots \to \mathcal{O}(E_1) \to \mathcal{O}(E_0). \tag{1.5}$$

Let R^ℓ denote the component of R that takes values in $\operatorname{Hom}(E_\ell, \oplus_k E_k)$. Our first main theorem states that (1.5) is exact if and only if $R^\ell = 0$ for $\ell \geq 1$ (Theorem 4.1). If this holds we say that $R = R^0$ is a Noetherian residue for the analytic subsheaf $J = \operatorname{Im}(\mathcal{O}(E_1) \to \mathcal{O}(E_0))$ of $\mathcal{O}(E_0)$ generated by f_1. Our second main result (Theorem 4.3) states that if R is Noetherian, and $\operatorname{ann}(\mathcal{O}(E_0)/J)$ is nonzero, in particular if J is a nonzero ideal sheaf (i.e., $\operatorname{rank} E_0 = 1$), then a holomorphic section ϕ of $\mathcal{O}(E_0)$ is in J if and only if the current $R\phi$ vanishes; this fact thus motivates the notion Noetherian residue current. In case $\operatorname{ann}(\mathcal{O}(E_0)/J) = 0$ a similar characterization holds except that an additional compatibility condition is needed.

If J is any coherent subsheaf of some locally free sheaf $\mathcal{O}(E_0)$, then at least locally $\mathcal{O}(E_0)/J$ admits a resolution (1.5), and if we equip the corresponding complex of vector bundles with any Hermitian metric we thus locally get a Noetherian residue current for J. In case $J \subset \mathcal{O}(E_0)$, $\operatorname{rank} E_0 = 1$, is defined by a complete intersection, the Koszul complex provides a resolution, and our Noetherian residue current so obtained is just the Coleff-Herrera product, see Section 5.

To some extent, the Noetherian residue current depends on the choice of resolution and of the Hermitian metrics chosen on (1.4). However, if $\mathcal{O}(E_0)/J$ is a sheaf of Cohen-Macaulay modules, then it turns out that the associated Noetherian current R is essentially canonical, see Section 6 for precise statements. In the Cohen-Macaulay case we can also define a cohomological residue for J, so that the duality principle extends (Theorem 6.2).

Combined with the framework of integral formulas developed in [6], we present in Section 7 a holomorphic decomposition formula

$$\phi(z) = f_1(z) \int T(\zeta, z)\phi(\zeta) + \int S(\zeta, z)(R\phi)(\zeta), \tag{1.6}$$

for holomorphic sections ϕ of a trivial bundle E_0, where T and S are certain integral kernels. Here we assume that $\operatorname{ann}(\mathcal{O}(E_0)/J)$ is nonzero; for the general case see Section 7. If R is Noetherian, then as soon as $\phi \in J$, (1.6) provides an explicit realization of the membership.

By means of a similar integral formula we also obtain a residue characterization (Theorem 7.2) of the sheaf $\mathcal{E}J$ of $\mathcal{E}$-modules generated by J; this is a generalization of the corresponding result for a complete intersection in [2].

In Section 9 we consider the module J over $\mathbb{C}[z_1, \ldots, z_n]$, generated by an $r_0 \times r_1$-matrix $F(z)$ of polynomials in $\mathbb{C}^n$ of generic rank r_0. We find a global Noetherian residue current R for J in $\mathbb{C}^n$. It is obtained from a resolution of the module over the graded ring $\mathbb{C}[z_0, \ldots, z_n]$ induced by a homogenization of F. For each natural number m we get a polynomial decomposition formula like (1.6), holding for all (r_0-tuples of) polynomials Φ of degree at most m; for Φ in J thus realizing the membership.

Finally we present a residue version of the fundamental principle: If F^T is the transpose of F, then any smooth solution to $F^T(i\partial/\partial t)\xi = 0$ on a smoothly bounded convex set in $\mathbb{R}^n$ can be written

$$\xi(t) = \int_{\mathbb{C}^n} R^T(\zeta)A(\zeta)e^{-i\langle t,\zeta\rangle},$$

for an appropriate (explicitly given matrix of smooth functions) A; here R^T is the transpose of R. Conversely, since R is Noetherian, any $\xi(t)$ given in this way is a homogeneous solution. This follows along the same lines as in [16], where Berndtsson and Passare obtained this result for a complete intersection F by means of the Coleff-Herrera product.

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2. Some preliminaries

Assume that E and Q are holomorphic Hermitian vector bundles over an n-dimensional complex manifold X and let $f\colon E \to Q$ be a holomorphic vector bundle morphism. If we consider f as a section of $E^* \otimes Q$, then for any positive integer q, $F = f_q = f^q/q!$ is a well-defined section of $\Lambda^q(E^* \otimes Q) \simeq \Lambda^q E^* \otimes \Lambda^q Q$, and it is easily seen that F is non-vanishing at a point z if and only if $\operatorname{rank} f(z) = \dim \operatorname{Im} f(z) \geq q$. Let us assume now that $\operatorname{rank} f(z) \leq q$ for all $z \in X$. Then $Z = \{z;\ \operatorname{rank} f(z) < q\}$ is equal to the analytic variety $\{F = 0\}$.

If ξ is a section of a Hermitian bundle, the section η of the dual bundle with minimal norm such that $\eta\xi = |\xi|^2$ will be called the dual section of ξ.

Let s be the section of $E \otimes Q^*$ that is dual to f, and let S be the section of $\Lambda^q E \otimes \Lambda^q Q^*$ that is dual to F. Notice that f induces a natural contraction (interior multiplication) mapping

$$\delta_f\colon \Lambda^{\ell+1} E \otimes \Lambda^{\ell+1} Q^* \to \Lambda^\ell E \otimes \Lambda^\ell Q^*$$

and let $(\delta_f)_\ell = \delta_f^\ell/\ell!$, $s_\ell = s^\ell/\ell!$ etc. Moreover, in $X \setminus Z$, let $\sigma\colon Q \to E$ be the minimal inverse of f, i.e., $f\sigma$ is the identity on $\operatorname{Im} f$, σ vanishes on $(\operatorname{Im} f)^\perp$, and $\operatorname{Im}\sigma$ is orthogonal to $\operatorname{Ker} f$.

Lemma 2.1. *In $X \setminus Z$ we have that*

$$S = s_q \ (= s^q/q!) \tag{2.1}$$

and

$$\sigma = (\delta_f)_{q-1} S/|F|^2. \tag{2.2}$$

Proof. Since the statements are pointwise we may assume that $f\colon E \to Q$ is just a linear mapping between finite-dimensional Hermitian vector spaces. Let ϵ_k be an ON-basis for Q such that $\operatorname{Im} f$ is spanned by $\epsilon_1, \dots, \epsilon_q$. Then $f = \sum_1^q f_k \otimes \epsilon_k$ with $f_k \in E^*$, and it is easy to see that

$$s = \sum_1^q s_k \otimes \epsilon_k^*,$$

where ϵ_k^* is the dual basis and s_k are the duals of f_k. Now $F = f_1 \wedge \dots \wedge f_q \otimes \epsilon_1 \wedge \dots \wedge \epsilon_q$, and since $s_1 \wedge \dots \wedge s_q$ is the dual of $f_1 \wedge \dots \wedge f_q$ it follows that

$s_q = s_1\wedge\ldots\wedge s_q \otimes \epsilon_1^*\wedge\ldots\wedge\epsilon_q^*$ is the dual of F, and thus (2.1) is shown. In particular,

$$|F|^2 = \delta_{f_q}\cdots\delta_{f_1}(s_1\wedge\ldots\wedge s_q), \tag{2.3}$$

where δ_{f_j} is interior multiplication with f_j. To see (2.2), first notice that

$$(\delta_f)_{q-1}S = \sum_{j=1}^{q}(-1)^{j+1}\delta_{f_q}\cdots\delta_{f_{j+1}}\delta_{f_{j-1}}\cdots\delta_{f_1}(s_1\wedge\ldots\wedge s_q)\otimes\epsilon_j^*.$$

Thus $\alpha = (\delta_f)_{q-1}S$, considered as an element in $\mathrm{Hom}\,(Q,E)$, vanishes on $(\mathrm{Im}\, f)^\perp$, and since each s_j is in $(\mathrm{Ker}\, f_j)^\perp \subset (\mathrm{Ker}\, f)^\perp$, α takes values in $(\mathrm{Ker}\, f)^\perp$. Finally, if we compose with f we get, cf., (2.3),

$$f\alpha = \sum_1^q \delta_{f_q}\cdots\delta_{f_1}(s_1\wedge\ldots\wedge s_q)\epsilon_j\otimes\epsilon_j^* = |F|^2\sum_1^q \epsilon_j\otimes\epsilon_j^*,$$

which shows that $\alpha/|F|^2$ is the identity on $\mathrm{Im}\, f$. Altogether this means that $\alpha/|F|^2 = \sigma$ by definition. □

Clearly σ is smooth outside Z. We also have

Proposition 2.2. *If $F = F^0F'$ in X, where F^0 is a holomorphic function and F' is non-vanishing, then $F^0\sigma$ is smooth across Z.*

Proof. Since $F = F^0F'$ we have that $S = \overline{F^0}S'$, where S' is the dual of F', and $|F|^2 = |F^0|^2|F'|^2$, where $|F'|^2$ is smooth and non-vanishing. Thus by Lemma 2.1,

$$F^0\sigma = F^0(\delta_f)_{q-1}S/|F|^2 = (\delta_f)_{q-1}S'/|F'|^2,$$

which is smooth across Z. □

Throughout this paper, $\mathcal{E}(X)$ denotes the space of smooth functions, $\mathcal{E}_\bullet(X)$ the space of smooth differential forms, $\mathcal{D}_\bullet$ the space of test forms, $\mathcal{D}'_\bullet(X)$ denotes the space of currents on X, and $\mathcal{E}(X,E)$ the space of smooth sections of E over X. Furthermore, $\mathcal{O}(E)$ denotes the analytic sheaf of holomorphic sections of E, and $\mathcal{E}(E)$ denotes the sheaf of smooth sections.

We will frequently use some basic facts of analytic sheaves, see, e.g., [30]. Let $\mathcal{F}$ be an analytic sheaf in X. Recall that $\mathcal{F}$ is coherent if it locally admits a presentation

$$\mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0) \to \mathcal{F} \to 0,$$

where E_1 and E_0 are holomorphic vector bundles. Given such a presentation in $X' \subset X$, the open subset of X' where $\mathcal{F}$ is locally free, i.e., a vector bundle, coincides with the open set where f_1, considered as a vector bundle morphism, has locally constant (i.e., optimal) rank; the complement of this set, cf., above, is an analytic variety Z with positive codimension.

The mapping f_1 is pointwise surjective, i.e., it has full rank, outside Z if and only if the the annihilator ideal $\mathrm{ann}\mathcal{F}_x$ in the local ring $\mathcal{O}_x$ is nonzero for each $x \in Z$. In this case, the set Z coincides with the zero locus of the ideal sheaf $\mathrm{ann}\mathcal{F}$.

3. Residue currents of generically exact complexes

Let

$$0 \to E_N \xrightarrow{f_N} E_{N-1} \xrightarrow{f_{N-1}} \dots \xrightarrow{f_{-M+2}} E_{-M+1} \xrightarrow{f_{-M+1}} E_{-M} \to 0 \tag{3.1}$$

be a holomorphic complex of Hermitian vector bundles over the n-dimensional complex manifold X, and assume that it is generically exact, i.e., pointwise exact outside an analytic set Z of positive codimension. Then for each k, $\mathrm{rank}\, f_k$ is constant in $X \setminus Z$ and equal to

$$\rho_k = \dim E_k - \dim E_{k+1} + \cdots \pm \dim E_N. \tag{3.2}$$

Since $z \mapsto \mathrm{rank}\, f_k(z)$ is lower semicontinuous it follows that $\mathrm{rank}\, f_k(z) \leq \rho_k$ everywhere in X.

We are now going to define a residue current associated to (3.1), and to this end we will need some algebraic formalism. The bundle $E = \oplus E_k$ has a natural superbundle structure, i.e., a $\mathbb{Z}_2$-grading, $E = E^+ \oplus E^-$, E^+ and E^- being the subspaces of even and odd elements, respectively, by letting $E^+ = \oplus_{2k} E_k$ and $E^- = \oplus_{2k+1} E_k$. The space of E-valued currents

$$\mathcal{D}'_\bullet(X, E) = \mathcal{D}'_\bullet(X) \otimes_{\mathcal{E}(X)} \mathcal{E}(X, E)$$

is a left $\mathcal{E}_\bullet(X)$-module, and it gets a natural grading by combining the gradings of $\mathcal{D}_\bullet(X)$ and $\mathcal{E}(X, E)$. We make $\mathcal{D}'_\bullet(X, E)$ into a right $\mathcal{E}_\bullet(X)$-module by letting $\xi\phi = (-1)^{\deg \xi \deg \phi} \phi\xi$ for sections ξ of $\mathcal{D}'_\bullet(X, E)$ and smooth forms ϕ.

The $\mathbb{Z}_2$-grading on E induces a $\mathbb{Z}_2$-grading $\mathrm{End}E = (\mathrm{End}E)^+ \oplus (\mathrm{End}E)^-$, where $(\mathrm{End}E)^-$ consists of odd mappings, i.e., mappings which, like $f = \sum f_j$, map $E^\pm \to E^\mp$, and $(\mathrm{End}E)^+$ is the subspace of even mappings. We get a $\mathbb{Z}_2$-grading of $\mathcal{D}'_\bullet(X, \mathrm{End}E)$ as well, and $\bar\partial$ extends to an odd mapping by the formula $\bar\partial A = \bar\partial \circ A - (-1)^{\deg A} A \circ \bar\partial$ for $A \in \mathcal{D}'_\bullet(X, \mathrm{End}E)$, i.e., so that $\bar\partial(A\xi) = (\bar\partial A)\xi + (-1)^{\deg A} A(\bar\partial \xi)$ for a section ξ of E. Since f is holomorphic, $\bar\partial f = 0$.

We now introduce the mapping $\nabla = f - \bar\partial$ on $\mathcal{D}'_\bullet(X, E)$. Actually, it is (minus) the $(0,1)$-part of the superconnection $D - f$ introduced by Quillen, [43], where D is the Chern connection on E. It is easy to see that ∇ is an odd mapping and that $\nabla^2 = 0$. Moreover, it extends to an odd mapping ∇_{End} on $\mathcal{D}'_\bullet(X, \mathrm{End}E)$ so that

$$\nabla(g\xi) = (\nabla_{\mathrm{End}} g)\xi + (-1)^{\deg g} g(\nabla\xi) \tag{3.3}$$

for sections $g \in \mathcal{D}'_\bullet(X, \mathrm{End}E)$ and $\xi \in \mathcal{E}_\bullet(X, E)$, and $\nabla_{\mathrm{End}}(gh) = (\nabla_{\mathrm{End}} g)h + (-1)^{\deg g} g(\nabla_{\mathrm{End}} h)$ for sections $g, h \in \mathcal{E}_\bullet(X, \mathrm{End}E)$. Moreover, $\nabla^2_{\mathrm{End}} = 0$.

In $X \setminus Z$ we have the minimal inverses $\sigma_k \colon E_{k-1} \to E_k$ of f_k, cf., Section 2, and we let $\sigma = \sigma_{-M+1} + \cdots + \sigma_N \colon E \to E$. If I denotes the identity endomorphism on E, then

$$f\sigma + \sigma f = I. \tag{3.4}$$

Moreover, it is easily checked that $\sigma\sigma = 0$, and thus we get

$$\sigma(\bar\partial\sigma) = (\bar\partial\sigma)\sigma. \tag{3.5}$$

In view of (3.3),

$$\nabla_{\mathrm{End}}\sigma = \nabla \circ \sigma + \sigma \circ \nabla = f\sigma + \sigma f - (\bar\partial \circ \sigma + \sigma \circ \bar\partial),$$

so we get

$$\nabla_{\mathrm{End}}\sigma = I - \bar\partial\sigma. \tag{3.6}$$

Notice that $\bar\partial\sigma$ has even degree. In $X \setminus Z$ we define the End$E$-valued form, cf., (3.6),

$$u = \sigma(\nabla_{\mathrm{End}}\sigma)^{-1} = \sigma(I - \bar\partial\sigma)^{-1} = \sigma + \sigma(\bar\partial\sigma) + \sigma(\bar\partial\sigma)^2 + \dots. \tag{3.7}$$

Now,

$$\nabla_{\mathrm{End}} u = \nabla_{\mathrm{End}}\sigma(\nabla_{\mathrm{End}}\sigma)^{-1} - \sigma\nabla_{\mathrm{End}}(\nabla_{\mathrm{End}}\sigma)^{-1},$$

and since $\nabla^2_{\mathrm{End}} = 0$ we thus have

$$\nabla_{\mathrm{End}} u = I. \tag{3.8}$$

Notice that

$$u = \sum_\ell \sum_{k \ge \ell+1} u^\ell_k$$

where

$$u^\ell_k = \sigma_k(\bar\partial\sigma_{k-1}) \cdots (\bar\partial\sigma_{\ell+1}) \tag{3.9}$$

is in $\mathcal{E}_{0,k-\ell-1}(X \setminus Z, \mathrm{Hom}\,(E_\ell, E_k))$. In view of (3.5) we also have

$$u^\ell_k = (\bar\partial\sigma_k)(\bar\partial\sigma_{k-1}) \cdots (\bar\partial\sigma_{\ell+2})\sigma_{\ell+1}. \tag{3.10}$$

Let

$$u^\ell = \sum_{k \ge \ell+1} u^\ell_k,$$

i.e., u^ℓ is u composed with the projection $E \to E_\ell$. Following [42] and [1] we can make a current extension of u across Z.

Proposition 3.1. *Let F be any holomorphic function (or tuple of holomorphic functions) that vanishes on Z. Then $\lambda \mapsto |F|^{2\lambda}u$, a priori defined for* $\operatorname{Re}\lambda >> 0$*, has a continuation as a current-valued analytic function to* $\operatorname{Re}\lambda > -\epsilon$*. Moreover,*

$$U := |F|^{2\lambda}u|_{\lambda=0}$$

is a current extension of u across Z that is independent of the choice of F.

Proof. The proof is very similar to the proof of Theorem 1.1 in [1] so we only provide an outline. For each j, following Section 2, we have a section F_j of $\Lambda^{\rho_j}E_j^* \otimes \Lambda^{\rho_j}E_{j-1}$, and its dual S_j such that $\sigma_j = (\delta_{f_j})_{\rho_j-1}S_j/|F_j|^2$. After a sequence of suitable resolutions of singularities we may assume that, for all j, $F_j = F_j^0 F_j'$, where F_j^0 is a monomial and F_j' is non-vanishing, and that also F is a monomial F^0 times a non-vanishing factor. By Proposition 2.2 therefore $\sigma_j = \alpha_j/F_j^0$, where α_j is smooth across Z. Since $\alpha_{j+1}\alpha_j = 0$ outside the set $\{F_{j+1}^0 F_j^0 = 0\}$, thus $\alpha_{j+1}\alpha_j = 0$ everywhere. Therefore, cf., (3.10), it is easy to see that

$$u_{\ell+k}^{\ell} = \frac{(\bar\partial\alpha_{\ell+k})(\bar\partial\alpha_{\ell+k-1})\cdots(\bar\partial\alpha_{\ell+2})\alpha_{\ell+1}}{F_{\ell+k}^0\cdots F_{\ell+1}^0}.$$

Since F_j only vanish on Z and F vanishes there, F^0 must contain each coordinate factor that occurs in any F_j^0. Therefore, cf., e.g., [1], the proposed analytic continuation exists and the value at $\lambda = 0$ is the natural principal value current extension. □

In the same way we can now define the residue current $R = R(E_\bullet)$ associated to (3.1) as

$$R = \bar\partial|F|^{2\lambda}\wedge u|_{\lambda=0}.$$

It clearly has its support on Z. If $R_k^\ell = \bar\partial|F|^{2\lambda}\wedge u_k^\ell|_{\lambda=0}$ and R^ℓ is defined analogously, then

$$R = \sum_\ell R^\ell = \sum_\ell \sum_{k\geq\ell+1} R_k^\ell.$$

Notice that R_k^ℓ is a $\operatorname{Hom}(E_\ell, E_k)$-valued $(0, k-\ell)$-current. The currents U^ℓ and U_k^ℓ are defined analogously. Notice that U has odd degree and R has even degree.

Proposition 3.2. *Let U and R be the currents associated to the complex* (3.1)*. Then*

$$\nabla_{\mathrm{End}}U = I - R, \qquad \nabla_{\mathrm{End}}R = 0. \tag{3.11}$$

Moreover, R_k^ℓ vanishes if $k-\ell <$ $\operatorname{codim} Z$, and $\bar\xi R = 0$ if ξ is holomorphic and vanishes on Z.

We can also write (3.11) as

$$\nabla \circ U + U \circ \nabla = I - R, \qquad \nabla \circ R = R \circ \nabla.$$

Proof. In fact, if $\operatorname{Re}\lambda$ is large,

$$\nabla_{\mathrm{End}}\big(|F|^{2\lambda}u\big) = |F|^{2\lambda}\nabla_{\mathrm{End}}u - \bar\partial|F|^{2\lambda}\wedge u = |F|^{2\lambda}I - \bar\partial|F|^{2\lambda}\wedge u.$$

By the uniqueness of analytic continuations this equality must hold for $\operatorname{Re}\lambda > -\epsilon$, and the first statement in (3.11) now follows by taking $\lambda = 0$. The second statement follows immediately since $\nabla^2_{\mathrm{End}} = 0$. The vanishing of R^ℓ_k for $k-\ell < \operatorname{codim} Z$ follows from the principle that a residue current of bidegree $(0,q)$ cannot have its support contained in a variety of codimension higher than q. For a precise argument for this fact, as well as for the last statement of the proposition, see, e.g., the proof of Theorem 1.1 in [42] or Theorems 1.1 and 1.2 in [1]. □

The following theorem suggests that the residue current $R = R(E_\bullet)$ measures to what extent the associated complex of sheaves of holomorphic sections of $E_\bullet$ is not exact. Notice that if ϕ is a holomorphic section of E_ℓ, then $R^\ell\phi$ is a E-valued current.

Theorem 3.3. *Let* (3.1) *be a generically exact holomorphic complex of Hermitian vector bundles, let $R = R(E_\bullet)$ be the associated residue current, and let ϕ be a holomorphic section of E_ℓ.*
(i) If $f_\ell\phi = 0$ and $R^\ell\phi = 0$, then locally there is a holomorphic section ψ of $E_{\ell+1}$ such that $f_{\ell+1}\psi = \phi$.
(ii) If moreover $R^{\ell+1} = 0$, then the existence of such a local solution ψ implies that $R^\ell\phi = 0$.

Proof. Let U be the associated current such that (3.11) holds. Then $\nabla(U\phi) = \phi - U(\nabla\phi) - R\phi$. Since $U\phi = U^\ell\phi$, $R\phi = R^\ell\phi$, and $\nabla\phi = f_\ell\phi - \bar\partial\phi$, it follows from the assumptions of ϕ that $\nabla(U^\ell\phi) = \phi$. Thus we have a current solution $v = U^\ell\phi$ to

$$f_{\ell+1}v_{\ell+1} = \phi, \quad f_{\ell+k+1}v_{\ell+k+1} = \bar\partial v_{\ell+k}, \ k > 1,$$

where $v_{\ell+k} \in D'_{0,k-1}(X, E_{k+\ell})$. By solving the sequence of $\bar\partial$-equations

$$\bar\partial w_{\ell+k} = v_{\ell+k} + f_{\ell+k+1}w_{\ell+k+1}$$

locally, we end up with the desired holomorphic solution $\psi = v_{\ell+1} + f_{\ell+2}w_{\ell+2}$, cf., [1]. For the second part, assume that $f_{\ell+1}\psi = \phi$. Then by (3.11),

$$R^\ell\phi = R\phi = R(\nabla\psi) = \nabla(R\psi) = \nabla(R^{\ell+1}\psi) = 0.$$

□

Now assume that

$$0 \to E_N \xrightarrow{f_N} \dots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \tag{3.12}$$

is a generically exact holomorphic complex of Hermitian bundles. Since $\operatorname{rank} f_1$ is generically constant, we can define σ_1 in an unambiguous way in $X \setminus Z$, and therefore the currents R^ℓ for $\ell \geq 0$ can be defined as above, and we have:

Corollary 3.4. *If* (3.12) *is a generically exact complex of Hermitian vector bundles, then Theorem 3.3 still holds (for $\ell \geq 0$), provided that $f_0\phi = 0$ is interpreted as ϕ belonging generically (outside Z) to the image of f_1.*

If f_1 is generically surjective, in particular if $\operatorname{rank} E_0 = 1$ and f_1 is not identically 0, then this latter condition is of course automatically fulfilled.

Proof. The corollary actually follows just from a careful inspection of the arguments in the proof of Theorem 3.3. However, a possibly more satisfactory way to derive the corollary is to extend (3.12) to a generically exact complex (3.1) and then refer directly to Theorem 3.3, still noting that the definition of R^ℓ for $\ell \geq 0$ as well as the condition $f_0\phi = 0$ are independent of such an extension.

The complex (3.12) can be extended in the following way. By assumtion the dual mapping $f_1^*\colon E_0^* \to E_1^*$ induces a sheaf mapping $\mathcal{O}(E_0^*) \to \mathcal{O}(E_1^*)$ which (at least locally) can be extended to an exact complex (a resolution)

$$0 \to \mathcal{O}(E_{-M}^*) \to \dots \to \mathcal{O}(E_0^*) \to \mathcal{O}(E_1^*).$$

In particular the corresponding complex of vector bundles is generically exact, and taking duals and combining with (3.12) we get a generically exact extension. □

4. Construction of Noetherian residue currents

We will now discuss how one can find a current whose annihilator coincides with a given ideal sheaf (or subsheaf of $\mathcal{O}^{\oplus r}$). Notice that the complex (3.12) corresponds to a complex of locally free analytic sheaves

$$0 \to \mathcal{O}(E_N) \to \cdots \to \mathcal{O}(E_1) \to \mathcal{O}(E_0), \tag{4.1}$$

that is exact outside Z; conversely, any such sequence of locally free sheaves that is exact outside some analytic set gives rise to a generically exact complex (3.12) of vector bundles. Our basic result is the following characterization of exactness of (4.1).

Theorem 4.1. *Assume that* (3.12) *is generically exact, let R be the associated residue current, and let* (4.1) *be the associated complex of sheaves. Then $R^\ell = 0$ for all $\ell \geq 1$ if and only if* (4.1) *is exact.*

For the proof we will use the following characterization of exactness due to Buchsbaum-Eisenbud, see [28] Theorem 20.9: The complex (4.1) is exact if and only if

$$\operatorname{codim} Z_j \geq j \tag{4.2}$$

for all j, where, cf., (3.2),

$$Z_j = \{z;\ \operatorname{rank} f_j < \rho_j\}.$$

Remark 1. To be precise we will only use the "only if"-direction. The other direction is actually a consequence of Corollary 3.4 and (the proof of) Theorem 4.1. □

Proof. From Corollary 3.4 it follows that (4.1) is exact if $R^\ell = 0$ for $\ell \geq 1$. For the converse, let us now assume that (4.1) is exact; by the Buchsbaum-Eisenbud theorem then (4.2) holds. We will prove that $R^1 = 0$; the case when $\ell > 1$ is handled in the same way. The intuitive idea in the proof is based on the somewhat vague principle that a residue current of bidegree $(0,q)$ cannot be supported on a variety of codimension $q+1$. Taking this for granted, we notice to begin with that $R^1_2 = \bar\partial|F|^{2\lambda}\wedge\sigma_2|_{\lambda=0}$ is a $(0,1)$-current and has its support on Z_2, which has codimension at least 2. Hence R^1_2 must vanish according to the vague principle. Now, σ_3 is smooth outside Z_3, and hence $R^1_3 = \bar\partial\sigma_3\wedge R^1_2 = 0$ outside Z_3; thus R^1_3 is supported on Z_3 and again, by the same principle, R^1_3 must vanish etc. To make this into a strict argument we will use the following simple lemma.

Lemma 4.2. *Suppose that $\gamma(s,\tau)$ is smooth in $\mathbb{C}\times\mathbb{C}^r$ and that moreover $\gamma(s,\tau)/\bar s$ is smooth where $\tau_1\cdots\tau_k \neq 0$. Then $\gamma(s,\tau)/\bar s$ is smooth everywhere.*

Proof. The assumption means that $\gamma(s,\tau) = \bar s\omega(s,\tau)$ where $\tau_1\cdots\tau_r \neq 0$ and ω is smooth outside $\tau_1\cdots\tau_r = 0$. It follows that, for each ℓ, $(\partial^\ell/\partial s^\ell)\gamma(0,\tau) = 0$ where $\tau_1\cdots\tau_r \neq 0$, and hence by continuity it holds also when $\tau_1\cdots\tau_r = 0$. It now follows from a Taylor expansion in s that $\gamma(s,\tau)/\bar s$ is smooth. □

After a sequence of resolutions of singularities the action of R^1_k on a test form ξ is a finite sum of integrals of the form

$$\int \bar\partial|F^0|^{2\lambda}\wedge\frac{(\bar\partial\alpha_k)(\bar\partial\alpha_{k-1})\cdots(\bar\partial\alpha_3)\alpha_2}{F^0_kF^0_{k-1}\cdots F^0_3F^0_2}\wedge\tilde\xi\Big|_{\lambda=0}$$

where F^0, F^0_i and α_i are as in the proof of Proposition 3.1 and where $\tilde\xi$ is the pullback of ξ. To be precise, there are also cutoff functions involved that we suppress for simplicity. Observe that $\bar\partial|F^0|^{2\lambda}$ is a finite sum of terms like $a\lambda|F^0|^{2\lambda}d\bar s/\bar s$, where a is a positive integer and s is just one of the coordinate functions that divide F^0. We need to show that all the corresponding integrals vanish when $\lambda = 0$, and to this end it is enough to show, see, e.g., Lemma 2.1 in [1], that

$$\eta = \frac{d\bar{s}}{\bar{s}}\wedge(\bar\partial\alpha_k)(\bar\partial\alpha_{k-1})\cdots(\bar\partial\alpha_3)\alpha_2\wedge\tilde{\xi}$$

is smooth ($(d\bar{s}/\bar{s})\wedge\beta$ being smooth for a smooth β, means that each term of β contains a factor $\bar{s}$ or $d\bar{s}$).

Let ℓ be the largest index among $2, \dots, k$ such that s is a factor in F^0_ℓ (possibly there is no such index at all; then ℓ below is to be interpreted as 1) and let $\tau_1, \dots, \tau_r$ denote the coordinates that divide $F^0_k \cdots F^0_{\ell+1}$. We claim that, outside $\tau_1 \cdots \tau_r = 0$, the form

$$\frac{d\bar{s}}{\bar{s}}\wedge\frac{(\bar\partial\alpha_k)\cdots(\bar\partial\alpha_{\ell+1})}{F^0_k\cdots F^0_{\ell+1}}\wedge\tilde{\xi}$$

is smooth. This follows by standard arguments, see, e.g., the proof of Lemma 2.2 in [42] or the proof of Theorem 1.1 in [1]; in fact, outside $Z_k \cap \ldots \cap Z_{\ell+1}$ the $(n, n-\ell+1)$-form $(\bar\partial\sigma_k)\ldots(\bar\partial\sigma_{\ell+1})\wedge\xi$ is smooth and it must vanish on Z_ℓ for degree reasons, since Z_ℓ has codimension at least ℓ. Thus the form

$$\tilde\eta = \frac{d\bar{s}}{\bar{s}}\wedge(\bar\partial\alpha_k)\cdots(\bar\partial\alpha_{\ell+1})\wedge\tilde{\xi}$$

is smooth outside $\tau_1 \cdots \tau_r = 0$. By Lemma 4.2, applied to

$$\gamma = d\bar{s}\wedge(\bar\partial\alpha_k)\cdots(\bar\partial\alpha_{\ell+1})\wedge\tilde{\xi},$$

$\tilde\eta$ is smooth everywhere, and therefore η is smooth. □

Definition 1. A current $R = R(E_\bullet)$ associated to a holomorphic complex of Hermitian vector bundles (3.12) such that the corresponding sheaf complex (4.1) is exact, or equivalently, $R^\ell = 0$ for all $\ell \geq 1$, will be called a Noetherian residue current.

If $R = R(E_\bullet)$ is Noetherian, with no ambiguity, we write R_k rather than R^0_k. The definition is motivated by the following result and its corollaries. Recall that if $\mathcal{F}$ is a coherent sheaf, then (4.1) is a (locally free) resolution of $\mathcal{F}$ if (4.1) is exact and $\mathcal{F} = \mathcal{O}(E_0)/\mathrm{Im}\,(\mathcal{O}(E_1) \to \mathcal{O}(E_0))$.

Theorem 4.3. *Let* (4.1) *be a locally free resolution of the coherent analytic sheaf* $\mathcal{F}$*, and assume that* E_k *are equipped with some Hermitian metrics. Then the associated Noetherian residue current* R *has support on the analytic set* Z *where* $\mathcal{F}$ *is not locally free. Furthermore, a holomorphic section* ϕ *of* $\mathcal{O}(E_0)$ *is mapped to zero in* $\mathcal{F}$ *if and only if* ϕ *is mapped to zero in* $\mathcal{F}$ *outside* Z *and the current* $R\phi$ *vanishes.*

Proof. A free resolution of a locally free sheaf is pointwise exact. Therefore u^0 is smooth outside Z and thus the support of R must be contained in Z. Since $R^1 = 0$ the second assertion follows from Corollary 3.4. $\square$

Notice that if

$$J = \mathrm{Im}\,(\mathcal{O}(E_1) \to \mathcal{O}(E_0)),$$

then $\mathcal{F} = \mathcal{O}(E_0)/J$, so the theorem can be rephrased as: A holomorphic section ϕ of $\mathcal{O}(E_0)$ is in J if and only if ϕ is in J outside Z and $R\phi = 0$. This fact motivates the notion of Noetherian current for J.

Recall that the condition $\phi = 0$ in $\mathcal{F}$ outside Z can be expressed as $f_0\phi = 0$ for an appropriately chosen mapping f_0. Of course, the condition is automatically fulfilled if $\mathcal{F} = 0$ outside Z:

Corollary 4.4. *Assume that* $\mathrm{rank}\, E_0 = 1$ *and that the ideal sheaf* $J \subset \mathcal{O}(E_0)$ *is nonzero. Then* R *has support on the zero locus of* J *and a holomorphic section* ϕ *of* E_0 *is in* J *if and only if* $R\phi = 0$.

More generally, assume that $\mathrm{rank}\, E_0 \geq 1$ *and* $\mathrm{ann}(\mathcal{O}(E_0)/J)$ *is nonzero. Then* R *has support on the zero locus of* $\mathrm{ann}(\mathcal{O}(E_0)/J)$ *and a section* ϕ *of* E_0 *is in* J *if and only if* $R\phi = 0$.

Remark 2. Let J be any ideal sheaf with zero locus Z. It could have been natural to allow a wider definition and say that any (vector-valued) current T with support on Z such that $\mathrm{ann} T = J$ is a Noetherian residue current for J. For example, one could take an appropriate tuple of Coleff-Herrera currents, cf., the introduction. Here is another example: If we take a resolution (4.1) and extend it on the left with a non-exact sequence, then R^ℓ can be non-vanishing for large ℓ but since $R^1 = 0$, the current $T = R^0$ will be Noetherian in this wider sense. We do not know if R^0 being Noetherian in the wider sense, implies that R^ℓ vanish for "small" ℓ; even not for $\ell = 1$. However, for simplicity we keep the more restrictive notion of Noetherian residue current; it will cover all currents of this type that we consider in this paper. $\square$

Given any coherent sheaf $\mathcal{F}$ in a Stein manifold X and compact subset $K \subset X$, one can always find a resolution

$$\cdots \to \mathcal{O}^{\oplus r_2} \to \mathcal{O}^{\oplus r_1} \to \mathcal{O}^{\oplus r_0} \tag{4.3}$$

of $\mathcal{F}$ in a neighborhood of K, e.g., by iterated use of Theorem 7.2.1 in [32]. The key stone in the proof of Theorem 4.1, the Buchsbaum-Eisenbud theorem, in general requires that the resolution (4.3) starts with 0 somewhere on the left. However, by the Syzygy theorem and Oka's lemma, $\mathrm{Ker}\,(\mathcal{O}^{\oplus r_\ell} \to \mathcal{O}^{\oplus r_{\ell-1}})$ is (locally) free for large ℓ, so we can replace such a module $\mathcal{O}^{\oplus r_\ell}$ with this kernel and 0 before that. Therefore Theorem 4.1 holds and we have

Proposition 4.5. *Let J be a coherent subsheaf of $\mathcal{O}^{\oplus r_0}$ in a Stein manifold X. For each compact subset $K \subset X$ there is a Noetherian residue current R for J defined in a neighborhood of K.*

Notice that in this case $R = (R_k)$, where R_k is an $r_k \times r_0$-matrix of scalar-valued residue currents. If ϕ is an r_0-column of functions in $\mathcal{O}(K)$ then $R_k\phi$ is an r_k-column of currents in a neighborhood of K. We can also choose a matrix f_0 such that ϕ is generically in the image of f_1 if and only if $f_0\phi = 0$ and we have, cf., the proofs of Theorem 3.3 and Corollary 3.4:

The column $\phi \in \mathcal{O}(K)^{r_0}$ of holomorphic functions is in the image of $\mathcal{O}(K)^{r_1} \to \mathcal{O}(K)^{r_0}$ if and only if $f_0\phi = 0$ and all the residue currents $R_k\phi$ vanish.

The degree of explicitness of such a Noetherian residue current R is of course directly depending on the degree of explicitness of a resolution of the sheaf $\mathcal{F}$.

5. Examples

We will now consider some explicit examples of the residue currents defined above.

Example 1 (The Koszul complex). Let H be a Hermitian bundle over X of rank m and let h be a non-trivial holomorphic section of the dual bundle H^*. It can be considered as a morphism $H \to \mathbb{C} \times X$, and if we let δ denote contraction (interior multiplication) with h we have the Koszul complex

$$0 \to \Lambda^m H \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Lambda^2 H \xrightarrow{\delta} H \xrightarrow{\delta} \mathbb{C} \times X,$$

which is exact where h is non-vanishing. Notice that in this case the super-bundle structure on the total bundle $E = \oplus \Lambda^k H = \Lambda H$ is obtained from the natural grading on ΛH. Moreover, the desired $\mathcal{E}_\bullet(X)$-module structure of $\mathcal{D}'_\bullet(X, E)$ is obtained from the wedge product in $\Lambda(H \oplus T^*(X))$. Of course we assume that E has the Hermitian structure induced by H. Let ξ be the section of H over $X \setminus Z$ with minimal norm such that $h \cdot \xi = 1$. It is easy to verify that $\sigma_k \eta = \xi \wedge \eta$ for sections η of $\Lambda^k H$. Therefore,

$$u_k^\ell = \xi \wedge (\bar\partial \xi)^{k-\ell-1},$$

acting on $\Lambda^\ell H$ via wedge multiplication, and hence

$$R_k^\ell = \bar\partial |h|^{2\lambda} \wedge \xi \wedge (\bar\partial \xi)^{k-\ell-1}|_{\lambda=0}.$$

This is precisely the current(s) considered in [1].

The associated complex of sheaves is exact if (and only if) $\operatorname{codim} Z = m$; this is very well-known, and follows, e.g., from the Buchsbaum-Eisenbud theorem, cf., also Remark 1. In this case R is a Noetherian residue current for the ideal sheaf $J = \text{Im}\,(\mathcal{O}(H) \to \mathcal{O})$. Since $\Lambda^m H$ has rank 1, $R = R^0_m$ has just one entry. If $h = h_1 e_1^* + \cdots + h_m e_m^*$ in some local holomorphic frame e_j^* for H^*, then R is precisely the Coleff-Herrera product (1.2) times $e_1 \wedge \ldots \wedge e_m$, where e_j is the dual frame, see [1]. $\square$

We now consider some generalizations of the Koszul complex, but for simplicity we only discuss the most interesting component R^0 of the associated residue current.

Lemma 5.1. *If the mapping f_1 in* (3.12) *is generically surjective, then u^0_k are the unique forms in $X \setminus Z$ with values in* $\text{Hom}\,(E_0, (\text{Ker}\, f_k)^\perp)$ *such that*

$$f_1 u^0_1 = I_{E_0}, \quad f_{k+1} u^0_{k+1} = \bar\partial u^0_k, \quad k \geq 1. \tag{5.1}$$

Proof. Since $u^0_{k+1} = \sigma_{k+1}(\bar\partial u^0_k)$, cf., (3.9), it is clear that u^0_{k+1} takes values in $\text{Hom}\,(E_0, (\text{Ker}\, f_k)^\perp)$. Since f_1 is generically surjective we can extend (3.12) with $f_0 = 0$ on the right. From (3.8) we get, restricting to the action on E_0, that

$$f u^0 - \bar\partial u^0 + u^0 f_0 = I_{E_0},$$

which is precisely (5.1) since $f_0 = 0$. $\square$

Example 2 (The Eagon-Northcott and Buchsbaum-Rim complexes). Suppose that H and Q are Hermitian bundles of ranks m and r respectively, and $h\colon H \to Q$ is a generically surjective holomorphic morphism. We then have a natural morphism

$$\det h\colon \Lambda^r H \to \det Q,$$

cf., Section 2. Let $S^\ell Q^*$ denote the bundle of symmetric tensors of degree ℓ. If $E_k = \Lambda^{r+k-1} H \otimes S^{k-1} Q^*$ for $k \geq 1$, and δ is contraction with h, considered as a section of $E^* \otimes Q$, we get the Eagon-Northcott complex

$$0 \to E_{m-r+1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} E_3 \xrightarrow{\delta} E_2 \xrightarrow{\delta} \Lambda^r H \xrightarrow{\det h} \det Q, \tag{5.2}$$

which is exact (precisely) where h is surjective, see, e.g., [28]. Moreover, the corresponding complex of sheaves is exact in the generic case, i.e., when $\text{codim}\, Z = m - r + 1$; this, e.g., follows from the Buchsbaum-Eisenbud theorem. In this case thus R is a Noetherian current for the ideal sheaf $J(\det h) = \text{Im}\, \det h$.

Let us now give a more explicit description of the current R^0, that will also show that it coincides with the residue current constructed in [5]. Let ϵ_j be a local holomorphic frame for Q with dual frame ϵ_j^*. Then $h = \sum_1^r h_j \otimes \epsilon_j$ where h_j are sections of H^*. Let $\xi = \sum_1^r \xi_j \otimes \epsilon_j^*$ be the section of $\text{Hom}\,(Q, H) = H \otimes Q^*$ with minimal norm such that $h\xi = \sum_j \epsilon_j \otimes \epsilon_j^* = I_Q$. This means that ξ_j are the sections of H over $X \setminus Z$ with minimal norms such that $\xi_j \cdot h_k = \delta_{jk}$. In particular, ξ_j take values in $(\text{Ker}\, h)^\perp$.

Proposition 5.2. *With the notation above,*

$$u_k^0 = \Xi\wedge\Big(\sum_j \bar\partial\xi_j\otimes\epsilon_j^*\Big)^{\dot\otimes(k-1)}\otimes\epsilon^*/(k-1)!,\quad k\geq 1,$$

where $\Xi=\xi_1\wedge\ldots\wedge\xi_r$ *and* $\epsilon^*=\epsilon_1^*\wedge\ldots\wedge\epsilon_r^*$.

Here $\dot\otimes$ shall be interpreted as wedge product on the first factors, i.e., $\bar\partial\xi_j$, and symmetric tensor product on the second factors, i.e., the ϵ_j.

Sketch of proof. Clearly, the u_k^0 in the proposition are sections of $E_k\otimes\det Q^*=\Lambda^{r+k-1}H\otimes S^{k-1}Q^*\otimes\det Q^*$ and it is readily verified that they satisfy (5.1), see [5] for details. In view of Lemma 5.1 we therefore just have to verify that the image of u_1^0 is orthogonal to the kernel of $\det h$, and the image of u_k^0 is orthogonal to the kernel of δ for $k>1$.

Since h is surjective in $X\setminus Z$, ξ_j are linearly independent, and hence they span $(\operatorname{Ker} h)^\perp$. Moreover, notice that decomposable elements in $\Lambda^\nu H$ with different number of factors ξ_j are orthogonal. Since Ker $\det h$ is spanned by decomposable elements with fewer than r factors ξ_j it follows that $\Xi\otimes\epsilon^*$ is orthogonal to Ker $\det h$ in $\Lambda^r\otimes\det Q^*$.

We now use induction over k. Since $\bar\partial\xi_j$ is in $(\operatorname{Ker} h)^\perp$, (each term in) the minimal solution to $\delta\eta=\bar\partial u_k$ must contain all r factors ξ_j. However, we claim that $\eta=u_{k+1}$ is the unique such solution. In fact, we can consider η as a homogeneous polynomial in ϵ_j with coefficients in $\Lambda^\nu E$, and $\delta\eta=0$ then means that the gradient of the polynomial vanishes, which in turn implies that $\eta=0$. □

Now let instead $E_1=H$ and $E_0=Q$. There is a closely related complex, the Buchsbaum-Rim complex, where

$$E_k=\Lambda^{r+k-1}H\otimes S^{k-2}Q^*\otimes\det Q^*,\quad k\geq 2,$$

$f_1=h$, $f_2=\det h$ and f_k is interior multiplication with h for $k\geq 3$, see [4]. Again, if $\operatorname{codim} Z=m-r+1$, the induced complex of sheaves is exact and hence $R=R^0$ is a Noetherian residue current for the sheaf $J=\operatorname{Im} h$. By similar arguments as above one can verify that

$$u_1^0=\Xi,\quad u_k^0=\Xi\wedge\Big(\sum_j\bar\partial\xi_j\otimes\epsilon_j\Big)^{\dot\otimes(k-2)}\otimes\epsilon^*\otimes\bar\partial\xi/(k-2)!,\quad k\geq 2.$$

One now sees that R^0 coincides with the current introduced in [4]. □

There has recently been a lot of work done on finding free resolutions of monomial ideals, see for example [36], [8] or [10]. In case the monomial ideal fulfills a certain genericity condition, there are explicit algorithms that produce a minimal resolution.

We now consider a simple example of a non-complete intersection ideal.

Example 3. Consider the ideal $J = (z_1^2, z_1 z_2)$ in $\mathbb{C}^2$ with zero variety $\{z_1 = 0\}$. It is easy to see that

$$0 \to \mathcal{O} \xrightarrow{f_2} \mathcal{O}^{\oplus 2} \xrightarrow{f_1} \mathcal{O}, \tag{5.3}$$

where

$$f_1 = \begin{bmatrix} z_1^2 & z_1 z_2 \end{bmatrix} \text{ and } f_2 = \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix},$$

is a (minimal) resolution of $\mathcal{O}/J$. We equip the corresponding vector bundles with the trivial Hermitian metrics. Since the associated residue current R is Noetherian and Z has codimension 1, R consists of the two parts $R_2 = \bar\partial |F|^{2\lambda} \wedge u_2^0|_{\lambda=0}$ and $R_1 = \bar\partial |F|^{2\lambda} \wedge u_1^0|_{\lambda=0}$, where $u_2^0 = \sigma_2 \bar\partial \sigma_1$ and $u_1^0 = \sigma_1$, respectively. Notice that $\sigma_1 = f_1^*(f_1 f_1^*)^{-1}$ and $\sigma_2 = (f_2^* f_2)^{-1} f_2^*$. To compute R we consider the proper mapping $\Pi : \widetilde{\mathcal{U}} \to \mathcal{U}$, where $\mathcal{U}$ is a neighborhood of the origin and $\widetilde{\mathcal{U}}$ is the blow up at the origin of $\mathcal{U}$. We cover $\widetilde{\mathcal{U}}$ by the two coordinate neighborhoods

$$\Omega_1 = \{t;\ (t_1 t_2, t_1) = z \in \mathcal{U}\} \text{ and } \Omega_2 = \{s;\ (s_1, s_1 s_2) = z \in \mathcal{U}\}.$$

In Ω_1 we get $\Pi^* f_1 = t_1^2 t_2 \begin{bmatrix} t_2 & 1 \end{bmatrix}$ so

$$\Pi^* \sigma_1 = \frac{1}{t_1^2 t_2 (1 + |t_2|^2)} \begin{bmatrix} \bar t_2 \\ 1 \end{bmatrix}.$$

Moreover

$$\Pi^* f_2 = t_1 \begin{bmatrix} 1 \\ -t_2 \end{bmatrix}$$

which gives

$$\Pi^* \sigma_2 = \frac{1}{t_1 (1 + |t_2|^2)} \begin{bmatrix} 1 & -\bar t_2 \end{bmatrix}.$$

It follows that

$$u_2^0 = \frac{d\bar t_2}{t_1^3 t_2 (1 + |t_2|^2)^2}.$$

To compute R_2, take a test form $\phi = \varphi(z) dz_1 \wedge dz_2$. In Ω_1, $\Pi^* dz_1 \wedge dz_2 = -t_1 dt_1 \wedge dt_2$ and thus

$$R_2.\phi = -\int \bar\partial \left[\frac{1}{t_1^2}\right] \wedge \left[\frac{1}{t_2}\right] \frac{d\bar t_2}{(1 + |t_2|^2)^2} \varphi(t_1 t_2, t_1)\, dt_1 \wedge dt_2, \tag{5.4}$$

where the brackets denote one-variable principal value currents. To be precise one has to check that no extra contributions appear from Ω_2, but we omit that simple verification, cf., [46]. In view of the one-variable formula

$$\bar\partial \left[\frac{1}{s}\right] \wedge ds = 2\pi i [s = 0]$$

($[V]$ denotes the current of integration over V), a Taylor expansion of φ and symmetry considerations reveal that (5.4) is equal to

$$2\pi i\int_{t_2}\frac{d\bar{t}_2\wedge dt_2}{(1+|t_2|^2)^2}\varphi_{1,0}(0,0)=(2\pi i)^2\varphi_{1,0}(0,0),$$

where $\varphi_{1,0}=\partial\varphi/\partial z_1$. Thus

$$R_2=\bar\partial\Big[\frac{1}{z_1^2}\Big]\wedge\bar\partial\Big[\frac{1}{z_2}\Big].$$

A similar computation yields that

$$R_1=\begin{bmatrix}0\\1\end{bmatrix}\Big[\frac{1}{z_2}\Big]\bar\partial\Big[\frac{1}{z_1}\Big].$$

We see that $\operatorname{ann}R_2=(z_1^2,z_2)$ and $\operatorname{ann}R_1=(z_1)$, and hence $\operatorname{ann}R=(z_1^2,z_2)\cap(z_1)=J$ as expected.

Notice that the Koszul complex associated with the ideal J is like (5.3) but with

$$f_2=\begin{bmatrix}z_1z_2\\-z_1^2\end{bmatrix},$$

i.e., with an extra factor z_1. Clearly, then it is no longer a resolution. The current R_1^0 is of course the same as before, but

$$R_2^0=\frac{1}{2}\bar\partial\Big[\frac{1}{z_1^3}\Big]\wedge\bar\partial\Big[\frac{1}{z_2}\Big].$$

In this case $\operatorname{ann}R^0=\operatorname{ann}R_2^0\cap\operatorname{ann}R_1^0=(z_1^3,z_2)\cap(z_1)$ which is strictly smaller than J. As expected thus R is not Noetherian. Roughly speaking, the annihilator of R_2^0 is too small, since the singularity of σ_2 and hence of u_2^0 is too big, due to the extra factor z_1 in f_2. □

We proceed with somewhat more involved zero-dimensional example.

Example 4. Consider the ideal $J=(z_1^5,z_1^3z_2,z_2^4)$ with variety $Z=\{0\}$ in $\mathbb{C}^2$. We have the (minimal) resolution

$$0\to\mathcal{O}^{\oplus 2}\xrightarrow{f_2}\mathcal{O}^{\oplus 3}\xrightarrow{f_1}\mathcal{O}\tag{5.5}$$

of $\mathcal{O}/J$, where

$$f_1=\begin{bmatrix}z_1^5 & z_1^3z_2 & z_2^4\end{bmatrix}\quad\text{and}\quad f_2=\begin{bmatrix}0 & z_2\\ z_2^3 & -z_1^2\\ -z_1^3 & 0\end{bmatrix}.$$

Since Z is of dimension 0, $R=R_2=\bar\partial|F|^{2\lambda}\wedge u_2^0|_{\lambda=0}$. To compute R we consider the proper mapping $\Pi:\widetilde{\mathcal{U}}\to\mathcal{U}$, where $\widetilde{\mathcal{U}}$ is a toric variety covered by the three coordinate neighborhoods

$$\Omega_1 = \{t;\ (t_1t_2, t_2) = z \in \mathcal{U}\}\ ,\ \Omega_2 = \{s;\ (s_1s_2, s_1s_2^2) = z \in \mathcal{U}\}\ \text{ and } \Omega_3 = \{r;\ (r_1, r_1^2r_2) = z \in \mathcal{U}\}.$$

By considerations inspired by [46] it is enough to make the computation in Ω_2. We get

$$\Pi^* f_1 = s_1^4 s_2^5 \begin{bmatrix} s_1 & 1 & s_2^3 \end{bmatrix} \text{ and } \Pi^* f_2 = s_1 s_2^2 \begin{bmatrix} 0 & 1 \\ s_1^2 s_2^4 & -s_1 \\ -s_1^2 s_2 & 0 \end{bmatrix}.$$

It follows that

$$\Pi^*\sigma_1 = \frac{1}{s_1^4 s_2^5 \nu(s)} \begin{bmatrix} \bar{s}_1 \\ 1 \\ \bar{s}_2^3 \end{bmatrix},$$

where $\nu(s) = (1 + |s_1|^2 + |s_2^3|^2)$. A simple computation yields

$$\Pi^*\sigma_2 = \frac{1}{s_1^3 s_2^3 \nu(s)} \begin{bmatrix} s_1 \bar{s}_2^3 & \bar{s}_2^3 & -(1+|s_1|^2) \\ s_1^2 s_2 (1 + |s_2^3|^2) & -s_1^2 \bar{s}_1 s_2 & -s_1^2 \bar{s}_1 s_2^4 \end{bmatrix},$$

and thus

$$u_2^0 = \frac{1}{s_1^7 s_2^8 \nu(s)^2} \begin{bmatrix} s_1 \bar{s}_2^3 d\bar{s}_1 - 3\bar{s}_2^2 (1 - |s_1|^2) d\bar{s}_2 \\ s_1^2 s_2 (1 + |s_2^3|^2) d\bar{s}_1 - 3 s_1^2 \bar{s}_1 s_2^4 \bar{s}_2^2 d\bar{s}_2 \end{bmatrix}.$$

Let us compute the action of R_2 on a test form $\phi = \varphi dz_1 \wedge dz_2$. In Ω_2, $\Pi^* dz_1 \wedge dz_2 = s_1 s_2^2 ds_1 \wedge ds_2$, and so

$$R_2.\phi = \int \bar\partial \left[\frac{1}{s_1^6}\right] \wedge \left[\frac{1}{s_2^6}\right] \frac{1}{\nu^2} \begin{bmatrix} -3\bar{s}_2^2 d\bar{s}_2 \\ 0 \end{bmatrix} \varphi(s_1 s_2, s_1 s_2^2)\ ds_1 \wedge ds_2 + \int \bar\partial \left[\frac{1}{s_2^5}\right] \wedge \left[\frac{1}{s_1^4}\right] \frac{1}{\nu^2} \begin{bmatrix} 0 \\ d\bar{s}_1 \end{bmatrix} \varphi(s_1 s_2, s_1 s_2^2)\ ds_1 \wedge ds_2.$$

Let us start by considering the first term. Evaluating the s_1-integral, the "upper" integral becomes

$$2\pi i \int \frac{3|s_2|^4}{(1+|s_2|^6)^2}\ \varphi_{2,3}(0,0)\ d\bar{s}_2 \wedge ds_2 = 2!3!\ \bar\partial\left[\frac{1}{z_1^3}\right] \wedge \bar\partial\left[\frac{1}{z_2^4}\right].\phi;$$

indeed, for symmetry reasons everything else vanish as in Example 3. Continuing with the second term, the "lower" integral is equal to

$$2\pi i \int \frac{1}{(1+|s_1|^2)^2}\ \varphi_{4,0}(0,0)\ d\bar{s}_1 \wedge ds_1 = 4!\ \bar\partial\left[\frac{1}{z_1^5}\right] \wedge \bar\partial\left[\frac{1}{z_2}\right].\phi$$

Thus $\operatorname{ann} R = (z_1^3, z_2^4) \cap (z_1^5, z_2) = J$ as expected. □

We conclude with a simple example where $\operatorname{ann}(\mathcal{O}(E_0)/J) = 0$.

Example 5. Consider the submodule J of $\mathcal{O}^{\oplus 2}$ generated by

$$f_1 = \begin{bmatrix} z_1 z_2 \\ -z_1^2 \end{bmatrix} \tag{5.6}$$

and the resolution

$$0 \to \mathcal{O} \xrightarrow{f_1} \mathcal{O}^{\oplus 2}. \tag{5.7}$$

Notice that $Z = \{z_1 = 0\}$ is the associated set where where $\mathcal{O}^{\oplus 2}/J$ is not locally free, or equivalently where f_1 is not locally constant. Moreover, notice that $\operatorname{ann}(\mathcal{O}^{\oplus 2}/J) = 0$. It is easily seen that (5.7) is the minimal resolution. The associated Noetherian residue current $\bar\partial|F|^{2\lambda} \wedge u_1^0|_{\lambda=0}$, where $u_1^0 = \sigma_1$, can be computed as Example 3, and we get that

$$R = R_1 = \Big[\frac{1}{z_2}\Big]\bar\partial\Big[\frac{1}{z_1}\Big] \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

If we extend (5.7) with the mapping $f_0 = \begin{bmatrix} z_1 & z_2 \end{bmatrix}$ the new complex is still exact outside Z. Observe that $\operatorname{ann} R$ is generated by $z_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and moreover that $\ker f_0$ is generated by $\begin{bmatrix} z_2 \\ -z_1 \end{bmatrix}$. Thus $\operatorname{Ker} f_0 \cap \operatorname{ann} R = J$ as expected. □

6. Cohen-Macaulay ideals and modules

Let $\mathcal{F}_x$ be a $\mathcal{O}_x^r$-module. The minimal length ν_x of a resolution of $\mathcal{F}_x$ is precisely $n - \operatorname{depth} \mathcal{F}_x$, and $\operatorname{depth} \mathcal{F}_x \le \dim \mathcal{F}_x$, so the length of the resolution is at least equal to $\operatorname{codim} \mathcal{F}_x$. Recall that the $\mathcal{F}_x$ is Cohen-Macaulay if $\operatorname{depth} \mathcal{F}_x = \dim \mathcal{F}_x$, or equivalently, $\nu_x = \operatorname{codim} \mathcal{F}_x$, see [28]. As usual we say that an ideal $J_x \subset \mathcal{O}_x$ is Cohen-Macaulay if $\mathcal{F}_x = \mathcal{O}_x/J_x$ is a Cohen-Macaulay module.

A coherent analytic sheaf $\mathcal{F}$ is Cohen-Macaulay if $\mathcal{F}_x$ is Cohen-Macaulay for each x. If we have any locally free resolution of $\mathcal{F}$ and $\operatorname{codim} \mathcal{F} = p$, then at each point $\operatorname{Ker}(\mathcal{O}(E_{p-1}) \to \mathcal{O}(E_{p-2}))$ is free by the uniqueness theorem, see below, so by Oka's lemma the kernel is locally free; hence we can modify the given resolution to a locally free resolution of minimal length p.

Notice that if h is a complete intersection, cf, Example 1, then $J(h) \subset \mathcal{O}$ is a Cohen-Macaulay ideal sheaf, i.e., $\mathcal{O}/J(h)$ is Cohen-Macaulay. Moreover, in Example 2, if $h\colon H \to Q$ and $\operatorname{codim} Z = m - r + 1$, then $J(\det h) \subset \mathcal{O}$ is a Cohen-Macaulay ideal sheaf, and $\mathcal{O}(Q)/J$ is Cohen-Macaulay, where $J = \operatorname{Im} h \subset \mathcal{O}(Q)$.

Notice that the Noetherian residue current associated with a resolution of minimal length p just consists of the single term $R = R_p^0$, which locally is a $r_p \times r_0$-matrix of currents.

The definition of a Noetherian current in general depends slightly on the chosen Hermitian metric. However, we have the following generalization of the corresponding result (in [1]) for a complete intersection.

Proposition 6.1. *Suppose that $\mathcal{F}$ is a coherent analytic sheaf with codimension $p>0$ that is Cohen-Macaulay, and assume that*

$$0 \to \mathcal{O}(E_p) \to \cdots \to \mathcal{O}(E_1) \to \mathcal{O}(E_0) \tag{6.1}$$

is a locally free resolution of $\mathcal{F}$ of minimal length p. Then the associated Noetherian current is independent of the Hermitian metric.

Notice that since $p>0$, i.e, $\operatorname{ann}\mathcal{F} \neq 0$, the right-most mapping in (6.1) is pointwise surjective outside Z.

Proof. Assume that u and u' are the forms in $X \setminus Z$ constructed by means of two different choices of metrics on E. Then $\nabla_{\text{End}} u = I$ and $\nabla_{\text{End}} u' = I$ in $X \setminus Z$, and hence

$$\nabla_{\text{End}}(uu') = (\nabla_{\text{End}} u)u' - u\nabla_{\text{End}} u' = u' - u,$$

where the minus sign occurs since u has odd order. For large $\operatorname{Re}\lambda$ we thus have, cf., the proof of Proposition 3.2,

$$\nabla_{\text{End}}\big(|F|^{2\lambda}uu'\big) = |F|^{2\lambda}u' - |F|^{2\lambda}u - \bar\partial|F|^{2\lambda}\wedge uu'.$$

As before one can verify that each term admits an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$, and evaluating at $\lambda = 0$ we get $\nabla_{\text{End}} W = U' - U - M$, where $W = |F|^{2\lambda}uu'|_{\lambda=0}$, and M is the residue current

$$M = \bar\partial|F|^{2\lambda}\wedge uu'|_{\lambda=0}. \tag{6.2}$$

Since $\nabla^2_{\text{End}} = 0$, by Proposition 3.2 we therefore get

$$R - R' = \nabla_{\text{End}} M. \tag{6.3}$$

However, since the complex ends up at p, each term in uu' has at most bidegree $(0, p-2)$ and hence the current M has at most bidegree $(0, p-1)$. Since it is supported on Z with codimension p, it must vanish, cf., the proof of Proposition 3.2. □

When $\mathcal{F} = \mathcal{O}(E_0)/J$ is Cohen-Macaulay we can also define a cohomological residue that characterizes the module sheaf $J = \operatorname{Im}(\mathcal{O}(E_1) \to \mathcal{O}(E_0))$ locally. Suppose that we have a fixed resolution (6.1) of minimal length and let us assume that $p>1$. If u is any solution to $\nabla_{\text{End}} u = I$ in $X \setminus Z$, then u^0_p is a $\bar\partial$-closed $\operatorname{Hom}(E_0, E_p)$-valued $(0, p-1)$-form. Moreover if u' is another solution, then it follows from the preceding proof that $\bar\partial(uu')^0_p = u^0_p - u'^0_p$. Therefore u^0_p defines a Dolbeault cohomology class $\omega \in H^{0,p-1}(X \setminus Z, \operatorname{Hom}(E_0, E_p))$. If ϕ is a holomorphic section of E_0 then $\omega\phi = [u^0_p\phi]$ is an element in $H^{0,p-1}(X \setminus Z, E_p)$. Moreover, if v is any solution in $X \setminus Z$ to $\nabla v = \phi$, then v_p defines the class $\omega\phi$. In fact, $\nabla(uv) = v - u\phi = v - u^0\phi$ so that $\bar\partial(uv)_p = u_p^0\phi - v_p$.

Precisely as for a complete intersection, [25] and [40], we have the following duality principle.

Theorem 6.2. *Let X be a Stein manifold and let (6.1) be a resolution of minimal length p of the Cohen-Macaulay sheaf $\mathcal{O}(E_0)/J$ over X, and assume that $p > 1$. Moreover, let ω be the associated class in $H^{0,p-1}(X \setminus Z, \mathrm{Hom}\,(E_0, E_p))$. For a holomorphic section ϕ of E_0 the following conditions are equivalent:*

(i) ϕ is a section of J.

(ii) The class $\omega\phi$ in $X \setminus Z$ vanishes.

(iii) $\int \omega\phi\wedge\bar\partial\xi = 0$ for all $\xi \in \mathcal{D}_{n,n-p}(X, E_p^*)$ such that $\bar\partial\xi = 0$ in a neighborhood of Z.

Notice that if R is the associated Noetherian current, then $\bar\partial U_p^0 = R_p$, so by Stokes' theorem, *(iii)* is equivalent to that $\int R_p\phi\wedge\xi = 0$ for all $\xi \in \mathcal{D}_{n,n-p}(X, E_p^*)$ such that $\bar\partial\xi = 0$ in a neighborhood of Z.

It is important that $p > 1$ in the theorem. If $p = 1$, then f_1 is an isomorphism outside Z, so its inverse $\omega = \sigma_1$ is a holomorphic $(0,0)$-form in $X \setminus Z$. Thus a holomorphic section ϕ of E_0 belongs to J if and only if $\omega\phi$ has a holomorphic extension across Z.

Proof. If *(i)* holds, then $\phi = f_1\psi$ for some holomorphic ψ; thus $\nabla\psi = \phi$. However, since $p > 1$, ψ has no component in E_p, and hence by definition the class $\omega\phi$ vanishes. The implication *(ii)* $\to$ *(iii)* follows from Stokes' theorem.

Let us now assume that *(iii)* holds, and choose a point x on Z. Let $v_k = u_k^0\phi$. If X' is an appropriate small neighborhood of x, then, since Z has codimension p and v_p is a $\bar\partial$-closed $(0,p)$-current, one can verify that the condition *(iii)* ensures that $\bar\partial w_p = v_p$ has a solution in $X' \setminus \overline{W}$, where W is a small neighborhood of Z in X'. Then, successively, all the lower degree equations $\bar\partial w_k = v_k + f_{k+1}w_{k+1}$, $k \geq 2$, can be solved in similar domains. Finally, we get a holomorphic solution $\psi = v_1 + f_2w_2$ to $f_1\psi = \phi$, in such a domain. By Hartogs' theorem ψ extends across Z in X'. Alternatively, one can obtain such a local holomorphic solution ψ, using the decomposition formula (7.5) below and mimicking the proof of the corresponding statement for a complete intersection in [40]; cf., also the proof of Proposition 7.1 in [6]. Since X is Stein, one can piece together to a global holomorphic solution to $f_1\psi = \phi$, and hence ϕ is a section of J. $\square$

Example 6. Let J be an ideal in $\mathcal{O}_0$ of dimension zero. Then it is Cohen-Macaulay and for each germ ϕ in $\mathcal{O}_0$, $\omega\phi$ defines a functional on $\mathcal{O}_0(E_n^*) \simeq$

$\mathcal{O}_0^{r_n}$. If J is defined by a complete intersection, then we may assume that (6.1) is the Koszul complex. Then $r_n = 1$, and in view of the Dolbeault isomorphism, see, e.g., Proposition 3.2.1 in [40], $\omega\phi$ is just the classical Grothendieck residue. $\square$

For the rest of this section we will restrict our attention to modules over the local ring $\mathcal{O}_0$, and we let $\mathcal{O}(E_k)$ denote the free $\mathcal{O}_0$-module of germs of holomorphic sections at 0 of the vector bundle E_k. Given a free resolution

$$0 \to \mathcal{O}(E_N) \xrightarrow{f_p} \cdots \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0) \tag{6.4}$$

of a module $\mathcal{F}_0$ over $\mathcal{O}_0$ and given metrics on E_k we thus get a germ R of a Noetherian residue current at 0. Recall that the resolution (6.4) is *minimal* if for each k, f_k maps a basis of $\mathcal{O}(E_k)$ to a minimal set of generators of $\operatorname{Im} f_k$. The uniqueness theorem, see, e.g., Theorem 20.2 in [28], states that any two minimal (free) resolutions are equivalent, and moreover, that any (free) resolution has a minimal resolution as a direct summand.

For a Cohen-Macaulay module $\mathcal{F}_0$ over $\mathcal{O}_0$ we have the following uniqueness of Noetherian currents.

Proposition 6.3. *Let $\mathcal{F}_0$ be a Cohen-Macaulay module over $\mathcal{O}_0$ of codimension p. If we have two minimal free resolutions $\mathcal{O}(E_\bullet)$ and $\mathcal{O}(E'_\bullet)$ of $\mathcal{F}_0$, then there are holomorphic invertible matrices g_p and g_0 (local holomorphic isomorphism $g_p\colon E'_p \simeq E_p$ and $g_0\colon E'_0 \simeq E_0$) such that such that $R = g_p R' g_0^{-1}$.*

Since minimal resolutions have minimal length p, the currents are independent of the metrics, in view of Proposition 6.1.

Proof. By the uniqueness theorem there are holomorphic local isomorphisms $g_k\colon E'_k \to E_k$ such that

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}(E'_p) & \xrightarrow{f'_p} & \cdots & \xrightarrow{f'_2} & \mathcal{O}(E'_1) & \xrightarrow{f'_1} & \mathcal{O}(E'_0) \\
 & & g_p \downarrow & & & & g_1 \downarrow & & g_0 \downarrow \\
0 & \to & \mathcal{O}(E_p) & \xrightarrow{f_p} & \cdots & \xrightarrow{f_2} & \mathcal{O}(E_1) & \xrightarrow{f_1} & \mathcal{O}(E_0)
\end{array}$$

commutes. Let g denote the induced isomorphism $E \to E'$. Choose any metric on E and equip E' with the induced metric, i.e., such that $|\xi| = |g^{-1}\xi|$ for a section ξ of E'. If $\sigma\colon E \to E$ and $\sigma'\colon E' \to E'$ are the associated endomorphisms over $X \setminus Z$, cf., Section 3, then $\sigma' = g\sigma g^{-1}$ in $X \setminus Z$, and therefore

$$u' = \sigma' + (\bar\partial\sigma')\sigma' + \cdots = g(\sigma + (\bar\partial\sigma)\sigma + \cdots)g^{-1} = gug^{-1}.$$

Therefore, $(u')^0_p = g_p u^0_p g_0^{-1}$, and hence the statement follows since $R = R_p = R^0_p$. $\square$

We shall now consider the Noetherian current associated to a general free resolution.

Theorem 6.4. *Let $\mathcal{F}_0$ be a Cohen-Macaulay module over $\mathcal{O}_0$ of codimension p. If R is the Noetherian residue current associated to an arbitrary free resolution* (6.4) *(and given metrics on E_k) and $R' = R'_p$ is associated to a minimal resolution*

$$0 \to \mathcal{O}(E'_p) \xrightarrow{f'_p} \cdots \xrightarrow{f'_2} \mathcal{O}(E'_1) \xrightarrow{f'_1} \mathcal{O}(E'_0),$$

then

$$R_p = h_p R'_p \beta_0, \tag{6.5}$$

where $\beta_0 \colon E_0 \to E'_0$ is a local holomorphic pointwise surjective morphism and h_p is a local smooth pointwise injective morphism $h_p \colon E'_p \to E_p$. Moreover, for each $\ell > 0$,

$$R_{p+\ell} = \alpha_\ell R_p,$$

where α_ℓ is a smooth $\mathrm{Hom}\,(E_p, E_{p+\ell})$*-valued* $(0,\ell)$*-form.*

Proof. By the uniqueness theorem for resolutions, the resolution $E'_\bullet$ is isomorphic to a direct summand in $E_\bullet$, and in view of the preceding proposition, we may assume that

$$\mathcal{O}(E_k) = \mathcal{O}(E'_k \oplus E''_k) = \mathcal{O}(E'_k) \oplus \mathcal{O}(E''_k)$$

and $f_k = f'_k \oplus f''_k$, so that

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}(E'_p) & \xrightarrow{f'_p} & \cdots & \xrightarrow{f'_2} & \mathcal{O}(E'_1) & \xrightarrow{f'_1} & \mathcal{O}(E'_0) \\
i_{p+1} \downarrow & & i_p \downarrow & & & & i_1 \downarrow & & i_0 \downarrow \\
\to \mathcal{O}(E_{p+1}) & \xrightarrow{f_{p+1}} & \mathcal{O}(E_p) & \xrightarrow{f_p} & \cdots & \xrightarrow{f_2} & \mathcal{O}(E_1) & \xrightarrow{f_1} & \mathcal{O}(E_0),
\end{array}$$

where $i_k \colon E'_k \to E'_k \oplus E''_k$ are the natural injections, and

$$\to \mathcal{O}(E''_{p+1}) \xrightarrow{f''_{p+1}} \mathcal{O}(E''_p) \xrightarrow{f''_p} \cdots \xrightarrow{f''_2} \mathcal{O}(E''_1) \xrightarrow{f''_1} \mathcal{O}(E''_0)$$

is a resolution of 0. In particular,

$$\to E_{p+1} \xrightarrow{f''_{p+1}} E''_p \xrightarrow{f''_p} \cdots \xrightarrow{f''_2} E''_1 \xrightarrow{f''_1} E''_0 \to 0$$

is a pointwise exact sequence of vector bundles, and therefore the set Z_k where $\mathrm{rank}\, f_k$ is not optimal coincides with the set Z'_k where $\mathrm{rank}\, f'_k$ is not optimal. In particular, $Z_k = \emptyset$ for $k > p$. If we choose, to begin with, Hermitian metrics on E_k that respect this direct sum, and let σ_k, σ'_k, and σ''_k be the corresponding minimal inverses, then

$$\sigma_k = \sigma'_k \oplus \sigma''_k$$

and hence

$$u_k^0 = (\bar\partial\sigma'_k \oplus \bar\partial\sigma''_k)(\bar\partial\sigma'_{k-1} \oplus \bar\partial\sigma''_{k-1}) \cdots (\bar\partial\sigma'_2 \oplus \bar\partial\sigma''_2)(\sigma'_1 \oplus \sigma''_1) = (u')_k^0 \oplus (u'')_k^0$$

for all k. However, $(u'')_k^0$ is smooth, and hence

$$R_p = R'_p \oplus 0, \quad R_k = 0 \text{ for } k \neq p.$$

For this particular choice of metric thus (6.5) holds with h_p as the natural injection $i_p\colon E'_p \to E_p$ and β_0 as the natural projection.

Without any risk of confusion we can therefore from now on let R'_p denote the residue current with respect to this particular metric on E, and moreover let σ' denote the minimal inverse of f with respect to this metric etc. We now choose other metrics on E_k and let R_k from now on denote the Noetherian residue current associated with this new metric. Following the notation in the proof of Proposition 6.1 we again have (6.3), and for degree reasons still $M_p^0 = 0$; here M_k^ℓ denotes the component of M that takes values in $\text{Hom}\,(E_\ell, E_k)$. Thus

$$R_p - R'_p = f_{p+1} M_{p+1}^0.$$

Moreover, if we expand uu', we get

$$M_{p+1}^0 = \bar\partial|F|^{2\lambda}\wedge\big[\sigma_{p+1}\sigma'_p(\bar\partial\sigma'_{p-1})\cdots(\bar\partial\sigma'_1) + \\ \sigma_{p+1}(\bar\partial\sigma_p)\sigma'_{p-1}(\bar\partial\sigma'_{p-2})\cdots(\bar\partial\sigma'_1) + \cdots\big]|_{\lambda=0}.$$

However, $\sigma_{p+1}(\bar\partial\sigma_p) = (\bar\partial\sigma_{p+1})\sigma_p$ and σ_{p+1} is smooth since Z_{p+1} is empty, so

$$M_{p+1}^0 = -\sigma_{p+1}R'_p + (\bar\partial\sigma_{p+1})M_p^0 = -\sigma_{p+1}R'_p.$$

Thus,

$$R_p = R'_p - f_{p+1}\sigma_{p+1}R'_p = (I_{E_p} - f_{p+1}\sigma_{p+1})R'_p.$$

Since f_{p+1} has constant rank, $H = \text{Im}\, f_{p+1}$ is a smooth subbundle of E_p. Notice that $\Pi = I_{E_p} - f_{p+1}\sigma_{p+1}$ is the orthogonal projection of E_p onto the orthogonal complement of H with respect to the new metric. In this case therefore h in (6.5) becomes the natural injection $i_p\colon E'_p \to E_p$ composed by Π, and since $E'_p \cap H = 0$, h is pointwise injective.

Since Z_k is empty for $k > p$, σ_k is smooth for $k > p$ and hence for $\ell > p$,

$$R_\ell = \bar\partial|F|^{2\lambda}\wedge(\bar\partial\sigma_\ell)\cdots(\bar\partial\sigma_{p+1})u_p^0 = (\bar\partial\sigma_\ell)\cdots(\bar\partial\sigma_{p+1})\bar\partial|F|^{2\lambda}\wedge u_p^0 = \alpha_\ell R_p$$

where $\alpha_\ell = (\bar\partial\sigma_\ell)\cdots(\bar\partial\sigma_{p+1})$. □

7. Division and interpolation formulas

Explicit formulas for division and interpolation were introduced by Berndtsson [15], and have been used by many authors since then, notably for instance [12], [13], [41], [40]; see also [14] and the references given there. To obtain such formulas that involve our currents R and U we will use the general scheme developed in [6], that we first recall briefly.

Let z be a fixed point in $\mathbb{C}^n$, let $\delta_{\zeta-z}$ denote interior multiplication by the vector field

$$2\pi i \sum_1^n (\zeta_j - z_j)\frac{\partial}{\partial \zeta_j},$$

and let $\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar\partial$. Moreover, let $g = g_{0,0} + \cdots + g_{n,n}$ be a smooth form with compact support such that $\nabla_{\zeta-z} g = 0$ and $g_{0,0}(z) = 1$; here lower indices denote bidegree; such a form will be called a weight with respect to the point z. The basic observation is that if g is a weight, then

$$\phi(z) = \int g\phi \tag{7.1}$$

holds for each function ϕ that is holomorphic in a neighborhood of the support of g, see [6].

Example 7. Let D be a ball with center at the origin in $\mathbb{C}^n$ and let

$$s = \frac{\partial|\zeta|^2}{2\pi i(|\zeta|^2 - \langle\bar\zeta, z\rangle)}.$$

Then $\delta_{\zeta-z} s = 1$ and

$$s\wedge(\bar\partial s)^{k-1} = \frac{1}{(2\pi i)^k}\frac{\partial|\zeta|^2\wedge(\bar\partial\partial|\zeta|^2)^{k-1}}{(|\zeta|^2 - \langle\bar\zeta, z\rangle)^k}.$$

If χ is a cutoff function that is 1 in a neighborhood of $\overline{D}$, then for each z in a neighborood of $\overline{D}$,

$$g = \chi - \bar\partial\chi\wedge\frac{s}{\nabla_{\zeta-z}s} = \chi - \bar\partial\chi\wedge[s + s\wedge\bar\partial s + s\wedge(\bar\partial s)^2 + \cdots + s\wedge(\bar\partial s)^{n-1}]$$

is a weight, and it depends holomorphically on z.

For other choices of weights, e.g., in strictly pseudoconvex domains, see [6]. □

Let (3.1) be a complex of (trivial) bundles over a neighborhood of the closed unit ball $\overline{D}$ in $\mathbb{C}^n$, and let $J = \operatorname{Im} f_1$. Let us fix global frames for the bundles E_k. Then $E_k \simeq \mathbb{C}^{\operatorname{rank} E_k}$ and the morphisms f_k are just matrices of holomorphic functions. One can find (see [6] for explicit choices) $(k-\ell, 0)$-form-valued holomorphic Hefer morphisms, i.e., matrices, $H_k^\ell\colon E_k \to E_\ell$

depending holomorphically on z and ζ, such that $H_k^\ell = 0$ for $k < \ell$, $H_\ell^\ell = I_{E_\ell}$, and in general,

$$\delta_{\zeta-z} H_k^\ell = H_{k-1}^\ell f_k - f_{\ell+1}(z) H_k^{\ell+1}; \tag{7.2}$$

here f stands for $f(\zeta)$. Let

$$H^{\ell+1}U = \sum_k H_k^{\ell+1} U_k^\ell, \quad H^\ell R = \sum_k H_k^\ell R_k^\ell.$$

Thus $H^{\ell+1}U$ takes a section of E_ℓ depending on ζ into a (current-valued) section of $E_{\ell+1}$ depending on both ζ and z, and similarly, $H^\ell R$ takes a section of E_ℓ into a section of E_ℓ. We let $HU = \sum_\ell H^\ell U$ and $HR = \sum_\ell H^\ell R$.

Proposition 7.1. *The form*

$$g' = f(z)HU + HUf + HR$$

maps E_ℓ into E_ℓ for each ℓ,

$$\nabla_{\zeta-z} g' = 0 \quad \textit{and} \quad g'_{0,0} = I_E. \tag{7.3}$$

Proof. For any $\mathrm{End}(E)$-valued current $A = \sum_{k,\ell} A_k^\ell$, where A_k^ℓ takes values in $\mathrm{Hom}\,(E_\ell, E_k)$, let temporarily HA denote the composition with $H = \sum_{k,\ell} H_k^\ell$. Then g' is the part of

$$f(z)HU + HUf + HR \tag{7.4}$$

that maps E_ℓ to E_ℓ for each ℓ. In view of (7.2) and Proposition 3.2, (7.4) is equal to

$$\begin{aligned} &HfU - \delta_{\zeta-z} HU + HUf + HR = \\ &\qquad H[fU + Uf - \bar\partial U + R] - \delta_{\zeta-z} HU + H\bar\partial U = H - \nabla_{\zeta-z}(HU). \end{aligned}$$

Recalling that $H_\ell^\ell = I_{E_\ell}$, therefore

$$g' = I_E - \nabla_{\zeta-z}\big(\sum_{k,\ell} H_k^\ell U_k^\ell\big),$$

and from this (7.3) follows. □

If ϕ is a holomorphic section of E, then $g'\phi\wedge g$ has compact support, $\nabla_{\zeta-z} g'\phi\wedge g = 0$, and $(g'\phi\wedge g)_{0,0}|_{\zeta=z} = \phi(z)$. By a slightly more general form of (7.1), we therefore have, cf. Proposition 5.4 in [6], the representation

$$\phi(z) = \int_\zeta g'\phi\wedge g$$

for z in a neighborhood of $\overline{D}$. Expressed in another way,

$$\phi(z) = f(z)(T\phi)(z) + T(f\phi)(z) + S\phi(z), \tag{7.5}$$

where

$$T\phi(z) = \int_\zeta HU\phi\wedge g, \qquad S\phi(z) = \int_\zeta HR\phi\wedge g.$$

Thus we get an explicit realization (in terms of U) of a solution $\psi = T\phi$ of $f\psi = \phi$, if $f\phi = 0$ and $R\phi = 0$. By translation and dilation, we achieve an explicit proof of Theorem 3.3 (i).

Now suppose that we have a complex (3.12) over a neighborhood of $\overline{D}$, and assume that either f_1 is generically surjective or we have an extension to a generically exact complex ending at E_{-1}. Then for $\phi \in \mathcal{O}(X, E_0)$ we have

$$\phi(z) = f_1(z)\int_\zeta H^1U\phi\wedge g + \int_\zeta H^0Uf_0\phi\wedge g + \int_\zeta H^0R\phi\wedge g. \tag{7.6}$$

If R is Noetherian, then the last two terms vanish if and only if ϕ is in J. We thus obtain an explicit realization of the membership of J.

In the same way as in [2] one can extend these formulas slightly, to obtain a characterization of the module $\mathcal{E}J$ of smooth tuples of functions generated by J, i.e., the set of all $\phi = f_1\psi$ for smooth ψ. For simplicity we assume that $\mathcal{O}(E_0)/J$ has positive codimension so that $f_0 = 0$. Let R be a Noetherian current for J. First notice that if $\phi = f_1\psi$, then, cf., Proposition 3.2, $R\phi = R^0\phi = R^0 f_1\psi - R^1\bar\partial\psi = R\nabla\psi = \nabla R^1\psi = 0$, so that $R\phi = 0$. Since each partial derivative $\partial/\partial\bar z_j$ commutes with f_1, we get that

$$R(\partial^\alpha\phi/\partial\bar z^\alpha) = 0 \tag{7.7}$$

for all multiindices α. The converse can be proved by integral formulas precisely as in [2], and thus we have

Theorem 7.2. *Assume that $J \subset \mathcal{O}^{\oplus r_0}$ is a coherent subsheaf such that $\mathcal{O}^{\oplus r_0}/J$ has positive codimension, and let R be a Noetherian residue current for J. Then an r_0-tuple $\phi \in \mathcal{E}^{\oplus r_0}$ of smooth functions is in $\mathcal{E}J$ if and only if (7.7) holds for all α.*

One can also obtain analogous results for lower regularity as in [2] and [6], as well as a version where the codimension of $\mathcal{O}^{\oplus r_0}/J$ is zero; one then must add the compatibility condition $f_0\phi = 0$.

In the case of a complete intersection, Björk, [20], has recently given a simple proof of Theorem 7.2 based on a deep criterion for membership of ideals of smooth functions in terms of formal power series due to Malgrange, [34]. It extends to a general ideal if our current R is replaced by a tuple of Coleff-Herrera currents γ_j such that $I = \cap \mathrm{ann}\gamma_j$.

Remark 3. One should notice that the corresponding statement, where "smooth" is replaced by "real-analytic" easily follows from the holomorphic case. In fact, if $\phi(\zeta)$ is real-analytic, then $\phi(\zeta)=\tilde\phi(\zeta,\bar\zeta)$, where

$$\tilde\phi(\zeta,\omega)=\sum_\alpha \frac{\partial^\alpha\phi}{\partial\bar\zeta^\alpha}(\zeta)(\omega-\bar\zeta)^\alpha/\alpha!$$

is holomorphic in a neighborhood of $(\zeta,\bar\zeta)$ in $\mathbb{C}^n\times\mathbb{C}^n$. Notice that $R\otimes 1$ is a Noetherian current for $J\otimes 1$ in $\mathbb{C}^n\times\mathbb{C}^n$. If (7.7) holds, it follows that $R\otimes 1\tilde\phi=0$; hence $f_1(\zeta)\psi(\zeta,\omega)=\tilde\phi(\zeta,\omega)$ and thus $f_1(\zeta)\psi(\zeta,\bar\zeta)=\phi(\zeta)$. $\square$

Let J be a coherent Cohen-Macaulay ideal sheaf of codimension p over some pseudoconvex set X and let μ be an analytic functional that annihilates J. In [26] was proved (Theorem 4.4) that μ can be represented by an (n,n)-current $\tilde\mu$ with compact support of the form $\tilde\mu=\alpha\wedge R$, where α is a smooth $(n,n-p)$-form with compact support and R is the Coleff-Herrera product of a complete intersection ideal contained in J. In particular, $\tilde\mu$ vanishes on $\mathcal{E}J$. As another application of our integral formulas we prove the following more general result.

Theorem 7.3. *Let X be a pseudoconvex set in $\mathbb{C}^n$ and let J be a coherent subsheaf of $\mathcal{O}(E_0)\simeq\mathcal{O}^{\oplus r_0}$ such that $\mathcal{O}(E_0)/J$ has positive codimension. If $\mu\in\mathcal{O}'(X,E_0^*)$ is an analytic functional that vanishes on J, then there is an (n,n)-current $\tilde\mu$ with compact support that represents μ, i.e.,*

$$\mu.\xi=\tilde\mu.\xi,\quad \xi\in\mathcal{O}(X,E_0), \tag{7.8}$$

and such that $\tilde\mu$ vanishes on $\mathcal{E}J$. More precisely we can choose $\tilde\mu$ of the form

$$\tilde\mu=\sum_k \alpha_k R_k,$$

where R is a Noetherian residue current for J and $\alpha_k\in\mathcal{D}_{n,n-k}(X,E_k^*)$.

Here E_k refers to the trivial vector bundles associated to a free resolution of $\mathcal{O}(E_0)/J$.

Proof. Assume that μ is carried by the $\mathcal{O}(X)$-convex compact subset $K\subset X$ and let V be an open neighborhood of K. For each $z\in V$ we can choose a weight g^z with respect to z, such that $z\mapsto g^z$ is holomorphic in V and all g^z have support in some compact $\tilde K\subset X$, see Example 10 in [1]. Let R be a Noetherian residue current for J, associated to a free resolution of $\mathcal{O}(E_0)/J$ in a neighborhood of $\tilde K$, cf, Proposition 4.5. Now consider the corresponding decomposition (7.6) that holds for $z\in V$, with $g=g^z$; notice that $f_0=0$ by the assumption on J. The analytic functional μ has a continuous extension to $\mathcal{O}(K,E_0)$ and since $\mathcal{O}(X)$ is dense in $\mathcal{O}(K)$ μ will vanish on the first term on the right hand side in (7.6). If we define the (n,n)-current

$$\tilde\mu=\mu_z(g^z\wedge H^0)R=\sum_k \mu_z(g^z_{n-k,n-k}\wedge H^0_k)R_k=\sum_k\alpha_k R_k,$$

then α_k have compact support and (7.8) holds. Since R is Noetherian, $\tilde{\mu}$ annihilates $\mathcal{E}J$. $\square$

8. Homogeneous residue currents

We will now make a construction of homogeneous Noetherian residue currents in $\mathbb{C}^{n+1}$. It is the key to find global Noetherian currents for polynomial ideals in $\mathbb{C}^n$ by homogenization in the next section. Let $S = \mathbb{C}[z_0, z_1, \dots, z_n]$ be the graded ring of polynomials in $\mathbb{C}^{n+1}$. Moreover, let $S(-d)$ be equal to S considered as an S-module, but with the grading shifted by $-d$, so that the constants have degree d, the linear forms have degree $d+1$ etc. Assume that

$$0 \to M_N \to \cdots \to M_1 \to M_0 \tag{8.1}$$

is a complex of free graded S-modules,

$$M_k = S(-d_1^k) \oplus \cdots \oplus S(-d_{r_k}^k). \tag{8.2}$$

Then the (degree preserving) mappings are given by matrices of homogeneous elements in S. We can associate to (8.1) a complex of vector bundles over $\mathbb{P}^n$,

$$0 \to E_N \xrightarrow{f_N} \dots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0, \tag{8.3}$$

in the following way. Let $\mathcal{O}(\ell)$ be the holomorphic line bundle over $\mathbb{P}^n$ whose sections are (naturally identified with) ℓ-homogeneous functions in $\mathbb{C}^{n+1}$. Moreover, let E_j^i be disjoint trivial line bundles over $\mathbb{P}^n$ and let

$$E_k = \big(E_1^k \otimes \mathcal{O}(-d_1^k)\big) \oplus \cdots \oplus \big(E_{r_k}^k \otimes \mathcal{O}(-d_{r_k}^k)\big).$$

The mappings in (8.1) induce vector bundle morphisms $f_k \colon E_k \to E_{k-1}$. We equip E_k with the natural Hermitian metric, i.e., such that

$$|\xi(z)|^2_{E_k} = \sum_{j=1}^{r_k} |\xi_j(z)|^2 |z|^{2d_j^k},$$

if $\xi = (\xi_1, \dots, \xi_{r_k})$. If (8.3) is generically exact, which, e.g., holds if (8.1) is exact, then we can define the associated currents U and R as before, following the general scheme in Section 3.

Example 8. For each j,k let ϵ_j^k be a global frame element for the bundle E_j^k. Then

$$R_k^\ell = \sum_{i=1}^{r_\ell} \sum_{j=1}^{r_k} (R_k^\ell)_{ij} \otimes \epsilon_i^k \otimes (\epsilon_j^\ell)^*,$$

where each $(R_k^\ell)_{ij}$ is a $(0, k-\ell)$-current on $\mathbb{P}^n$, taking values in $\mathrm{Hom}\,(\mathcal{O}(-d_j^\ell), \mathcal{O}(-d_i^k)) \simeq \mathcal{O}(d_j^\ell - d_i^k)$; alternatively $(R_k^\ell)_{ij}$ can be viewed as a $(d_j^\ell - d_i^k)$-homogeneous current on $\mathbb{C}^{n+1}\setminus\{0\}$. In the affine part $\mathcal{U}_0 = \{[z]\in\mathbb{P}^n;\ z_0\neq 0\}$ we have, for each k, a holomorphic frame

$$e_j^k = z_0^{-d_j^k}\epsilon_j^k, \quad j=1,\dots,r_k,$$

for the bundle E_k. In these frames

$$R_k^\ell = \sum_{i=1}^{r_\ell}\sum_{j=1}^{r_k}(\hat{R}_k^\ell)_{ij}\otimes e_i^k\otimes(e_j^\ell)^*, \tag{8.4}$$

where $(\hat{R}_k^\ell)_{ij}$ are (scalar-valued) currents in $\mathcal{U}_0\simeq\mathbb{C}^n$. Since $(\hat{R}_k^\ell)_{ij}$ are the dehomogenizations of $(R_k^\ell)_{ij}$, and $d_j^\ell - d_i^k\leq 0$, it is easily seen that $(\hat{R}_k^\ell)_{ij}$ have current extensions to $\mathbb{P}^n$. □

In analogy with Theorem 4.1 the exactness of (8.1) is related to the vanishing of R:

Proposition 8.1. *Let* (8.1) *be a graded complex of free S-modules, $N\leq n+1$, let* (8.3) *be the corresponding complex of Hermitian vector bundles over $\mathbb{P}^n$, and let R be the associated residue current on $\mathbb{P}^n$. Then R^ℓ vanishes for all $\ell\geq 1$ if and only if* (8.1) *is exact.*

The restriction on the length of the resolution is needed for the "if"-direction to avoid cohomologous obstructions; we now need global solutions to the $\bar\partial$-equation, and we will use the following lemma; for a proof, see, e.g., [24].

Lemma 8.2. $H^{0,q}(\mathbb{P}^n,\mathcal{O}(\nu))=0$ *for all ν if $0<q<n$, whereas* $H^{0,n}(\mathbb{P}^n,\mathcal{O}(\nu))=0$ *if $\nu\geq -n$.*

Also recall that $H^{0,0}(\mathbb{P}^n,\mathcal{O}(\nu))$, i.e., the global holomorphic sections of $\mathcal{O}(\nu)$ are naturally identified with the ν-homogeneous polynomials in $\mathbb{C}^{n+1}$.

Proof of Proposition 8.1. First assume that (8.1) is exact for $\ell\geq 1$. According to the Buchsbaum-Eisenbud theorem for graded rings, see [29], the set in $\mathbb{C}^{n+1}$ (or equivalently in $\mathbb{P}^n$) where the rank of f_k is strictly less than the generic rank ρ_k, has at least codimension k. Precisely as in the proof of Theorem 4.1 it follows that $R^\ell=0$ for $\ell\geq 1$.

Conversely, assume that $R^\ell=0$ for all $\ell\geq 1$. Consider a homogeneous element ϕ in M_ℓ, $\ell\geq 1$, of degree r that is mapped to zero in $M_{\ell-1}$. Then ϕ corresponds to a global section of $E_\ell\otimes\mathcal{O}(r)$ that we also will denote ϕ, and $f_\ell\phi=0$. Notice that R and U can just as well be considered as the currents associated with the complex

$$0\to E_N\otimes\mathcal{O}(r)\xrightarrow{f_N}\dots\xrightarrow{f_2}E_1\otimes\mathcal{O}(r)\xrightarrow{f_1}E_0\otimes\mathcal{O}(r).$$

Since $R^\ell = 0$ we therefore have that $\nabla(U^\ell\phi) = \phi$. We want to find a holomorphic solution by solving a sequence of $\bar\partial$-equations, cf., the proof of Theorem 3.3. The first $\bar\partial$-equation to be solved is $\bar\partial w = U_N^\ell\phi$. However, since $N \leq n+1$ and $\ell \geq 1$ the right hand side is a $(0,q)$-current with $q \leq n-1$ and thus solvable by Lemma 8.2. Neither for the remaining equations there are any cohomological obstructions and hence we obtain a holomorphic section ψ of $E_{\ell+1} \otimes \mathcal{O}(r)$ such that $f_{\ell+1}\psi = \phi$; this section ψ corresponds to the desired element in $M_{\ell+1}$. □

Let $J \subset M_0$ be a homogeneous submodule of M_0. Let us choose a graded resolution (8.1) of M_0/J; in view of Proposition 8.1 the associated residue current R is then Noetherian for the subsheaf of $\mathcal{O}(E_0)$ generated by J. We also assume that the resolution has minimal length, which is $n+1-\operatorname{depth}(M_0/J)$ by the Auslander-Buchsbaum theorem, see [28]. Let ϕ be a holomorphic section of $E_0 \otimes \mathcal{O}(r)$ that is generically in the image of f_1 and such that $R\phi = 0$. Then $\nabla(U^0\phi) = \phi$, cf., the proof of Corollary 3.4. Arguing as in the preceding proof we can find a global solution $f_1\psi = \phi$ provided that either the complex terminates at (at most) level n, or if the $\bar\partial$-equation of top degree is solvable, which it indeed is in view of Lemma 8.2 if $r - d_j^{n+1} \geq -n$ for all j. Summing up we have the following partial analogue of Theorem 4.3:

Proposition 8.3. *Assume that $J \subset M_0$ is a homogeneous submodule of the free graded S-module M_0, and let R be the Noetherian residue current associated with a resolution of M_0/J of minimal length N. Let ϕ be a holomorphic section of $E_0 \otimes \mathcal{O}(r)$ that lies generically in the image of $f_1 \colon E_1 \otimes \mathcal{O}(r) \to E_0 \otimes \mathcal{O}(r)$. If either*

(i) $N \leq n$

or

(ii) $r \geq \max_j(d_j^{n+1}) - n$,

then $f_1\psi = \phi$ has a global holomorphic solution if (and only if) $R\phi = 0$.

Remark 4. The condition (i) is equivalent to that $\operatorname{depth}(M_0/J) \geq 1$ which means that M_0/J contains a nontrivial nonzerodivisor. If J is defined by a complete intersection, then the condition (i) is fulfilled. Also if Z is discrete and all the zeros are of first order, then $\operatorname{depth} S/J = 1$, see [29], so that (i) holds.

The least possible value of r in (ii), i.e., $\max_j(d_j^{n+1}) - n$ is closely related to the degree of regularity of J, see, e.g., [29]. An estimate of the regularity for zerodimensional ideals is given in [44]. See [9] for a general criterion for a given degree of regularity. See also Remark 5 below. □

9. Noetherian residue currents for polynomial ideals

We will now use the results from the previous section to obtain Noetherian residue currents for (sheaves induced by) polynomial modules in $\mathbb{C}^n$. Let $z' = (z_1, \dots, z_n)$ be the standard coordinates in $\mathbb{C}^n$ that we identify with $\mathcal{U}_0 = \{[z] \in \mathbb{P}^n;\ z_0 \neq 0\}$, where $[z] = [z_0, \dots, z_n]$ are the usual homogeneous coordinates on $\mathbb{P}^n$. Let F_1 be a $\mathrm{Hom}\,(\mathbb{C}^{r_1}, \mathbb{C}^{r_0})$-valued polynomial in $\mathbb{C}^n$, whose columns $F^1, \dots, F^{r_1}$ have (at most) degrees $d_1^1, \dots, d_{r_1}^1$ and let J be the submodule of $\mathbb{C}[z_1, \dots, z_n]^{r_0}$ generated by $F^1, \dots, F^{r_1}$. After the homogenizations $f^k(z) = z_0^{d_k^1} F^k(z'/z_0)$ we get an $r_0 \times r_1$-matrix f_1 whose columns are d_k^1-homogeneous forms in $\mathbb{C}^{n+1}$; thus a graded mapping

$$f_1 \colon S(-d_1^1) \oplus \cdots \oplus S(-d_{r_1}^1) \to S^{\oplus r_0}.$$

Extending to a graded resolution (of minimal length) (8.1) we obtain a Noetherian residue current R for the sheaf generated by f_1 and an associated current U. In the trivializations in $\mathbb{C}^n \simeq \mathcal{U}_0$, described in Example 8, the component R_k of R is the matrix $(\hat{R}_k^0)_{ij}$. In the same trivializations U_k^ℓ corresponds to a matrix $(\hat{U}_k^\ell)_{ij}$. Moreover, the mappings f_k correspond to the matrices F_k that are just the dehomogenizations of the matrices f_k in (8.3).

If Φ is an r_0-tuple of polynomials in $\mathbb{C}^n$ and there is a tuple Ψ of polynomials such that $\Phi = F_1 \Psi$ in $\mathbb{C}^n$ then clearly $R\Phi = 0$. Conversely, if $R\Phi = 0$ in $\mathbb{C}^n$ (and the equation is locally solvable generically) we know that Φ is in the sheaf generated by F_1 and hence by Cartan's theorem there is a polynomial solution to $F_1 \Psi = \Phi$. However, we now have a procedure to find such a Ψ: Take a homogenization $\phi(z) = z_0^r \Phi(z'/z_0)$ for some $r \geq \deg \Phi$. The condition $R\Phi = 0$ in $\mathbb{C}^n$ means that $R\phi = 0$ outside the hyperplane at infinity, so if r is large enough, $R\phi = 0$ on $\mathbb{P}^n$. Now Proposition 8.3 applies if either r is so large that condition (ii) is fulfilled, or if the length of the resolution is less than $n+1$. If r is chosen large enough we thus have a holomorphic section ψ of $E_1 \otimes \mathcal{O}(r)$ such that $f_1 \psi = \phi$. After dehomogenization we get the desired polynomial solution $\Psi = (\Psi^j)$ to $F_1 \Psi = \sum F^j \Psi^j = \Phi$, and $\deg F^j \Psi^j \leq r$. It is well-known that in the worst case the final degree has to be doubly exponential; at least $d^{2^{(n/10)}}$, if d is the degree of F_1, see [35].

Remark 5. The final degree is essentially depending on the maximal polynomial degree in the resolution, and it is known to be at worst like $(2d)^{2^n - 1}$ if d is the degree of the generators, see [7]. $\square$

We proceed with a result where we have optimal control of the degree of the solution; it is a generalization of Max Noether's classical theorem, [37]; see also [30].

Theorem 9.1. *Let $F^1, \dots, F^{r_1}$ be r_0-columns of polynomials in $\mathbb{C}^n$ and let J be the homogeneous submodule of $M_0 = S^{\oplus r_0}$ defined by the homogenized forms $f^1, \dots, f^{r_1}$. Furthermore, assume that the quotient module M_0/J is Cohen-Macaulay and that no irreducible component of Z is contained in the hyperplane at infinity. If Φ belongs to the submodule $\tilde{J} \subset \mathbb{C}[z']^{r_0}$ generated by $F^1, \dots, F^{r_1}$, then there are tuples of polynomials Ψ^j with deg $(F^j\Psi^j) \leq$ deg Φ such that $F^1\Psi^1 + \cdots + F^{r_1}\Psi^{r_1} = \Phi$.*

Sketch of proof. We follow the procedure described above. Assume that $\operatorname{codim} M_0/J = p$. The Cohen-Macaulay assumption means that $\dim M_0/J = \operatorname{depth} M_0/J = n + 1 - \operatorname{codim} M_0/J$. By the Auslander-Buchsbaum theorem therefore we can choose a resolution (8.1) of M_0/J of length p, see [29]. Moreover all irreducible components of Z have codimension p. We choose $r = \deg \Phi$. Since Φ is in the ideal in $\mathbb{C}^n$ we have that $R\phi = 0$ in $\mathbb{C}^n$. By Proposition 3.2, $R = R_p$ and since Z has no component contained in the hyperplane at infinity, we can copy the argument in the proof of Theorem 1.2 in [3] and conclude that $R\phi = 0$ in $\mathbb{P}^n$. Since $p < n+1$, cf., Proposition 8.3, we can find a holomorphic section ψ of $E_1 \otimes \mathcal{O}(r)$ such that $f_1\psi = \phi$. After dehomogenization we get the desired solution Ψ. $\square$

We conclude this section with an explicit integral formula that provides a realization of the membership of Φ in $J \subset \mathbb{C}[z_1, \dots, z_n]^{r_0}$; for simplicity we assume that the matrix $F_1 = (F^1, \dots, F^{r_1})$ is generically surjective, i.e., has generic rank r_0. From now on we write z rather than z'.

Lemma 9.2. *One can choose Hefer matrices of forms H_k^ℓ satisfying* (7.2) *(with f_k replaced by F_k) that are polynomials in both z and ζ.*

Sketch of proof. Following the proof of Proposition 5.3 in [6] one can construct H_k^ℓ inductively, using the following two statements:

(i) If p is a polynomial, then there is a polynomial-valued $(1,0)$-form h such that $\delta_{\zeta - z} h = p(\zeta) - p(z)$.

(ii) If $p(\zeta, z)$ is a polynomial-valued $(q,0)$-form in $d\zeta$, $q \geq 1$, such that $\delta_{\zeta - z} p = 0$, then there is a polynomial-valued $(q+1,0)$-form h such that $\delta_{\zeta - z} h = p$.

The first one is easy and the second one is also quite elementary. In fact, notice that $\eta_j = \zeta_j - z_j$ is a complete intersection in $\mathbb{C}^n_\zeta \times \mathbb{C}^n_z$, so the sheaf complex induced by the Koszul complex is exact above level 0, and so there are local holomorphic solutions in $\mathbb{C}^n_\zeta \times \mathbb{C}^n_z$. One can obtain a global polynomial solution for instance from Proposition 8.1 by homogenization. $\square$

Notice that

$$g = \frac{1 + \langle \bar{\zeta}, z \rangle}{1 + |\zeta|^2} + \frac{i}{2\pi} \partial\bar{\partial} \log(1 + |\zeta|^2)$$

is a weight in $\mathbb{C}^n$ with respect to the point z, cf., Section 7. Indeed, g is equal to

$$1 - \nabla_{\zeta - z} \partial \log(1 + |\zeta|^2)/2\pi i.$$

Since

$$g^\mu = \mathcal{O}\Big(\frac{1}{|\zeta|^\mu}\Big)$$

for fixed z and H^ℓ consists of polynomials, it follows that

$$g^\mu \wedge H^0 R, \quad g^\mu \wedge H^1 U \tag{9.1}$$

have current extensions to $\mathbb{P}^n$ if μ is large enough, cf., Example 8. Let $\chi_k(\zeta) = \chi(|\zeta|/k)$, where $\chi(t)$ is a cutoff function that is 1 for $t < 1$ and 0 for $t > 2$. If μ is sufficiently large, depending on the order at infinity of R and U, we have that

$$\begin{aligned} &\chi_k g^\mu \wedge H^0 R \to g^\mu \wedge H^0 R, \quad \bar{\partial}\chi_k \wedge g^\mu \wedge H^0 R \to 0, \\ &\qquad \chi_k g^\mu \wedge H^1 U \to g^\mu \wedge H^1 U, \quad \bar{\partial}\chi_k \wedge g^\mu \wedge H^1 U \to 0, \quad k \to \infty. \end{aligned} \tag{9.2}$$

Let

$$g_k = \chi_k - \bar{\partial}\chi_k \wedge \frac{s}{\nabla_{\zeta - z} s},$$

where s is the $(1,0)$-form in Example 7 in Section 7. Then $g_k \wedge g^{\mu+m}$ is a compactly supported weight with respect to z if $k > |z|$, cf., Section 7, and hence we have the representation (writing F rather than F_1)

$$\Phi(z) = F(z) \int g_k \wedge g^{\mu+m} \wedge H^1 U \Phi + \int g_k \wedge g^{\mu+m} \wedge H^0 R \Phi.$$

Notice that

$$\Big(\frac{1 + \langle \bar{\zeta}, z \rangle}{1 + |\zeta|^2}\Big)^m P(\zeta)$$

is smooth on $\mathbb{P}^n$ for fixed z if P is a polynomial with $\deg P \le m$. If we let $k \to \infty$ we therefore obtain

Theorem 9.3. *Let F be a $r_0 \times r_1$-matrix of polynomials in $\mathbb{C}^n$ with generic rank r_0 and let J be the submodule of $\mathbb{C}[z_1, \ldots, z_n]^{r_0}$ generated by the columns of F. For each given integer m, with the notation above and for a large enough μ, we have the polynomial decomposition*

$$\Phi(z) = F(z) \int g^{\mu+m} \wedge H^1 U \Phi + \int g^{\mu+m} \wedge H^0 R \Phi \tag{9.3}$$

of r_0-columns Φ of polynomials with degree at most m, and the last term vanishes as soon as $\Phi \in J$.

The integrals here are to be interpreted as the action of currents on test functions on $\mathbb{P}^n$. If Φ belongs to J thus (9.3) provides a realization of the membership, expressed in terms of the current U and the Hefer forms.

10. The fundamental principle

Let E_1 and E_0 be trivial bundles, let F be a $\operatorname{Hom}(E_1, E_0)$-valued polynomial of generic rank $r_0 = \operatorname{rank} E_0$ and let F^T be the transpose of F. Furthermore, let K be the closure of an open strictly convex bounded domain with smooth boundary in $\mathbb{R}^n$ containing the origin. The fundamental principle of Ehrenpreis and Palamodov states that every homogeneous solution to the system of equations $F^T(D)\xi = 0$, $D = i\partial/\partial t$, on K is a superposition of exponential solutions with frequencies in the algebraic set $Z = \{z;\ \operatorname{rank} F(z) < r\}$. Following the ideas in [16] we can produce a residue version of the fundamental principle.

Let $\rho(\eta)$ be the support function

$$\sup_{t\in K}\langle \eta, t\rangle$$

for K but smoothened out in a neighborhood of the origin in $\mathbb{R}^n$. Since ρ is smooth i $\mathbb{R}^n$ and 1-homogeneous outside a neighborhood of the origin, all its derivatives are bounded. Let

$$\rho'(\eta) = (\partial\rho/\partial\eta_1, \dots, \partial\rho/\partial\eta_n).$$

We extend to complex arguments $\zeta = \xi + i\eta$ by letting $\rho(\zeta) = \rho(\eta)$ and $\rho'(\zeta) = \rho'(\eta)$. Then ρ' maps $\mathbb{C}^n$ onto K, see [16]. The convexity of ρ implies that

$$e^{\rho(\zeta)}\left|e^{i\langle \rho'(\zeta), \zeta - z\rangle}\right| \le e^{\rho(z)}. \tag{10.1}$$

We are to modify the decomposition (9.3) to allow entire functions h with values in E_0 satisfying an estimate like

$$|h(z)| \le C(1+|z|)^M e^{\rho(z)} \tag{10.2}$$

for some, from now on, fixed natural number M. We will use the same notation as in the previous section. First we introduce a new weight.

Lemma 10.1. *The form*

$$g' = e^{i\langle\rho'(\zeta),\zeta - z\rangle + \frac{i}{\pi}\partial\bar\partial\rho} = e^{i\langle\rho'(\zeta),\zeta - z\rangle}\sum_{\ell\ge 0}\Big(\frac{i}{\pi}\partial\bar\partial\rho\Big)^{\ell}/\ell!$$

is a weight for each fixed $z \in \mathbb{C}^n$.

Proof. Notice that

$$\frac{\partial\rho}{\partial\zeta_k} = -\frac{i}{2}\rho_k'(\zeta).$$

Therefore,

$$\gamma = i\langle\rho'(\zeta), \zeta - z\rangle + \frac{i}{\pi}\partial\bar\partial\rho(\zeta) = \nabla_{\zeta-z}\frac{-\partial\rho}{\pi i}$$

is $\nabla_{\zeta-z}$-closed and $\gamma_{0,0}(z) = 0$. Thus e^γ is a weight. □

It follows from (10.1) that

$$g^\mu\wedge g'\wedge H^1Uh, \quad g^\mu\wedge g'\wedge H^0Rh$$

will vanish to a given finite order at infinity if μ is large enough and $h(\zeta)$ satisfies (10.2). Therefore, if μ is large enough, using the compactly supported weights g_k and arguing as in the proof of Theorem 9.3, we obtain the decomposition

$$h(z) = F(z)\int g'\wedge g^\mu\wedge H^1Uh + \int g'\wedge g^\mu\wedge H^0Rh = FTh + Sh \tag{10.3}$$

for all entire h satisfying (10.2). Furthermore, Sh vanishes if $h = Fq$ for some holomorphic q, and in view of (10.1), both Th and Sh satisfy (10.2) for some other large number M' instead of M.

Let $\mathcal{E}'(K)$ be the space of distributions in $\mathbb{R}^n$ with support contained in K and let $\mathcal{E}^{',M}(K)$ denote the subspace of distributions of order at most M. For $\omega\in\mathcal{E}'(K)$ let $\hat\omega(\zeta) = \omega(e^{-i\langle\zeta,\cdot\rangle})$ be its Fourier-Laplace transform. The Paley-Wiener-Schwartz theorem, see [33] Thm 7.3.1, states that if $\nu\in\mathcal{E}^{',M}(K)$, then

$$|\hat\nu(\zeta)| \le C(1+|\zeta|)^M e^{\rho(\eta)}, \tag{10.4}$$

and conversely: if h is an entire function that satisfies such an estimate then $h = \hat\nu$ for some $\nu\in\mathcal{E}'(K)$.

From (10.3), applied to $\hat\nu$ for $\nu\in\mathcal{E}^{',M}(K,E_0)$, we therefore get mappings

$$\mathcal{T}\colon \mathcal{E}^{',M}(K,E_0)\to\mathcal{E}'(K,E_1), \qquad \mathcal{S}\colon \mathcal{E}^{',M}(K,E_0)\to\mathcal{E}'(K,E_0),$$

such that

$$\nu = F(-D)\mathcal{T}\nu + \mathcal{S}\nu,$$

and $\mathcal{S}\nu = 0$ if $\nu = F(-D)\omega$ for some $\omega\in\mathcal{E}'(K,E_1)$. By duality we have mappings

$$\mathcal{T}^*\colon \mathcal{E}(K,E_1^*)\to C^M(K,E_0^*), \quad \mathcal{S}^*\colon \mathcal{E}(K,E_0^*)\to C^M(K,E_0^*)$$

and they satisfy

$$\xi = \mathcal{T}^*F^T(D)\xi + \mathcal{S}^*\xi, \quad \xi\in\mathcal{E}(K,E_0^*). \tag{10.5}$$

Theorem 10.2. *Suppose that $M \geq \deg F$. If $\xi \in \mathcal{E}(K, E_0^*)$, then $\mathcal{S}^*\xi \in C^M(K, E_0^*)$ satisfies $F^T(D)\mathcal{S}^*\xi = 0$. If in addition $F^T(D)\xi = 0$, then $S^*\xi = \xi$. Moreover, we have the explicit formula*

$$\mathcal{S}^*\xi(t) = \int_\zeta R^T(\zeta)\alpha^T(\zeta, D)\xi(\rho')e^{-i\langle\zeta, t-\rho'\rangle}\wedge e^{\frac{i}{\pi}\partial\bar\partial\rho}, \tag{10.6}$$

where $\alpha^T(\zeta, D)\xi(\rho')$ is the result when replacing each occurrence of z in $\alpha^T(\zeta, z)$ by D, letting it act on $\xi(t)$ and evaluating at the point $\rho'(\zeta)$.

Thus $\mathcal{S}^*$ is a projection onto the space of homogeneous solutions.

Recall that $\rho' \in K$. Also notice that $\mathrm{Re}\, -i\langle\zeta, t\rangle = \langle\eta, t\rangle \leq \rho(\eta)$ if $t \in K$, so combined with (10.1) we get that

$$\mathrm{Re}\, -i\langle\zeta, t - \rho'(\zeta)\rangle \leq 0, \quad t \in K$$

(for ζ outside a neighborhood of 0). Therefore the integral in (10.6) has meaning if μ is large enough.

Proof. Suppose that $M \geq \deg F$. Then for $\omega \in \mathcal{E}'^{,M-\deg F}(K, E_1)$ we have

$$\omega.F^T(D)\mathcal{S}^*\xi = F(-D)\omega.\mathcal{S}^*\xi = \mathcal{S}(F(-D)\omega).\xi = 0 \tag{10.7}$$

since $\tau = F(-D)\omega \in \mathcal{E}'^{,M}(K, E_0)$ so that $\mathcal{S}\tau = 0$. From (10.7) the first statement now follows. The second one follows immediately from (10.5).

It remains to prove (10.6). The argument is very similar to the proof of Theorem 2 in [16] so we only sketch it. To begin with we have

$$S\hat\nu(z) = \int_\zeta \alpha(\zeta, z)R(\zeta)\hat\nu(\zeta)e^{i\langle\zeta - z, \rho'(\zeta)\rangle}\wedge e^{\frac{i}{\pi}\partial\bar\partial\rho} \tag{10.8}$$

where $\alpha(\cdot, z) = g^\mu\wedge H^0$ is a polynomial in z. Let δ_t be the Dirac measure at $t \in K$. Then, letting T denote transpose of matrices, we have

$$\mathcal{S}^*\xi(t) = \delta_t.\mathcal{S}^*\xi = (\mathcal{S}\delta_t.\xi)^T =$$
$$\frac{1}{(2\pi)^n}\int_x\int_\zeta R^T(\zeta)\alpha^T(\zeta, x)e^{-i\langle x, \rho'\rangle}\hat\xi(-x)e^{-i\langle\zeta, t-\rho'\rangle}\wedge e^{\frac{i}{\pi}\partial\bar\partial\rho}.$$

As in [16] one can verify that it is legitimate to interchange the order of integration, and then (10.6) follows by Fourier's inversion formula. □

Corollary 10.3. *For any solution $\xi \in \mathcal{E}(K, E_0^*)$ of $F^T(D)\xi = 0$, there are smooth forms $A_k(\zeta)$ with values in E_k^* such that*

$$\xi(t) = \int_\zeta \sum_k R_k^T(\zeta)A_k(\zeta)e^{-i\langle\zeta, t-\rho'(\zeta)\rangle}. \tag{10.9}$$

Conversely, for any such smooth forms $A_k(\zeta)$ with sufficient polynomial decay at infinity the integral (10.9) *defines a homogeneous solution.*

The last statement follows just by applying $F^T(D)$ to the integral and using that $F^T(\zeta)R^T = 0$.

Remark 6. In case F defines a complete intersection, formulas similar to (10.9) were obtained in [16] and [41]. In [16] is assumed, in addition, that $F^T(D)$ is hypoelliptic; then one can avoid the polynomial weight factor g^μ and so the resulting formula is even simpler. See also [11] and [14]. $\square$

Example 9 (A final example). The ideal (z_1^2, z_1z_2) corresponds to the system

$$\frac{\partial^2}{\partial t_1^2}\xi(t) = 0, \ \frac{\partial^2}{\partial t_1 \partial t_2}\xi(t) = 0.$$

In view of (10.9) and Example 3, the solutions are precisely the functions that can be written

$$\xi(t) = \int_z \Big[\frac{1}{z_2}\Big]\bar\partial\Big[\frac{1}{z_1}\Big] \wedge A_1(z)\ dz_2 \wedge d\bar z_1 \wedge d\bar z_2\ e^{-i(z_1t_1+z_2t_2)} + \\ \int_z \bar\partial\Big[\frac{1}{z_1^2}\Big] \wedge \bar\partial\Big[\frac{1}{z_2}\Big] \wedge A_2(z)\ d\bar z_1 \wedge d\bar z_2\ e^{-i(z_1t_1+z_2t_2)},$$

for smooth functions A_1 and A_2 with appropriate growth. It is easily checked directly to be the general solution, since the first integral is a quite arbitrary function $C(t_2)$ whereas the second integral is an arbitrary polynomial $C_1 + C_2t_1$. $\square$

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Paper IV

RESIDUE CURRENTS CONSTRUCTED FROM RESOLUTIONS OF MONOMIAL IDEALS

ELIZABETH WULCAN

Abstract. Given a free resolution of an ideal $\mathcal{J}$ of holomorphic functions, one can construct a vector valued residue current R, whose annihilator is precisely $\mathcal{J}$. In this paper we compute R in case $\mathcal{J}$ is a monomial ideal and the resolution is a cellular resolution in the sense of Bayer and Sturmfels. A description of R is given in terms of the underlying polyhedral cell complex and it is related to irreducible decompositions of $\mathcal{J}$.

1. Introduction

Given a free resolution of an ideal $\mathcal{J}$ of holomorphic functions, in [2] a vector valued so called Noetherian residue current R was constructed, which has the property that the ideal of holomorphic functions that annihilate R is precisely $\mathcal{J}$. This relation generalizes the well known duality theorem for Coleff-Herrera currents for complete intersection ideals, due to Dickenstein and Sessa, [7], and Passare, [16].

The degree of explicitness of the current R of course directly depends on the degree of explicitness of the resolution. In case $\mathcal{J}$ is a complete intersection the Koszul complex is exact and the corresponding current is the classical Coleff-Herrera current, [6], as shown in [17] and [1]. In general, though, explicit resolutions are hard to find. In this paper we will focus on monomial ideals, for which there has recently been a lot of work done, see for example the book [13] and the references mentioned therein. Because of their simplicity and nice combinatorial description monomial ideals serve as a good toy model for illustrating general ideas and results in commutative algebra and algebraic geometry, see [19] for examples, which make them a natural first example to consider. In [21] residue currents of Bochner-Martinelli type, in the sense of [17], were computed for monomial ideals, and in [2], there are presented some explicit computations of Noetherian residue currents of certain simple monomial ideals. On the other hand many results for general ideals can be proved by specializing to monomial ideals. In fact, recall that the existence of Bochner-Martinelli as well as Noetherian residue currents is proved by reducing to a monomial situation by resolving singularities.

The aim of this paper is to compute Noetherian residue currents associated with monomial ideals and by that also illustrate the extended duality theorem. We will consider so called cellular resolutions, which were introduced by Bayer and Sturmfels in [3], and which can be nicely encoded into polyhedral cell complexes. The construction will be described in Section 2.

Our results, which are presented in Section 3, concern Artinian monomial ideals, that is (monomial) ideals with zero-dimensional variety. A priori the Noetherian residue current R corresponding to a cellular resolution has one entry R_F for each $(n-1)$-dimensional face F of the underlying polyhedral cell complex. The main technical result in this paper, Proposition 3.1, asserts that each R_F is a certain nice Coleff-Herrera current:

$$c\,\bar\partial\Big[\frac{1}{z_1^{\alpha_1}}\Big]\wedge\ldots\wedge\bar\partial\Big[\frac{1}{z_n^{\alpha_n}}\Big],$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$ can be read off from the cell complex and c is a constant. In particular, if $c\neq 0$ the ideal of functions annihilating R_F, $\operatorname{Ann} R_F$, is $(z_1^{\alpha_1},\ldots,z_n^{\alpha_n})$. A monomial ideal of this form, where the generators are powers of variables, is called *irreducible*. One can show that every monomial ideal can be written as a finite intersection of irreducible ideals; this is called an *irreducible decomposition* of the ideal. Since one has to annihilate all entries R_F to annihilate R, $\bigcap \operatorname{Ann} R_F$ yields an irreducible decomposition of the ideal $\operatorname{Ann} R$, which by the duality theorem equals $\mathcal{J}$, and so the (nonvanishing) entries of R can be seen to correspond to components in an irreducible decomposition. In particular, the number of nonvanishing entries are bounded from below by the minimal number of components in an irreducible decomposition.

In general, we can not extract enough information from our computations to determine which entries R_F that are nonvanishing. Still, for "most" monomial ideals we can; if the monomial ideal $\mathcal{J}$ is generic, which means that the exponents in the set of minimal generators fulfill a certain genericity condition (see Section 2 for a precise definition), then Theorem 3.3 states that R_F is nonvanishing precisely when F is a facet of the Scarf complex introduced by Bayer, Peeva and Sturmfels, [4]. In particular, if the underlying cell complex is the Scarf complex, then all entries of R are nonvanishing. The cellular resolution so obtained is in fact a minimal resolution of the generic ideal $\mathcal{J}$. Theorem 3.5 asserts that whenever the cellular resolution is minimal, the corresponding Noetherian residue current has only nonvanishing entries. Also, the number of entries is equal to the minimal number of components in an irreducible decomposition.

The technical core of this paper is the proof of Proposition 3.1, which is given in Section 4. It is very much inspired by [21], where similar results were obtained for currents of Bochner-Martinelli type corresponding to the Koszul complex. When considering general cellular resolutions the computations get more involved though; in particular, they involve finding inverses of all mappings in the resolution. As in [21], the proof amounts to computing currents in a certain toric manifold constructed from the generators of the ideal, using ideas originally due to Khovanskii [11] and Varchenko [20]. Once Proposition 3.1 is proved, Theorems 3.3 and 3.5 follow easily by invoking results from [4] and [12].

2. Preliminaries and background

Let us start by briefly recalling the construction of residue currents in [2]; for details we refer to this paper. Consider an arbitrary complex of Hermitian holomorphic vector bundles over a complex manifold Ω,

$$0 \to E_N \xrightarrow{f_N} \dots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0, \tag{2.1}$$

that is exact outside an analytic variety Z of positive codimension, and suppose that the rank of E_0 is 1. In $\Omega \setminus Z$, let σ_k be the minimal inverse of f_k, with respect to some Hermitian metric, let $\sigma = \sigma_0 + \ldots + \sigma_N$, $u = \sigma(I-\bar\partial\sigma)^{-1} = \sigma+\sigma(\bar\partial\sigma)+\sigma(\bar\partial\sigma)^2+\ldots$, and let R be the analytic continuation of $\bar\partial|F|^{2\lambda} \wedge u$ to $\lambda = 0$, where F is any tuple of holomorphic functions that vanishes on Z. It turns out that R is a well defined current taking values in $\operatorname{End}(E)$, where $E = \oplus_k E_k$, which has support on Z, and which in a certain way measures the lack of exactness of the associated complex of locally free sheaves of $\mathcal{O}$-modules $\mathcal{O}(E_k)$ of holomorphic sections of E_k,

$$0 \to \mathcal{O}(E_N) \xrightarrow{f_N} \cdots \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0). \tag{2.2}$$

In particular, if $\mathcal{J}$ is the ideal sheaf $\operatorname{Im}(\mathcal{O}(E_1) \to \mathcal{O}(E_0))$ and $\varphi \in \mathcal{O}(E_0)$ fulfills that the (E-valued) current $R\varphi = 0$, then locally $\varphi \in \mathcal{J}$.

Moreover, letting R^ℓ_k denote the component of R that takes values in $\operatorname{Hom}(E_\ell, E_k)$ and $R^\ell = \sum_k R^\ell_k$, it turns out that $R^\ell = 0$ for $\ell \geq 1$ is equivalent to that (2.2) is exact, in other words that it is a resolution of $\mathcal{O}(E_0)/\mathcal{J}$, see Theorem 4.1 in [2]. In this case, $R\varphi = 0$ precisely when $\varphi \in \mathcal{J}$ (Theorem 4.3 in [2]), and we say that R is *Noetherian*. The notion comes from the analogy with Noetherian operators (introduced in [10] and [15]), which are differential operators that can be used to characterize ideals.

Let us continue with the construction of cellular complexes from [3]. Let S be the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$ and let $\deg m$ denote the multidegree of a monomial m in S. When nothing else is mentioned we will assume that monomials and ideals are in S.

Next, a polyhedral cell complex X is a finite collection of convex polytopes (in a real vector space $\mathbb{R}^d$ for some d), the *faces* of X, that fulfills that if $F \in X$ and G is a face of F (for the definition of a face of a polytope, see for example [22]), then $G \in X$, and moreover if F and G are in X, then $F \cap G$ is a face of both F and G. The dimension of a face F, $\dim F$, is defined as the dimension of its affine hull (in $\mathbb{R}^d$) and the dimension of X, $\dim X$, is defined as $\max_{F\in X} \dim F$. Let X_k denote the set of faces of X of dimension $(k-1)$ (X_0 should be interpreted as $\{\emptyset\}$). Faces of dimension 0 are called *vertices*. We will frequently identify $F \in X$ with its set of vertices. Maximal faces (with respect to inclusion) are called *facets*. A face F is a *simplex* if the number of vertices, $|F|$, is equal to $\dim F + 1$. If all faces of X are simplices, we say that X is a *simplicial complex*. A polyhedral cell complex $X' \subset X$ is said to be a *subcomplex* of X.

Moreover, we say that X is *labeled* if there is monomial m_i in S associated to each vertex i. An arbitrary face F of X is then labeled by the least common multiple of the labels of the vertices of F, that is $m_F = \text{lcm}\{m_i | i \in F\}$. Let $\mathbb{N}^n \ni \alpha_F = \deg(m_F)$. By $\mathbb{N}$ we mean $0, 1, 2, \ldots$. We will sometimes be sloppy and not differ between the faces of labeled complex and their labels.

Now, let M be a monomial ideal in S with minimal generators $\{m_1, \ldots, m_r\}$ (recall that the set of minimal generators of a monomial ideal is unique). Throughout this paper M will be supposed to be of this form if nothing else is mentioned. Moreover, let X be a polyhedral cell complex with vertices $\{1, \ldots, r\}$ endowed with some orientation and labeled by $\{m_i\}$. We will associate with X a graded complex of free S-modules: for $k = 0, \ldots, \dim X + 1$, let A_k be the free S-module with basis $\{e_F\}_{F\in X_k}$ and let the differential $f_k : A_k \to A_{k-1}$ be defined by

$$f_k : e_F \mapsto \sum_{\text{facets } G \subset F} \text{sgn}\,(G, F)\, \frac{m_F}{m_G}\, e_G, \tag{2.3}$$

where the sign sgn (G, F) $(= \pm 1)$ comes from the orientation on X. Note that m_F/m_G is a monomial. The complex

$$\mathbb{F}_X : 0 \longrightarrow A_{\dim X - 1} \xrightarrow{f_{\dim X-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

is the *cellular complex supported on* X, which was introduced in [3]. It is exact if the labeled complex X fulfills a certain acyclicity condition. More precisely, for $\beta \in \mathbb{N}^n$ let $X_{\preceq\beta}$ denote the subcomplex of X consisting of all faces F for which $\alpha_F \leq \beta$ with respect to the usual ordering in $\mathbb{Z}^n$. Then $\mathbb{F}_X$ is exact if and only if $X_{\preceq\beta}$ is acyclic, which means that it is empty or has zero reduced homology, for all $\beta \in \mathbb{N}^n$, see Proposition 4.5 in [13]. We then say that $\mathbb{F}_X$ is a *cellular resolution* of S/M.

In particular, if X is the $(r-1)$-simplex this condition is fulfilled and we obtain the classical *Taylor resolution*, introduced by Diana Taylor, [18]. Note that if M is a complete intersection, then the Taylor resolution coincides with the Koszul complex. If X is an arbitrary simplicial complex, $\mathbb{F}_X$ is the more general *Taylor complex*, introduced in [4]. Observe that if X is simplicial the orientation comes implicitly from the ordering on the vertices.

Recall that a graded free resolution $\cdots \longrightarrow A_k \xrightarrow{f_k} A_{k-1} \longrightarrow \cdots$ is *minimal* if and only if for each k, f_k maps a basis of A_k to a minimal set of generators of $\operatorname{Im} f_k$, see for example Corollary 1.5 in [9]. The Taylor complex $\mathbb{F}_X$ is a minimal resolution if and only if it is exact and for all $F \in X$, the monomials m_F and $m_{F\setminus i}$ are different, see Lemma 6.4 in [13].

Now, to put the cellular resolutions into the context of [2], let (2.1) be the vector bundle complex where ($N = \dim X + 1$ and) E_k is a trivial bundle over $\mathbb{C}^n$ of rank $|X_k|$, endowed with the trivial metric, and with a global frame $\{e_F\}_{F\in X_k}$, and where the differential is given by (2.3). Alternatively, we can regard f_k as a section of $E_k^* \otimes E_{k-1}$, that is

$$f_k = \sum_{F\in X_k} \sum_{\text{facets } G\subset F} \operatorname{sgn}(G, F)\, \frac{m_F}{m_G}\, e_F^* \otimes e_G.$$

We will frequently say that the corresponding residue current R is associated with X, and we will use R_F to denote the coefficient of $e_F \otimes e_\emptyset^*$.

It is well known that the induced sheaf complex (2.2) is exact if and only if $\mathbb{F}_X$ is. (For example it can be seen from the Buchsbaum-Eisenbud theorem, Theorem 20.9 in [8], and residue calculus - the proof of Theorem 4.1 in [2].)

Observe that the elements in S (holomorphic polynomials) can be regarded as holomorphic sections of E_0. In this paper, by the annihilator ideal of a current T, $\operatorname{Ann} T$, we will mean the ideal in S which consists of the elements $\varphi \in S$ for which $R\varphi = 0$.

For $b = (b_1, \ldots, b_n) \in \mathbb{N}^n$ we will use the notation $\mathfrak{m}^b$ for the irreducible ideal $(z_1^{b_1}, \ldots, z_n^{b_n})$. If $M = \cap_{i=1}^q \mathfrak{m}^{b^i}$, for some $b^i \in \mathbb{N}^n$, is an irreducible decomposition of the monomial ideal M, such that no intersectand can be omitted the decomposition is said to be *irredundant*, and the ideals $\mathfrak{m}^{b^i}$ are then called the *irreducible components* of M. One can prove that each monomial ideal M in S has a unique irredundant irreducible decomposition. Giving the irreducible components is in a way dual to giving the generators of the ideal (see Chapter 5 on Alexander duality in [13]), and the uniqueness of the irredundant irreducible decomposition corresponds to the uniqueness of the set of minimal generators of a monomial ideal. This duality will be illustrated in Example 1.

We will be particularly interested in so called generic monomial ideals. A monomial $m' \in S$ *strictly divides* another monomial m if m' divides m/z_i for all variables z_i dividing m. We say that a monomial ideal M is *generic* if whenever two distinct minimal generators m_i and m_j have the same positive degree in some variable, then there exists a third generator m_k that strictly divides the least common multiple of m_i and m_j. In particular M is generic if no two generators have the same positive degree in any variable. Almost all monomial ideals are generic in the sense that those which fail to be generic lie on finitely many hyperplanes in the matrix space of exponents, see [4].

We will use the notation $\bar\partial[1/f]$ for the analytic continuation of $\bar\partial|f|^{2\lambda}/f$ to $\lambda=0$, and analogously by $[1/f]$ we will mean $|f|^{2\lambda}/f|_{\lambda=0}$, that is, just the principal value of $1/f$. By iterated integration by parts we have that

$$\int_z \bar\partial\Big[\frac{1}{z^p}\Big]\wedge\varphi dz = \frac{2\pi i}{(p-1)!}\frac{\partial^{p-1}}{\partial z^{p-1}}\varphi(0). \tag{2.4}$$

In particular, the annihilator of $\bar\partial[1/z^p]$ is (z^p).

3. Residue currents constructed from cellular resolutions

We are now ready to present our results, which concern residue currents R associated with cellular complexes of Artinian monomial ideals. We are interested in the component R^0, which takes values in $\mathrm{Hom}\,(E_0, E)$. In fact, when (2.2) is exact $R=R^0$. From Proposition 3.2 in [2] we know that if M is Artinian, then $R^0=R^0_n$, where R^0_n is a $\mathrm{Hom}\,(E_0,E_n)$-valued current. Thus, a priori we know that R^0 consists of one entry $R_F\ e_F\otimes e^*_\emptyset$ for each $F\in X_n$. We will suppress the factor $e^*_\emptyset$ in the sequel.

Proposition 3.1. *Let M be an Artinian monomial ideal, and let R be the residue current associated with the polyhedral cell complex X. Then*

$$R^0=\sum_{F\in X_n} R_F\ e_F, \tag{3.1}$$

where

$$R_F = c_F\ \bar\partial\Big[\frac{1}{z_1^{\alpha_1}}\Big]\wedge\ldots\wedge\bar\partial\Big[\frac{1}{z_n^{\alpha_n}}\Big]. \tag{3.2}$$

Here c_F is a constant and $(\alpha_1,\ldots,\alpha_n)=\alpha_F$. If any of the entries of α_F is 0*,* (3.2) *should be interpreted as* 0*.*

Note that Proposition 3.1 gives a complete description of R^0 except for the constants c_F. We are particularly interested in whether the c_F are zero or not. Indeed, note that

$$\mathrm{Ann}\,\bar\partial\Big[\frac{1}{z_1^{\alpha_1}}\Big]\wedge\ldots\wedge\bar\partial\Big[\frac{1}{z_n^{\alpha_n}}\Big]=\mathfrak{m}^{\alpha_F},$$

so that $\mathrm{Ann}\,R_F=\mathfrak{m}^{\alpha_F}$ if $c_F\neq 0$. Note in particular that $\mathrm{Ann}\,R_F$ depends only on c_F and m_F and not on the particular vertices of F nor the remaining faces in X. Furthermore, to annihilate R^0 one has to annihilate each entry R_F and therefore

$$\mathrm{Ann}\,R^0=\bigcap_{F\in X;\ c_F\neq 0}\mathfrak{m}^{\alpha_F}.$$

Now, suppose that the cellular complex $\mathbb{F}_X$ is exact. Then, $R = R^0$, and from Theorems 4.1 and 9.3 in [2] we know that

$$\operatorname{Ann} R = M.$$

Thus a necessary condition for c_F to be nonvanishing is that $M \subset \mathfrak{m}^{\alpha_F}$. In general though, Proposition 3.1 does not give enough information to give a sufficient condition, as will be illustrated in Example 2.

Below we will discuss two situations, however, in which we can determine exactly which c_F that are nonzero.

First we will consider generic monomial ideals. To this end let us introduce the *Scarf complex* Δ_M of M, which is the collection of subsets $I \subset \{1, \dots, r\}$ whose corresponding least common multiple m_I is unique, that is,

$$\Delta_M = \{I \subset \{1, \dots, r\} | m_I = m_{I'} \Rightarrow I = I'\}.$$

One can prove that the Scarf complex is a simplicial complex, and that its dimension is a most $n-1$. In fact, when M is Artinian, Δ_M is a regular triangulation of $(n-1)$-simplex. For details, see for example [13]. In [4] (Theorem 3.2) it was proved that if M is generic, then the cellular complex supported on Δ_M gives a resolution of S/M, which is moreover minimal. Furthermore, if M in addition is Artinian, then

$$M = \bigcap_{F \text{ facet of } \Delta_M} \mathfrak{m}^{\alpha_F}, \tag{3.3}$$

yields the unique irredundant irreducible decomposition of M, as follows as a special case of their Theorem 3.7. To be precise, originally in [4], a less inclusive definition of generic ideals was used, but the results above were extended in [14] to the more general definition of generic ideals we use.

We can now deduce the following.

Proposition 3.2. *Let M be an Artinian generic monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is exact. Then c_F in* (3.2) *is non-zero if and only if $F \in X_n$ is a facet of the Scarf complex Δ_M.*

Proof. Suppose that $F \in X_n$ is not a facet of Δ_M. We show that $M \not\subset \mathfrak{m}^{\alpha_F}$, which forces c_F to be zero.

Let J be the largest subset of $\{1, \dots, r\}$ such that $m_J = m_F$. Then for some $j \in J$ it holds that $m_{J \setminus j} = m_F$, as follows from the definition of Δ_M. If m_j strictly divides m_F then clearly $m_j \notin \mathfrak{m}^{\alpha_F}$ and we are done. Otherwise, it must hold for some $k \in J \setminus j$ that m_k and m_j have the same positive degree in one of the variables. Then, since M is generic, there is a generator m_ℓ that strictly divides the least common multiple of m_j and m_k and consequently also strictly divides m_F. Hence $m_\ell \notin \mathfrak{m}^{\alpha_F}$.

On the other hand, since (3.3) is irredundant, c_F has to be nonzero whenever F is a facet of Δ_M. □

Thus, to sum up, Proposition 3.1 and Proposition 3.2 yield the following description of the Noetherian residue current of a generic monomial ideal.

Theorem 3.3. *Let M be an Artinian generic monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is exact. Then*

$$R = \sum_{F \text{ facet of } \Delta_M} R_F \; e_F,$$

where Δ_M is the Scarf complex of M, R_F is given by (3.2)*, and the constant c_F there is nonvanishing.*

In particular if we choose X as the Scarf complex Δ_M we get that all coefficients c_F are nonzero.

Remark 1. Observe that it follows from Theorem 3.3 that X must contain the Scarf complex as a subcomplex. Compare to Proposition 6.12 in [13]. □

An immediate consequence is the following.

Corollary 3.4. *Let M be an Artinian generic monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is exact. Then*

$$M = \bigcap_{F \in X} Ann\, R_F$$

yields the irredundant irreducible decomposition of M.

Another situation in which we can determine the set of nonvanishing constants c_F is when $\mathbb{F}_X$ is a minimal resolution of S/M. Indeed, in [12] (Theorem 5.12, see also Theorem 5.42 in [13]) was proved a generalization of Theorem 3.7 in [4]; if M is Artinian and $\mathbb{F}_X$ is a minimal resolution of S/M, then the irredundant irreducible decomposition is given by

$$M = \bigcap_{F \text{ facet of } X} \mathfrak{m}^{\alpha_F}. \tag{3.4}$$

Hence, from (3.4) and Proposition 3.1 we conclude that in this case all c_F are nonvanishing.

Theorem 3.5. *Let M be an Artinian generic monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is a minimal resolution of S/M. Then*

$$R = \sum_{F \text{ facet of } X} R_F \; e_F,$$

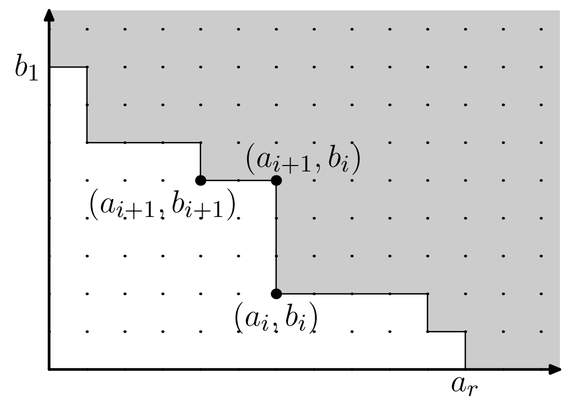

FIGURE 1. The staircase diagram of M in Example 1.

where R_F *is given by* (3.2) *and the constant* c_F *there is nonvanishing.*

Finally, we should remark, that even though we can not determine the set of non-vanishing entries of a Noetherian residue current associated with an arbitrary cell complex, we can still estimate the number of nonvanishing entries from below by the number or irreducible components of the corresponding ideal.

Let us now illustrate our results by some examples. First observe that the ideal $(z^A) = (z^a = z_1^{a_1} \cdots z_n^{a_n} | a \in A \subset \mathbb{N}^n)$ in S is precisely the set of functions that have support in $\bigcup_{a \in A}(a + \mathbb{R}^n_+)$, where

$$\operatorname{supp} \sum_{a \in \mathbb{Z}^n} c_a z^a = \{a \in \mathbb{Z}^n | c_a \neq 0\},$$

and thus we can represent the ideal by this set, see Figure 1. Such pictures of monomial ideals are usually referred to as *staircase diagrams.* The generators $\{z^a\}$ should be identified as the "inner corners of" the staircase, whereas the "outer corners" correspond to the exponents in the irredundant irreducible decomposition.

Example 1. Let us consider the case when $n = 2$. Note that then all monomial ideals are generic. If M is an Artinian monomial ideal, we can write

$$M = (w^{b_1}, z^{a_2} w^{b_2}, \ldots, z^{a_{r-1}} w^{a_{r-1}}, z^{a_r}),$$

for some integers $a_2 < \ldots < a_r$ and $b_1 > \ldots > b_{r-1}$. Now Δ_M is one-dimensional and its facets are the pairs of adjacent generators in the staircase. Moreover $m_{\{i,i+1\}} = z^{a_{i+1}} w^{b_i}$, which corresponds precisely to the ith outer corner of the staircase. Thus, according to Theorem 3.3 the Noetherian

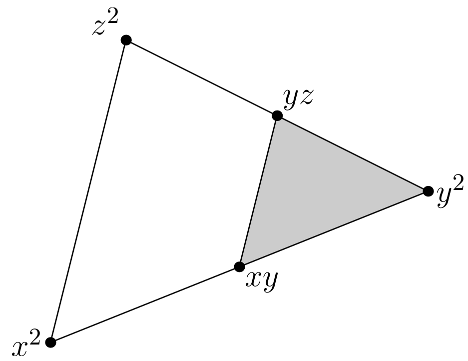

FIGURE 2. The Scarf complex Δ_M of the ideal M in Example 2.

residue current R associated with a cellular resolution of M is of the form

$$R = \sum_{i=1}^{n-1} c_i \ \bar\partial \left[\frac{1}{z^{a_{i+1}}}\right] \wedge \bar\partial \left[\frac{1}{w^{b_i}}\right] e_{\{i,i+1\}},$$

for some nonvanishing constants c_i. The annihilator of the ith entry is the irreducible component $(z^{a_{i+1}}, z^{b_i})$.

Figure 1 illustrates the two ways of thinking of M, either as a staircase with inner corners (a_i, b_i), corresponding to the generators, or as a staircase with outer corners (a_{i+1}, b_i), corresponding to the irreducible components or equivalently the annihilators of the entries of R. □

Let us also give an example that illustrates how we in general fail to determine the set of nonzero c_F when the ideal is not generic.

Example 2. Consider the non-generic ideal

$$M = (x^2, xy, y^2, yz, z^2) =: (m_1, \dots, m_5).$$

The Scarf complex Δ_M, depicted in Figure 2, consists of the 2-simplex $\{2,3,4\}$ together with the one-dimensional "handle" made up from the edges $\{1,2\}, \{1,5\}$ and $\{4,5\}$. Moreover the irredundant irreducible decomposition is given by $M = (x, y^2, z) \cap (x^2, y, z^2)$.

Let X be the full 4-simplex with vertices $\{1, \dots, 5\}$ corresponding to the Taylor resolution. It is then easily checked that for the associated Noetherian residue current, $c_{\{2,3,4\}}$ and at least one of $c_{\{1,2,5\}}$ and $c_{\{1,4,5\}}$ have to be zero, whereas $c_{\{1,2,4\}}$ and $c_{\{2,4,5\}}$ can be either zero or nonzero. The remaining c_F has to be zero since for them $M \not\subset \mathfrak{m}^{\alpha_F}$. Thus, in general Proposition 3.1 does not provide enough information to determine which of the coefficients c_F that vanish.

However, let instead X' be the polyhedral cell complex consisting of the two facets $\{2,3,4\}$ and $\{1,2,4,5\}$, that is the triangle and the quadrilateral in Figure 2. The resolution obtained from X', which is in fact the so called Hull resolution introduced in [3], is minimal. Thus, according to Theorem 3.5 the two entries of the associated residue current, which correspond to the two facets of X' are both nonvanishing, with annihilators (x, y^2, z) and (x^2, y, z^2) respectively. This could of course be seen directly since we already knew the irredundant irreducible decomposition of M. □

4. Proof of Proposition 3.1

The proof of Proposition 3.1 is very much inspired by the proof of Theorem 3.1 in [21]. We will compute R^0 as a push-forward of corresponding currents on a certain toric variety. To do this we will have use for the following simple lemma which is proved essentially by integration by parts.

Lemma 4.1. *Let v be a strictly positive smooth function in $\mathbb{C}$, φ a test function in $\mathbb{C}$, and p a positive integer. Then*

$$\lambda \mapsto \int v^\lambda |z|^{2\lambda} \varphi(z) \frac{dz \wedge d\bar{z}}{z^p}$$

and

$$\lambda \mapsto \int \bar{\partial}(v^\lambda |z|^{2\lambda}) \wedge \varphi(z) \frac{dz}{z^p}$$

both have meromorphic continuations to the entire plane with poles at rational points on the negative real axis. At $\lambda = 0$ they are both independent of v and equal to $[1/z^p]$ and $\bar{\partial}[1/z^p]$ respectively (acting on suitable test forms). Moreover, if $\varphi(z) = \bar{z}\psi(z)$ or $\varphi = d\bar{z} \wedge \psi$, then the value of the second integral at $\lambda = 0$ is zero.

Before presenting the proof of Proposition 3.1, let us just give a very brief overview of it. First, we will give a description of the current R^0 in terms of the cell complex X. After that we will introduce the toric variety mentioned above and show that R^0 equals the push-forward of certain currents on this variety. Finally, we will compute these currents.

Let us start by recalling from Section 3 in [2] that R^0_n is the analytic continuation to $\lambda = 0$ of $\bar{\partial}|F|^{2\lambda} \wedge u^0_n$, where F is a holomorphic function that vanishes at the origin and

$$u^0_n = (\bar{\partial}\sigma_n)(\bar{\partial}\sigma_{n-1}) \cdots (\bar{\partial}\sigma_2)\sigma_1.$$

By Lemma 2.1 in [2]

$$\sigma_k = \frac{\delta_{f_k}^{q_k - 1} S_k}{|F_k|^2}, \tag{4.1}$$

where q_k is the rank of f_k, δ_{f_k} is contraction with f_k, $F_k = (f_k)^{q_k}/q_k!$ and $S_k = (s_k)^{q_k}/q_k!$ is the dual section of F_k. For details, we refer to Section 2 in [2]. Furthermore, s_k is the section of $E_k \otimes E_{k-1}^*$ that is dual to f_k with respect to the trivial metric, that is,

$$s_k = \sum_{G \in X_k} \sum_{\text{facets } H \subset G} \text{sgn}\,(H,G)\, \frac{\overline{m_G}}{\overline{m_H}}\, e_G \otimes e_H^*.$$

Here $\overline{m_G}$ just denotes the conjugate of m_G. Notice that, since $\sigma_k \sigma_{k-1} = 0$, as follows by definition, it holds that only the terms obtained when the $\bar\partial$ fall in the numerator survive, and so

$$u_n^0 = \frac{\bar\partial(\delta_{f_n}^{q_n-1} S_n) \cdots \bar\partial(\delta_{f_2}^{q_2-1} S_2)\delta_{f_1}^{q_1-1} S_1}{|F_n|^2 \cdots |F_1|^2}.$$

Observe furthermore that the numerator of the right hand side of (4.1) is a sum of terms of the form

$$v_k = \pm|\omega_k|^2 \frac{\overline{m_G}}{\overline{m_H}}\, e_G \otimes e_H^*, \tag{4.2}$$

where $G \in X_k$ and $H \in X_k$ is a facet of G and

$$\omega_k = \frac{m_{G_1} \cdots m_{G_{q_k-1}}}{m_{H_1} \cdots m_{H_{q_k-1}}},$$

where for $1 \leq \ell \leq q_k - 1$, $G_\ell \in X_k$ and $H_\ell \in X_{k-1}$ is a facet of G_ℓ. The $\pm$ in front of $|\omega_k|$ depends on the orientation on X. Note that the coefficients are monomials. It follows that u_n^0 is a sum of terms of the form

$$u_v = u_{\{v_1,\dots,v_n\}} = \frac{(\bar\partial v_n) \cdots (\bar\partial v_2) v_1}{|F_n|^2 \cdots |F_1|^2},$$

where each v_k is of the form (4.2), and where

$$v_n \cdots v_1 = \pm|\omega_n \cdots \omega_1|^2\; \overline{m_F}\; e_F \otimes e_\emptyset^* \tag{4.3}$$

for some $F \in X_n$.

Observe that each F_k has monomial entries. By ideas originally from [11] and [20], one can show that there exists a toric variety $\mathcal{X}$ and a proper map $\widetilde{\Pi} : \mathcal{X} \to \mathbb{C}^n$ that is biholomorphic from $\mathcal{X} \setminus \widetilde{\Pi}^{-1}(\{z_1 \cdots z_n = 0\})$ to $\mathbb{C}^n \setminus \{z_1 \cdots z_n = 0\}$, such that locally, in a coordinate chart $\mathcal{U}$ of $\mathcal{X}$, it holds for all k that the pullback of one of the entries of F_k divides the pullbacks of all entries of F_k. In other words we can write $\widetilde{\Pi}^* F_k = F_k^0 F_k'$, where F_k^0 is a monomial and F_k' is nonvanishing, and analogously we have $\widetilde{\Pi}^* F = F^0 F'$. The construction is based on the so called Newton polyhedra associated with F_k and we refer to [5] and the references therein for details. The mapping $\widetilde{\Pi}$ is locally in the chart $\mathcal{U}$ given by

$$\begin{aligned} \Pi : \mathcal{U} &\to \mathbb{C}^n \\ t &\mapsto t^P, \end{aligned}$$

where $P = (\rho_{ij})$ is a matrix with determinant ± 1 and t^P is a shorthand notation for $(t_1^{\rho_{11}} \cdots t_n^{\rho_{n1}}, \ldots, t_1^{\rho_{1n}} \cdots t_n^{\rho_{nn}})$. Hence, the pullback Π^* transforms the exponent of monomials by the linear mapping P;

$$\Pi^* z^a = \Pi^* z_1^{a_1} \cdots z_n^{a_n} = t_1^{\rho_1 \cdot a} \cdots t_n^{\rho_n \cdot a} = t^{Pa},$$

where ρ_i denotes the ith row of P, so that the pullback of a monomial is itself a monomial.

Now, from Proposition 2.2 in [2] we know that $F_k^0 \Pi^* \sigma_k$ is smooth in $\mathcal{U}$. However,

$$F_k^0 \Pi^* \sigma_k = \sum_j \frac{\Pi^* v_k^j}{\bar{F}_k^0 |F_k'|^2} = \sum_{\alpha \in \mathbb{N}^n} \sum_{\deg \Pi^* v_k^j = \alpha} \frac{\Pi^* v_k^j}{\bar{F}_k^0 |F_k'|^2},$$

where v_k^j are just the different terms v_k that appear in the numerator of σ_k. Therefore clearly for each $\alpha \in \mathbb{N}^n$ the sum

$$\sum_{\deg \Pi^* v_k^j = \alpha} \frac{\Pi^* v_k^j}{\bar{F}_k^0 |F_k'|^2},$$

which is just equal to $Ct^\alpha / (\bar{F}_k^0 |F_k'|^2)$ for some constant C, has to be smooth and consequently $t^\alpha / (\bar{F}_k^0 |F_k'|^2)$ is smooth or $C = 0$. Hence, to compute R^0 we only need to consider terms u_v, where $v = (v_1, \ldots, v_n)$ is such that $\widetilde{\Pi}^* v_k / (\bar{F}_k^0 |F_k'|^2)$ is smooth on $\mathcal{X}$ for all k. For such a v let us define

$$R_v^0 := \bar{\partial} |F|^{2\lambda} \wedge u_v |_{\lambda=0} \quad \text{and} \quad \widetilde{R}_v^0 := \widetilde{\Pi}^* (\bar{\partial} |F|^{2\lambda} \wedge u_v)|_{\lambda=0}.$$

From below it follows that R_v^0 and $\widetilde{R}_v^0$ are well defined (globally defined) currents and moreover that $\widetilde{\Pi}_* \widetilde{R}_v^0 = R_v^0$. Furthermore, it is clear that $R^0 = \sum R_v^0$, where the sum is taken over all v. Next, observe that, in view of (4.2), the frame element of u_v is $e_F \otimes e_\emptyset^*$, where $F \in X_n$ is determined by v_n. Hence $R_F\, e_F$ in (3.1) will be the sum of currents R_v^0, where v is such that v_n contains the frame element e_F. Thus, to prove the proposition it suffices to show that R_v^0 is of the desired form.

Let us therefore consider $\widetilde{R}_v^0$ in $\mathcal{U}$. Observe that

$$\widetilde{R}_v^0 = \bar{\partial} |F^0 F'|^{2\lambda} \wedge \frac{\Pi^*((\bar{\partial} v_n) \cdots (\bar{\partial} v_2) v_1)}{|F_n^0 \cdots F_1^0|^2 \nu(t)} \bigg|_{\lambda=0}, \tag{4.4}$$

where $\nu(t) := (|F_n'|\cdots|F_1'|)^2$ is nonvanishing. For further reference, note that $\nu(t)$ only depends on $|t_1|,\dots,|t_n|$. Moreover, let us denote $\deg(F_n^0\cdots F_1^0)$ by $\mathbb{N}^n \ni \gamma = (\gamma_1,\dots,\gamma_n)$ and $\deg(\omega_n\cdots\omega_1)$ by β, and recall that $\deg m_F = \alpha_F$. By Leibniz' rule and Lemma 4.1, recalling (4.3), we see that (4.4) is equal to a sum of terms of the form a constant times

(4.5)
$$\bar\partial\Big[\frac{1}{t_i^{\gamma_i-\rho_i\cdot\beta}}\Big]\otimes\big[\prod_{j\neq i}|t_j|^{2(\rho_j\cdot\beta-\gamma_j)}\big]\wedge\frac{\bar t_i^{\rho_i\cdot(\alpha_F+\beta)-\gamma_i}\prod_{j\neq i}\bar t_j^{\rho_j\cdot\alpha_F-1}}{\nu(t)}\widehat{d\bar t_i}\; e_F\otimes e_\emptyset^*,$$

where t_i is one of the variables which fulfills that t_i divides the monomials F^0 and $F_n^0\cdots F_1^0$, whereas $t_1\cdots t_{i-1}t_{i+1}\cdots t_n$ divides $\Pi^* m_F$. In fact, it is not hard to check that, unless the latter requirement is fulfilled, the corresponding contribution will vanish for symmetry reasons. Here $\widehat{d\bar t_i}$ is just shorthand for $d\bar t_1\wedge\ldots\wedge d\bar t_{i-1}\wedge d\bar t_{1+1}\wedge\ldots\wedge d\bar t_n$. Note that since $\Pi^* v_k/(F_k^0|F_k'|^2)$ is smooth there will be no occurrences of any of the coordinate functions $\bar t_j$ in the denominator, except for them in $\nu(t)$, and in particular it follows that $\gamma_j-\rho_j\cdot\beta\geq 0$ when $j\neq i$. Moreover, due to Lemma 4.1, (4.5) vanishes whenever there is an occurrence of $\bar t_i$ in the numerator. Hence a necessary condition for (4.5) not to vanish is that

$$\rho_i\cdot(\alpha_F+\beta)-\gamma_i=0.$$

We will now compute the action of $\widetilde R_v^0$ on the pullback of a test form $\phi=\varphi(z)dz$ of bidegree $(n,0)$. Here $dz=dz_1\wedge\ldots\wedge dz_n$. Let $\{\mathcal U_\tau\}$ be the cover of $\mathcal X$ that naturally comes from the construction of $\mathcal X$ as described in the proof of Theorem 3.1 in [21], and let $\{\chi_\tau\}$ be a partition of unity on $\mathcal X$ subordinate $\{\mathcal U_\tau\}$. It is not hard to see that we can choose the partition in such a way that the χ_τ are circled, that is, they only depend on $|t_1|,\dots,|t_n|$. Now $\widetilde R_v^0=\sum_\tau\chi_\tau\widetilde R_v^0$. We will start by computing the contribution from our fixed chart $\mathcal U$)with corresponding cutoff function χ), where $\widetilde R_v^0$ is realized as a sum of terms (4.5).

Recall that R has support at the origin; hence it only depends on finitely many derivatives of φ at the origin. Moreover we know that $\bar h$ annihilates R if h is a holomorphic function which vanishes on Z, see Proposition 3.2 in [2]. For that reason, to determine R_v^0 it is enough to consider the case when φ is a holomorphic polynomial. We can write φ as a finite Taylor expansion,

$$\varphi=\sum_a\frac{\varphi_a(0)}{a!}z^a,$$

where $a=(a_1,\dots,a_n)$, $\varphi_a=\frac{\partial^{a_1}}{\partial z_1^{a_1}}\cdots\frac{\partial^{a_n}}{\partial z_n^{a_n}}\varphi$ and $a!=a_1!\cdots a_n!$, with pull-back to $\mathcal{U}$ given by

$$\Pi^*\varphi=\sum_a\frac{\varphi_a(0)}{a!}t^{Pa}=\sum_a\frac{\varphi_a(0)}{a!}t_1^{\rho_1\cdot a}\cdots t_n^{\rho_n\cdot a}.$$

Moreover a computation similar to the proof of Lemma 4.2 in [21] yields

$$\Pi^*dz=\det P\ t^{(P-I)\mathbf{1}}\ dt,$$

where $\mathbf{1}=(1,1,\dots,1)$.

Since $\det P\neq 0$, it follows that $\chi\widetilde{R}^0_v.\Pi^*\phi$ is equal to a sum of terms of the form a constant times

$$\int\bar\partial\Big[\frac{1}{t_i^{\rho_i\cdot\alpha_F}}\Big]\otimes\big[\prod_{j\neq i}|t_j|^{2(\rho_j\cdot\beta-\gamma_j)}\big]\wedge\frac{\prod_{j\neq i}\bar t_j^{\rho_j\cdot\alpha_F-1}}{\nu(t)}\widehat{d\bar t_i}\ e_F\otimes e_\emptyset^*\wedge$$
$$\chi(t)\sum_a\frac{\varphi_a(0)}{a!}t^{Pa}t^{(P-I)\mathbf{1}}dt=\sum_a I_a\ \wedge\frac{\varphi_a(0)}{a!}\ e_F\otimes e_\emptyset^*,$$

where

$$I_a=\int\bar\partial\Big[\frac{1}{t_i^{\rho_i\cdot(\alpha_F-a-\mathbf{1})+1}}\Big]\otimes[\mu_a]\wedge\frac{\chi(t)}{\nu(t)}\ \widehat{d\bar t_i}\wedge dt. \tag{4.6}$$

Here μ_a is the Laurent monomial

$$\mu_a=\prod_{j\neq i}t_j^{\rho_j\cdot(\beta+a+\mathbf{1})-\gamma_j-1}\ \bar t_j^{\rho_j\cdot(\beta+\alpha_F)-\gamma_j-1}.$$

Invoking (2.4) we evaluate the t_i-integral. Since ν and χ depend on $|t_1|,\dots,|t_n|$ it follows that $\frac{\partial^\ell}{\partial t_i^\ell}\frac{\chi}{\nu(t)}|_{t_i=0}=0$ for $\ell\geq 1$ and thus (4.6) is equal to

$$2\pi i\int_{\widehat{t_i}}\frac{\chi(t)|_{t_i=0}[\mu_a]}{\nu(t)|_{t_i=0}}\ \widehat{d\bar t_i}\wedge\widehat{dt_i}, \tag{4.7}$$

if

$$\rho_i\cdot(\alpha_F-a-\mathbf{1})+1=1, \tag{4.8}$$

and zero otherwise. Moreover, for symmetry reasons, (4.7) vanishes unless

$$\rho_j\cdot(\alpha_F-a-\mathbf{1})=0 \tag{4.9}$$

for $j\neq i$, that is, unless μ_a is real.

Thus, since P is invertible, the system of equations (4.8) and (4.9) has the unique solution $a=\alpha_F-\mathbf{1}$ if $\alpha_F\geq\mathbf{1}$. Otherwise there is no solution, since a has to be larger than $(0,\dots,0)$. With this value of a the Laurent monomial μ_a is nonsingular and so the integrand of (4.7),

$$\frac{\chi(t)|_{t_i=0}\ \prod_{j\neq i}|t_j|^{2(\rho_j\cdot(\beta+\alpha_F)-\gamma_j-1)}}{\nu(t)|_{t_i=0}},$$

becomes integrable. Hence I_a is equal to some finite constant if $a=\alpha_F-\mathbf{1}$ and zero otherwise.

Now, recall that the chart $\mathcal{U}$ was arbitrarily chosen. Thus adding contributions from all charts reveals that R_v^0 and thus R_F is of the desired form (3.2), and so Proposition 3.1 follows.

Remark 2. We should compare Proposition 3.1 to Theorem 3.1 in [21]. It states that the residue current of Bochner-Martinelli type of an Artinian monomial ideal is a vector with entries of the form (3.2), but it also tells precisely which of these entries that are non-vanishing. If we had not cared about whether a certain entry was zero or not we could have used the proof of Proposition 3.1 above. Indeed, the Koszul complex, which gives rise to residue currents of Bochner-Martinelli type, can be seen as the cellular complex supported on the full $(r-1)$-dimensional simplex with labels $m_F=\{\prod_{i\in F}m_i\}$. It is not hard to see that the proof above goes through also with this non-conventional labeling. □

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