ON A MONGE-AMPÈRE OPERATOR FOR PLURISUBHARMONIC FUNCTIONS WITH ANALYTIC SINGULARITIES

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Abstract. We study continuity properties of generalized Monge-Ampère operators for plurisubharmonic functions with analytic singularities. In particular, we prove continuity for a natural class of decreasing approximating sequences. We also prove a formula for the total mass of the Monge-Ampère measure of such a function on a compact Kähler manifold.

1. Introduction

We say that a plurisubharmonic (psh) function u on a complex manifold X has *analytic singularities* if locally it can be written in the form

$$u = c \log |F| + b, \tag{1.1}$$

where $c \geq 0$ is a constant, $F = (f_1, \dots, f_m)$ is a tuple of holomorphic functions, and b is bounded. For instance, if f_j are holomorphic functions and a_j are positive rational numbers, then $\log(|f_1|^{a_1} + \cdots + |f_m|^{a_m})$ has analytic singularities.

By the classical Bedford-Taylor theory, [5, 6], if u is of the form (1.1), then in $\{F \neq 0\}$, for any k, one can define a positive closed current $(dd^c u)^k$ recursively as

$$(dd^c u)^k := dd^c\big(u(dd^c u)^{k-1}\big). \tag{1.2}$$

It was shown in [3] that $(dd^c u)^k$ has locally finite mass near $\{F = 0\}$ for any k and that the natural extension $\mathbf{1}_{\{F\neq 0\}}(dd^c u)^{k-1}$ across $\{F = 0\}$ is closed, cf. [3, Eq. (4.8)]. Moreover, by [3, Proposition 4.1], $u\mathbf{1}_{\{F\neq 0\}}(dd^c u)^{k-1}$ has locally finite mass as well, and therefore one can define the Monge-Ampère current

$$(dd^c u)^k := dd^c\big(u\mathbf{1}_{\{F\neq 0\}}(dd^c u)^{k-1}\big) \tag{1.3}$$

for any k.

Demailly, [17] extended Bedford-Taylor's definition (1.2) to the case when the unbounded locus of u is small compared to k in a certain sense; in particular, if u is as in (1.1), then $(dd^c u)^k$ is well-defined in this way as long as $k \leq \operatorname{codim}\{F = 0\} =: p$. Since, a positive closed current of bidegree (k, k) with support on a variety of codimension $> k$ vanishes,

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$\mathbf{1}_{\{F\neq 0\}}(dd^c u)^k = (dd^c u)^k$ for $k \leq p-1$, and it follows that (1.3) coincides with (1.2) for $k \leq p$.

Recall that the Monge-Ampère operators $(dd^c u)^k$ defined by Bedford-Taylor-Demailly have the following continuity property: if u_j is a decreasing sequence of psh functions converging pointwise to u, then $(dd^c u_j)^k \to (dd^c u)^k$ weakly. Moreover, a general psh function u is said to be in the domain of the Monge-Ampère operator $\mathcal{D}(X)$ if, in all open sets $\mathcal{U} \subset X$, $(dd^c u_j)^n$ converge to the same Radon measure for all decreasing sequences of smooth psh u_j converging to u in $\mathcal{U}$. The domain $\mathcal{D}(X)$ was characterized in [10, 11]; in case X is a hyperconvex domain in $\mathbb{C}^n$ $\mathcal{D}(X)$ coincides with the Cegrell class, [14].

In this paper we study continuity properties of the Monge-Ampère operators $(dd^c u)^k$ defined by (1.3). It is not hard to see that general psh functions with analytic singularities do not belong to $\mathcal{D}(X)$, cf. Examples 3.2 and 3.4 below, and therefore we do not have continuity for all decreasing sequences in general. Our main result, however, states that continuity does hold for a large class of natural approximating sequences. It thus provides an alternative definition of $(dd^c u)^k$, and at the same time gives further motivation for that this Monge-Ampère operator is indeed natural.

Theorem 1.1. *Let u be a negative psh function with analytic singularities on a manifold of dimension n. Assume that $\chi_j(t)$ is a sequence of bounded nondecreasing convex functions defined for $t \in (-\infty, 0)$ decreasing to t as $j \to \infty$. Then for every $k = 1, \dots, n$ we have weak convergence of currents*

$$\big(dd^c(\chi_j \circ u)\big)^k \longrightarrow (dd^c u)^k$$

as $j \to \infty$.

For instance, we can take $\chi_j = \max(t, -j)$ or $\chi_j = (1/2)\log(e^{2t} + 1/j)$. Applied to $u = \log|F|$ and $\chi_j = (1/2)\log(e^{2t} + 1/j)$ Theorem 1.1 says that

$$\big(dd^c (1/2)\log(|F|^2 + 1/j)\big)^k \to \big(dd^c \log|F|\big)^k,$$

which was in fact proved already in [2, Proposition 4.4].

By a resolution of singularities the proofs of various local properties of Monge-Ampère currents for psh functions with analytic singularities can be reduced to the case of psh functions with *divisorial singularities*, i.e., psh functions that locally are of the form $u = c\log|f| + v$, where $c \geq 0$, f is a holomorphic function and v is bounded. Since $\log|f|$ is pluriharmonic on $\{f \neq 0\}$, in fact, v is psh. In Section 3 we prove Theorem 1.1 for u of this form; in this case

$$(dd^c u)^k = dd^c\big(u(dd^c v)^{k-1}\big) = dd^c u \wedge (dd^c v)^{k-1}. \tag{1.4}$$

Note that, in light of the Poincaré-Lelong formula,

$$(dd^c u)^k = [f = 0] \wedge (dd^c v)^{k-1} + (dd^c v)^k,$$

where $[f = 0]$ is the current of integration along $\{f = 0\}$ counted with multiplicities.

Our definition of $(dd^c u)^k$ thus relies on the possibility to reduce to the quite special case with divisorial singularities. It seems like an extension to more general psh u must involve some further ideas, cf., Section 6.

We also study psh functions with analytic singularities on compact Kähler manifolds. Recall that if (X, ω) is such a manifold then a function $\varphi\colon X \to \mathbb{R} \cup \{-\infty\}$ is called *ω-plurisubharmonic (ω-psh)* if locally the function $g + \varphi$ is psh, where g is a local potential for ω, i.e., $\omega = dd^c g$. Equivalently, one can require that $\omega + dd^c\varphi \geq 0$. We say that an ω-psh function φ *has analytic singularities* if the functions $g + \varphi$ have analytic singularities. Note that such a φ is locally bounded outside an analytic variety $Z \subset X$ that we will refer to as the *singular set* of φ. If φ is an ω-psh function with analytic singularities, we can define a global positive current $(\omega + dd^c\varphi)^k$, by locally defining it as $(dd^c(g + \varphi))^k$, see Lemma 5.1. We will prove the following formula for the total Monge-Ampère mass:

Theorem 1.2. *Let φ be an ω-psh function with analytic singularities on a compact Kähler manifold (X, ω) of dimension n. Let Z be the singular set of φ. Then*

$$\int_X (\omega + dd^c\varphi)^n = \int_X \omega^n - \sum_{k=1}^{n-1} \int_X \mathbf{1}_Z (\omega + dd^c\varphi)^k \wedge \omega^{n-k}. \tag{1.5}$$

In particular,

$$\int_X (\omega + dd^c\varphi)^n \leq \int_X \omega^n. \tag{1.6}$$

Remark 1.3. Let φ be a general ω-psh function such that the Bedford-Taylor-Demailly Monge-Ampère operator $(\omega + dd^c\varphi)^n$ is well-defined; if φ has analytic singularities, this means that the singular set has dimension 0. Then it follows from Stokes' theorem that equality holds in (1.6). □

To see that in general there is not equality in (1.6) consider the following simple example:

Example 1.4. Let X be the projective space $\mathbb{P}^n$ with the Fubini-Study metric ω and let $n \geq 2$. Define

$$\varphi\big([z_0 : z_1 : \ldots : z_n]\big) := \log\Big(\frac{|z_1|}{|z|}\Big), \quad z \in \mathbb{C}^{n+1} \setminus \{0\}.$$

Since $(dd^c \log|z_1|)^n = 0$ in $\mathbb{C}^{n+1}$, cf. (1.3), it follows that $(\omega + dd^c\varphi)^n = 0$ on $\mathbb{P}^n$. □

In Section 5 we provide a geometric interpretation of Theorem 1.2 which in particular shows that inequality in (1.6) is not an "exceptional case".

The paper is organized as follows. In Section 2 we prove a continuity result for currents of the form

$$u\, dd^c v_1 \wedge \cdots \wedge dd^c v_k,$$

where u is psh and $v_1, \ldots, v_k$ are locally bounded psh, defined by Demailly [15], cf. (1.4). In Section 3 we prove Theorem 1.1 for functions with divisorial singularities and we also characterize when such functions are maximal. The general case of Theorem 1.1 is proved in Section 4. In Section 5 we prove Theorem 1.2. Finally in Section 6 we make some further remarks.

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2. Continuity of certain Monge-Ampère currents

In the seminal paper [6] Bedford and Taylor, see [6, Theorem 2.1], showed that, for $k = 1, \dots, n$ and locally bounded psh functions $u, v_1, \dots, v_k$ on a manifold X of dimension n, the current

$$u\, dd^c v_1 \wedge \cdots \wedge dd^c v_k \tag{2.1}$$

is well-defined and continuous for decreasing sequences. Demailly generalized their definition to the case when u is merely psh; he proved that the current (2.1) has locally finite mass, see [15, Theorem 1.8]. Here we prove the corresponding continuity result.

Theorem 2.1. *Assume that u^j is a sequence of psh functions decreasing to a psh function u and that for $\ell = 1, \dots, k$ the sequence v_ℓ^j of psh functions decreases to a locally bounded psh v_ℓ as $j \to \infty$. Then*

$$u^j\, dd^c v_1^j \wedge \cdots \wedge dd^c v_k^j \longrightarrow u\, dd^c v_1 \wedge \cdots \wedge dd^c v_k$$

weakly as $j \to \infty$.

Proof. By the Bedford-Taylor theorem we have weak convergence

$$S^j := dd^c v_1^j \wedge \cdots \wedge dd^c v_k^j \longrightarrow dd^c v_1 \wedge \cdots \wedge dd^c v_k =: S.$$

By [15, Theorem 1.8] the sequence $u^j S^j$ is locally weakly bounded and thus it is enough to show that, if $u^j S^j \to \Theta$ weakly, then $\Theta = uS$.

Take an elementary positive form α of bidegree $(n-k, n-k)$ and fix j_0 and $\varepsilon > 0$. Then for $j \geq j_0$ we have

$$u^j S^j \wedge \alpha \leq u^{j_0} S^j \wedge \alpha \leq u^{j_0} * \rho_\varepsilon\, S^j \wedge \alpha,$$

where $u^{j_0} * \rho_\varepsilon$ is a standard regularization of u^{j_0} by convolution, i.e., ρ_ϵ is a rotation invariant approximate indentity. Letting $j \to \infty$ we get $\Theta \wedge \alpha \leq u^{j_0} * \rho_\varepsilon\, S \wedge \alpha$ and thus $\Theta \leq uS$.

We will use the following lemma.

Lemma 2.2. *Let $u, v_0, v_1, \dots, v_n$ be psh functions defined in a neighborhood of $\overline{\Omega}$ where Ω is a bounded domain in $\mathbb{C}^n$. Suppose that all of these functions except possibly u are bounded and set $T := dd^c v_2 \wedge \cdots \wedge dd^c v_n$. Assume that $v_0 \leq v_1$ in Ω and $v_0 = v_1$ in $\Omega \cap U$, where U is a neighborhood of $\partial\Omega$. Then*

$$\int_\Omega u\, dd^c v_0 \wedge T \leq \int_\Omega u\, dd^c v_1 \wedge T.$$

Proof. We have

$$\int_\Omega u\, dd^c v_0 \wedge T - \int_\Omega u\, dd^c v_1 \wedge T = \lim_{\varepsilon\to 0}\int_\Omega u * \rho_\varepsilon\, dd^c(v_0 - v_1)\wedge T$$
$$= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_\Omega u*\rho_\varepsilon\, dd^c\big((v_0-v_1)*\rho_\delta\big)\wedge T$$
$$= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_\Omega (v_0-v_1)*\rho_\delta\, dd^c(u*\rho_\varepsilon)\wedge T \leq 0.$$

□

End of proof of Theorem 2.1. We may assume that all functions are defined in a neighborhood of a ball $\overline{B} = \overline{B}(z_0, r)$ and, similarly as in the proof of Bedford-Taylor's theorem, that $v_\ell^j = v_\ell = A(|z - z_0|^2 - r^2)$ near ∂B for some $A > 0$, cf., e.g., the proof of [15, Theorem 1.5]. Since $\Theta \leq uS$, it remains to prove that $\int_B (uS - \Theta)\wedge\omega^{n-k} \leq 0$, where $\omega = dd^c|z|^2$. By successive application of Lemma 2.2 we get

$$\int_B u\, dd^c v_1 \wedge \cdots \wedge dd^c v_k \wedge \omega^{n-k} \leq \int_B u^j dd^c v_1^j \wedge \cdots \wedge dd^c v_k^j \wedge \omega^{n-k}.$$

Therefore,

$$\int_B u\, S\wedge\omega^{n-k} \leq \liminf_{j\to\infty}\int_B u^j dd^c v_1^j\wedge\cdots\wedge dd^c v_k^j \wedge \omega^{n-k} \leq \int_B \Theta\wedge\omega^{n-k},$$

and thus the theorem follows. □

Theorem 2.1 generalizes a result of Demailly (see [18], Proposition III.4.9 on p. 155) who assumed in addition that a complement of the open set where $u, v_1, \ldots, v_k$ are locally bounded has vanishing $(2n-1)$-dimensional Hausdorff measure.

3. The case of divisorial singularities

In this section we first prove a special case of Theorem 1.1.

Theorem 3.1. *Assume that $u = \log|f| + v$ is negative, where f is holomorphic and v is a bounded psh function. Let χ_j be as in Theorem 1.1. Then*

$$\big(dd^c(\chi_j\circ u)\big)^k \longrightarrow dd^c u\wedge (dd^c v)^{k-1}$$

as $j\to\infty$.

Proof. We will use an idea from [8]. Notice that locally on $(-\infty, 0)$, the sequence χ_j' is bounded and tends to 1 uniformly when $j\to\infty$. For each j,

$$\gamma_j(t) := \int_{-1}^t (\chi_j'(s))^k ds + \chi_j(-1)$$

is bounded, convex and nondecreasing on $(-\infty, 0)$, and $\gamma_j' = (\chi_j')^k$, where the derivative exists. Moreover, the sequence γ_j is decreasing and tends to t.

Let us first assume that χ_j, and hence γ_j, are smooth. Since $\log|f|$ is pluriharmonic on $\{f\neq 0\}$ we have that

$$\begin{aligned}\big(dd^c(\chi_j\circ u)\big)^k &= \big(\chi_j''\circ u\,du\wedge d^cu+\chi_j'\circ u\,dd^cu\big)^k\\ &= \big(k\chi_j''\circ u\,du\wedge d^cu+\chi_j'\circ u\,dd^cu\big)\wedge\big(\chi'\circ u\,dd^cu\big)^{k-1}\\ &= d\big((\chi_j'\circ u)^k d^cu\big)\wedge(dd^cu)^{k-1}\\ &= dd^c(\gamma_j\circ u)\wedge(dd^cv)^{k-1}\\ &= dd^c\big(\gamma_j\circ u\,(dd^cv)^{k-1}\big)\end{aligned}$$

there. Since none of the above currents charges the set $\{f=0\}$, the equality

$$(dd^c(\chi_j\circ u))^k = dd^c\big(\gamma_j\circ u\,(dd^cv)^{k-1}\big) \tag{3.1}$$

holds everywhere. If χ_j is not smooth we make a regularization $\chi_{j,\epsilon}=\chi_j*\rho_\epsilon$. Then $\chi_{j,\epsilon}'\to\chi_j'$ in $L^1_{loc}(-\infty,0)$ and hence the associated $\gamma_{j,\epsilon}$ tend to γ_j locally uniformly. We conclude that (3.1) still holds. The theorem now follows from (3.1) and Theorem 2.1. □

The following example shows that $(dd^cu_j)^k$ does not converge to $(dd^cu)^k$ for general decreasing sequences of psh functions $u_j\to u$.

Example 3.2. Let

$$u(z)=\log|z_1|+|z_2|^2.$$

One easily checks that

$$(dd^cu)^2=[z_1=0]\wedge dd^c|z_2|^2\neq 0.$$

Thus, if $u_j=\chi_j\circ u$, where χ_j is chosen as Theorem 1.1, e.g., $u_j=(1/2)\log(|z_1|^2e^{2|z_2|^2}+1/j)$, then

$$(dd^cu_j)^2\to(dd^cu)^2.$$

However, $v_j:=(1/2)\log(|z_1|^2+1/j)+|z_2|^2$ are also smooth psh functions that decrease to u but

$$(dd^cv_j)^2\to 2[z_1=0]\wedge dd^c|z_2|^2=2(dd^cu)^2.$$

It follows that u does not belong to the domain of definition of the Monge-Ampère operator; in fact, this follows directly from [10, Theorem 1.1] since clearly $u\notin W^{1,2}_{loc}$. By [10, Theorem 4.1] one can find another approximating sequence of smooth psh functions decreasing to u whose Monge-Ampère measures do not have locally uniformly finite mass near $\{z_1=0\}$. □

Recall that a psh function u is called *maximal* in an open set Ω in $\mathbb{C}^n$ if for any other psh v in Ω satisfying $v\le u$ outside a compact set, we have $v\le u$ in Ω. We refer to [25, 9] for basic properties of maximal psh functions. In particular, u is maximal if and only if for each $\Omega'\Subset\Omega$ and psh v such that $v\le u$ on $\partial\Omega'$ one has $v\le u$ in Ω'. By Bedford-Taylor's theory [5, 6] a locally bounded psh u is maximal if and only if $(dd^cu)^n=0$.

The following result due to Rashkovsii, see [23, Theorem 1], gives a local characterization of maximal psh functions with divisorial singularities.

Proposition 3.3. *Let Ω be a domain in $\mathbb{C}^n$, $n \geq 2$, f a holomorphic function in Ω (not vanishing identically), and v a locally bounded psh function in Ω. Then $u = \log|f| + v$ is maximal in Ω if and only if v is maximal in Ω.*

One can rephrase Proposition 3.3 as follows: if a psh function u is globally of the form $\log|f| + v$, where f is a holomorphic function and v is psh and locally bounded, then u is maximal if and only if it is maximal outside the singular set. It would be interesting to verify whether such a characterization is true globally for psh functions with divisorial singularities.

Example 3.4. Proposition 3.3 implies that the psh function u in Example 3.2 is maximal (in any domain in $\mathbb{C}^2$). Thus it is not true in general for psh functions with analytic singularities u that $(dd^c u)^n = 0$ is equivalent to u being maximal.

Moreover in any bounded domain we can find a sequence of continuous maximal psh functions decreasing to u, or a sequence u_j of smooth psh functions decreasing to u such that $(dd^c u_j)^2 \to 0$ weakly, see e.g., [9, Proposition 1.4.9]. It follows that (the mass of) $\lim_j (dd^c u_j)^2$ when u_j is a decreasing sequence of bounded psh functions $u_j \to u$ can be both smaller and larger than (the mass of) $(dd^c u)^2$, cf. Example 3.2. □

Remark 3.5. In [12] it was shown that the psh function

$$u(z) := -\sqrt{\log|z_1| \log|z_2|} \tag{3.2}$$

is maximal in $\{|z_1| < 1, \ |z_2| < 1\} \setminus \{(0,0)\}$, and that the Monge-Ampère measure of $\max\{u, -j\}$, however, does not converge weakly to 0 as $j \to \infty$.

In view of Theorem 3.1 and Proposition 3.3 the function u in Examples 3.2 and 3.4 gives a new example of such a maximal psh function. □

Proposition 3.3 implies that for psh functions with divisorial singularities it suffices to check their maximality outside hypersurfaces. This is not true in general as the following example shows.

Example 3.6. The function given by (3.2) is psh in the unit bidisc, maximal away from the singular set, i.e. the hypersurface $\{z_1 z_2 = 0\}$, but not maximal in the entire bidisc Δ^2. In fact, the psh function

$$v(z) := -\sqrt{-\log|z_1|} - \sqrt{-\log|z_2|} + 1$$

coincides with u on the boundary of the bidisk $(\Delta(0,1/e))^2$, but $v > u$ on the diagonal inside $(\Delta(0,1/e))^2$. □

4. The general case of Theorem 1.1

We now give a proof of Theorem 1.1. Since the statement is local we may assume that $u = \log|F| + b$, where F is a tuple of holomorphic functions on an open set $X \subset \mathbb{C}^n$, and b is bounded.

Let Z be the common zero set of F. By Hironaka's theorem one can find a proper map $\pi \colon X' \to X$ that is a biholomorphism $X' \setminus \pi^{-1}Z \simeq X \setminus Z$, where $\pi^{-1}Z$ is a hypersurface, such that the ideal sheaf generated by the functions $\pi^* f_j$ is principal. Let D be the exceptional divisor and let $L \to X'$ be the associated line bundle that has a global holomorphic section

f^0 whose divisor is precisely D. It then follows that $\pi^*F = f^0F'$, where F' is a nonvanishing tuple of sections of L^{-1}. Given a local frame for L on X' we can thus write $F = f^0F'$ where f^0 is a holomorphic function and F' a nonvanishing tuple of holomorphic functions. Then

$$\pi^*u = \log|\pi^*F| + \pi^*b = \log|f^0| + \log|F'| + \pi^*b =: \log|f^0| + v,$$

and since π^*u is psh it follows that v is. Another local frame gives rise to the same local decomposition up to a pluriharmonic function. Notice that

$$dd^c \log|f^0| = [D],$$

where D is the divisor determined by f^0.

In view of Theorem 3.1,

$$\big(dd^c(\chi_j\circ\pi^*u)\big)^k \to (dd^c\pi^*u)^k = [D]\wedge(dd^cv)^{k-1} + (dd^cv)^k.$$

Assume that a is psh and bounded. Since neither $(dd^ca)^k$ nor $(dd^c\pi^*a)^k$ charge subvarieties it follows that

$$\pi_*(dd^c\pi^*a)^k = (dd^ca)^k.$$

Since $\pi^*(\chi_j\circ u) = \chi_j\circ\pi^*u$, thus

$$\big(dd^c(\chi_j\circ u)\big)^k = \pi_*\big(dd^c\big(\pi^*(\chi_j\circ u)\big)\big)^k = \pi_*\big(dd^c(\chi_j\circ\pi^*u)\big)^k \to \pi_*\big([D]\wedge(dd^cv)^{k-1} + (dd^cv)^k\big).$$

By [3, Equation (4.5)],

$$\pi_*\big([D]\wedge(dd^cv)^{k-1} + (dd^cv)^k\big) = (dd^cu)^k$$

and thus Theorem 1.1 follows.

Remark 4.1. The definition of $(dd^cu)^k$ as well as proof of Theorem 1.1 work just as well if X is a reduced, not necessarily smooth, analytic space, cf., e.g., [4]. □

5. Proof and discussion of Theorem 1.2

We start by showing that the Monge-Ampère operators $(\omega + dd^c\varphi)^k$ are well-defined whenever φ is an ω-psh function with analytic singularities.

Lemma 5.1. *Let φ be an ω-psh function with analytic singularities. Then $(dd^c(g+\varphi))^k$ is independent of the local potential g of ω.*

Proof. We need to prove that

$$\big(dd^c(g+h+\varphi)\big)^k = \big(dd^c(g+\varphi)\big)^k \tag{5.1}$$

if h is pluriharmonic. Clearly this is true for $k = 1$.

If T is a positive closed current and u and v are functions such that uT and vT have locally finite mass, then clearly so has $(u+v)T = uT + vT$. Assuming that (5.1) holds for $k = \ell$, it follows that

$$\big(dd^c(g+h+\varphi)\big)^{\ell+1} = dd^c\Big((g+h+\varphi)\mathbf{1}_{X\setminus Z}\big(dd^c(g+h+\varphi)\big)^\ell\Big) =$$
$$dd^c\Big((g+\varphi)\mathbf{1}_{X\setminus Z}\big(dd^c(g+\varphi)\big)^\ell\Big) + dd^c\Big(h\mathbf{1}_{X\setminus Z}\big(dd^c(g+\varphi)\big)^\ell\Big),$$

where Z is the singular set of $\varphi+g$. Since h is pluriharmonic the rightmost expression equals

$$\big(dd^c(g+\varphi)\big)^{\ell+1} + dd^c h \wedge \mathbf{1}_{X\setminus Z}\big(dd^c(g+\varphi)\big)^{\ell} = \big(dd^c(g+\varphi)\big)^{\ell+1}.$$

Thus (5.1) follows by induction. □

Proof of Theorem 1.2. For $k=0,\dots,n-1$ we let

$$T_k := \mathbf{1}_{X\setminus Z}(\omega + dd^c\varphi)^k;$$

note that T_0 is just the function 1. Locally we can define

$$\varphi T_k := (g+\varphi)T_k - gT_k, \tag{5.2}$$

cf. (1.3). This definition is independent of the local potential g of ω and, cf. the proof of Lemma 5.1, thus φT_k defines a global current on X. Applying dd^c to (5.2) we get

$$dd^c(\varphi T_k) = dd^c\big((g+\varphi)T_k\big) - dd^c(gT_k) = (\omega+dd^c\varphi)^{k+1} - \omega\wedge T_k. \tag{5.3}$$

Now

$$\int_X \omega^{n-k}\wedge T_k = \int_X \omega^{n-k-1}\wedge(\omega+dd^c\varphi)^{k+1} - \int_X \omega^{n-k-1}\wedge dd^c(\varphi T_k) = \int_X \omega^{n-k-1}\wedge \mathbf{1}_Z(\omega+dd^c\varphi)^{k+1} + \int_X \omega^{n-k-1}\wedge T_{k+1}. \tag{5.4}$$

Here we have used (5.3) for the second equality; the second term in the middle expression vanishes by Stokes' theorem. Applying (5.4) inductively to $\int_X \omega^n = \int_X \omega^n T_0$ we get (1.5). □

Given an ω-psh function φ, in [21, 13] was introduced the *non-pluripolar Monge-Ampère operators*

$$\big\langle(\omega+dd^c\varphi)^k\big\rangle := \lim_{j\to\infty} \mathbf{1}_{\{\varphi>-j\}}\big(\omega + dd^c\max(\varphi,-j)\big)^k;$$

the definition is based on the corresponding local consctruction in [7].

Assume that φ has analytic singularities with singular set Z. Then $\langle(\omega+dd^c\varphi)^k\rangle$ coincides with the classical Monge-Ampère operator outside Z and it does not charge Z. Hence

$$\big\langle(\omega+dd^c\varphi)^k\big\rangle = \mathbf{1}_{X\setminus Z}(\omega+dd^c\varphi)^k.$$

Following [3], cf. [4], we let

$$M_k^{\varphi} := \mathbf{1}_Z(dd^c\varphi+\omega)^k,\ \ k=1,\dots,n.$$

Using this notation we can rephrase Theorem 1.2 as

$$\int_X \big\langle(\omega+dd^c\varphi)^n\big\rangle = \int_X \omega^n - \sum_{k=1}^n \int_X M_k^{\varphi}\wedge\omega^{n-k}. \tag{5.5}$$

In fact, by applying (5.4) inductively to $\int_X \omega^n T_0$ as in the proof of Theorem 1.2, but stopping at $k=\ell-1$, we get:

Proposition 5.2. *Let φ be an ω-psh function with analytic singularities on a compact Kähler manifold (X,ω) of dimension n. Then, for $\ell = 1,\dots,n$,*

$$\int_X \big\langle (\omega+dd^c\varphi)^\ell \big\rangle \wedge \omega^{n-\ell} = \int_X \omega^n - \sum_{k=1}^{\ell} \int_X M_k^\varphi \wedge \omega^{n-k}. \tag{5.6}$$

From [13, Theorem 1.16] it follows that if φ, φ' are ω-psh with analytic singularities and φ is *less singular* than φ', i.e., $\varphi \geq \varphi' + \mathcal{O}(1)$, then

$$\int_X \big\langle (\omega+dd^c\varphi)^\ell \big\rangle \wedge \omega^{n-\ell} \geq \int_X \big\langle (\omega+dd^c\varphi')^\ell \big\rangle \wedge \omega^{n-\ell} \tag{5.7}$$

for each ℓ. From (5.7) and Proposition 5.2 we conclude that

$$\sum_{k=1}^{\ell} \int_X M_k^\varphi \wedge \omega^{n-k} \leq \sum_{k=1}^{\ell} \int_X M_k^{\varphi'} \wedge \omega^{n-k}$$

for each ℓ. It is not true in general, however, that $\int_X M_k^\varphi \wedge \omega^{n-k} \leq \int_X M_k^{\varphi'} \wedge \omega^{n-k}$ for each k, as is illustrated by the following example.

Example 5.3. Let $X = \mathbb{P}^2_{[z_0:z_1:z_2]}$ with the Fubini-Study metric ω, and let

$$\varphi = \log\Big(\frac{(|z_1|^2+|z_2|^2)^{1/2}}{|z|}\Big) \text{ and } \varphi' = \log\Big(\frac{|z_1|}{|z|}\Big),$$

cf. Example 1.4. Then φ and φ' are ω-psh with analytic singularities and clearly φ is less singular than φ'. Note that $M_2^\varphi = [z_1 = z_2 = 0]$ and $M_1^{\varphi'} = [z_1 = 0]$, whereas M_1^φ and $M_2^{\varphi'}$ vanish. In particular, $\int_X M_2^\varphi > \int_X M_2^{\varphi'}$. □

Remark 5.4. In general we cannot have a global continuity result like Theorem 1.1. Indeed, assume that φ is an ω-psh function with analytic singularities such that

$$\int_X (\omega+dd^c\varphi)^\ell \wedge \omega^{n-\ell} < \int_X \omega^n,$$

cf. (5.6); this holds, e.g., for φ' in Example 5.3 and $\ell = 2$. Moreover, assume that there is a sequence of locally bounded ω-psh, or smooth, functions φ_j converging to φ. By Stokes' theorem

$$\int_X (\omega+dd^c\varphi_j)^\ell \wedge \omega^{n-\ell} = \int_X \omega^n$$

for all j, and thus $(\omega+dd^c\varphi_j)^\ell$ cannot converge to $(\omega+dd^c\varphi)^\ell$.

□

Let X be a, possibly non-smooth, analytic space, cf. Remark 4.1, and let ω be a smooth positive $(1,1)$-form on X that locally has a smooth potential. Then we still have the notion of ω-psh function on X and the formulation and proof of Theorem 1.2, as well as the definitions of M_k^φ, work as in the smooth case.

There is a close connection between Theorem 1.2 and the currents M_k^φ and global (non-proper) intersection theory, that will be studied in a forthcoming paper by two of the authors.

In some sense the currents M_k^φ can be seen as generalized intersection cycles, cf. [4, Section 6]. Let us just give a simple example with a proper intersection here, cf. Example 1.4 above.

Example 5.5. Let $i\colon X \to \mathbb{P}^n$ be a projective variety of dimension p, and let f be a m-homogeneous form in $\mathbb{C}^{n+1}$ that does not vanish identically on any irreducible component of X; i.e., $Z(f)$ intersects X properly. If we consider f as a section of the line bundle $\mathcal{O}(m) \to \mathbb{P}^n$ then it has the natural norm $\|f\| = |f(z)|/|z|^m$. It follows that $u = \log\|f\|$ is $m\omega$-psh on X, where ω is the Fubiny-Study form. Notice that $\langle (m\omega + dd^c\varphi)^n\rangle = 0$. Moreover $M_k = 0$ for $k \geq 2$ and $M_1 = dd^c \log|f|$. Thus the equality (5.5) means that

$$\int_X dd^c \log|f| \wedge \omega^{p-1} = m\int_X \omega^p = \deg Z \cdot \deg X$$

and the rightmost expression is equal to

$$\int_{\mathbb{P}^n} [Z] \wedge [X] \wedge \omega^{p-1}. \tag{5.8}$$

Since $[Z]\wedge[X]$ is the Lelong current of the proper intersection $Z\cdot X$ of Z and X, (5.8) equals $\deg(Z\cdot X)$ and thus (5.5) in this case is just an instance of Bezout's formula. □

6. Some further comments

The Monge-Ampère operators (1.3) are also closely related to local intersection theory. Given a psh function of the form (1.1) on a possibly non-smooth analytic space X, we let

$$M_k^u := \mathbf{1}_Z (dd^c u)^k, \;\; k = 1, \dots, n,$$

where $Z = \{F = 0\}$. In [3, 4] it was proved that

$$\ell_x M_k^u = e_k(x), \tag{6.1}$$

where $\ell_x\mu$ denotes the Lelong number of the positive closed current μ at x, and $e_k(x)$ is the kth *Segre number* at x of the ideal $\mathcal{J}$ generated by F. Segre numbers were introduced independently by Gaffney-Gassler [20] and Tworzewski [26] as certain local intersection numbers, and in a purely algebraic way by Achilles-Manaresi [1]. In fact, if Z is discrete, then the only nonvanishing Segre number $e_n(x)$ equals the classical *Hilbert-Samuel multiplicity* of $\mathcal{J}$ at x. Thus (6.1) is a generalization of the well-known fact the Lelong number of $(dd^c \log|F|)^n$ is the Hilbert-Samuel multiplicity of $\mathcal{J}$ if Z is discrete.

Demailly's approximation theorem [16] asserts that any psh function u on a bounded pseudoconvex domain Ω can be approximated by psh functions with analytic singularities. Let

$$u_j := \frac{1}{2j}\log\sup\left\{|f|^2\colon f\in\mathcal{O}(\Omega),\; \int_\Omega |f|^2 e^{-2ju} d\lambda \leq 1\right\}.$$

Then $u_j \to u$ pointwise and in L^1_{loc} and there exists a sequence of positive constants ε_j decreasing to 0 such that the subsequence $u_{2^j} + \varepsilon_j$ is decreasing, see [19]; in view of [22] this cannot be done for the whole sequence u_j. Since u_j are in fact defined by weighted Bergman kernels, it is clear that locally they can be written in the form (1.1) where b is smooth. If u has an isolated analytic singularity (so that the Demailly definition of the Monge-Ampère operator applies), it is proved in [24] that there is continuity for the Monge-Ampère masses of the u_j. It would be interesting to investigate possible convergence properties of $(dd^c u_j)^k$ in more general cases; for example when the initial function u also has analytic singularities, or for more general psh u as a means to extend $(dd^c u)^k$ to such u.

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