

ADJUNCTION FOR THE GRAUERT-RIEMENSCHNEIDER CANONICAL SHEAF AND EXTENSION OF L^2 -COHOMOLOGY CLASSES

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ABSTRACT. In the present paper, we derive an adjunction formula for the Grauert-Riemenschneider canonical sheaf of a singular hypersurface V in a complex manifold M . This adjunction formula is used to study the problem of extending L^2 -cohomology classes of $\bar{\partial}$ -closed forms from the singular hypersurface V to the manifold M in the spirit of the Ohsawa-Takegoshi-Manivel extension theorem. We do that by showing that our formulation of the L^2 -extension problem is invariant under bimeromorphic modifications, so that we can reduce the problem to the smooth case by use of an embedded resolution of V in M .

1. INTRODUCTION

The canonical sheaf of holomorphic forms of top degree is a fundamental object associated with a complex manifold. It is meaningful not only in the context of Serre duality and various vanishing theorems of Kodaira type, but plays also an essential role in the classification of complex spaces as it appears e.g. in the definition of the Kodaira dimension.

On singular complex varieties, there are different reasonable notions of canonical sheaves generalizing the canonical sheaf of smooth spaces. On the one hand, there is the dualizing sheaf of Grothendieck which behaves well with respect to Serre duality and is used for the classification of normal spaces. On the other hand, there is the canonical sheaf of Grauert-Riemenschneider which satisfies Kodaira vanishing and behaves very well under bimeromorphic modifications.

One crucial tool in the study of canonical sheaves of smooth varieties is the adjunction formula which gives a link between the canonical sheaf of the ambient space and the canonical sheaf of a subspace. The adjunction formula sits at the core of e.g. the Ohsawa-Takegoshi-Manivel L^2 -extension theorem, a celebrated analytical result with numerous applications, particularly in algebraic geometry.

When it comes to singular spaces, there is a well-known adjunction formula for the Grothendieck dualizing sheaf, but to our knowledge no general adjunction formula is known for the Grauert-Riemenschneider canonical sheaf.

The main results of the present paper are as follows: we derive an adjunction formula for the Grauert-Riemenschneider canonical sheaf of a singular hypersurface V in a complex manifold M , and we show that our adjunction formula is the 'right' one to be used in the context of L^2 -extension problems of Ohsawa-Takegoshi-Manivel type. In fact, we prove that our formulation of L^2 -extension problems is invariant under bimeromorphic modifications.

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Let us explain this more precisely. We consider the following situation: let M be a compact complex manifold of dimension n and V a hypersurface in M . When V is smooth, the well-known adjunction formula reads as

$$K_V = (K_M \otimes [V])|_V = K_M|_V \otimes N_V, \quad (1.1)$$

where K_V and K_M denote the canonical bundles of V and M , respectively, $[V]$ is the line bundle associated with $\mathcal{O}(V)$ ¹, and $N_V = [V]|_V$ is the normal bundle of V in M .

In the language of sheaves, (1.1) can be formulated as the short exact sequence

$$0 \rightarrow \mathcal{K}_M \hookrightarrow \mathcal{K}_M \otimes \mathcal{O}(V) \xrightarrow{\Psi} \iota_* \mathcal{K}_V \rightarrow 0, \quad (1.2)$$

where \mathcal{K}_V and \mathcal{K}_M are the canonical sheaves of V and M , respectively, and $\iota_* \mathcal{K}_V$ is the trivial extension of \mathcal{K}_V to M . The mapping Ψ is defined by the local equation

$$\eta = \frac{df}{f} \wedge \Psi(\eta) \quad (1.3)$$

in $\mathcal{K}_M \otimes \mathcal{O}(V)$ over V where f is any local defining function for the divisor V . More concretely, if z are local holomorphic coordinates on M such that $V = \{z_1 = 0\}$, then $\Psi(gdz_1 \wedge \cdots \wedge dz_n/z_1) = g|_V dz_2 \wedge \cdots \wedge dz_n$.

When V is a singular hypersurface, the adjunction formula (1.2) remains valid if we replace \mathcal{K}_V by Grothendieck's dualizing sheaf $\omega_V = \mathcal{E}xt_{\mathcal{O}_M}^1(\mathcal{O}_V, \mathcal{K}_M)$, sometimes also called the Barlet sheaf (see e.g. [PR], §5.3). However, this adjunction is not compatible with the classical Ohsawa-Takegoshi-Manivel extension theorem as sections of ω_V are in general not square-integrable $(n-1)$ -forms on the regular part of V , but may have poles of higher order at the singular set $\text{Sing } V$ of V . In contrast, all the classical extension theorems (see [OT], [M], [D1]) are about extension of holomorphic L^2 -forms from a possibly singular divisor.

Thus, we propose to replace the canonical sheaf of V in the singular case by the Grauert-Riemenschneider canonical sheaf of holomorphic square-integrable $(n-1)$ -forms on (the regular part of) V . We denote that sheaf again by \mathcal{K}_V .

Our first main result is the following adjunction formula for the Grauert-Riemenschneider canonical sheaf, cf. Section 3.2.

Theorem 1.1. *Let V be a (possibly singular) hypersurface in a complex manifold M . Then there exists a unique multiplier ideal sheaf $\mathcal{J}(V)$ such that there is a natural short exact sequence*

$$0 \rightarrow \mathcal{K}_M \hookrightarrow \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \xrightarrow{\Psi} \iota_* \mathcal{K}_V \rightarrow 0, \quad (1.4)$$

where \mathcal{K}_M is the usual canonical sheaf of M , $\iota_* \mathcal{K}_V$ is the trivial extension of the Grauert-Riemenschneider canonical sheaf \mathcal{K}_V of V to M , and Ψ is the adjunction map defined by (1.3) on $V^* = \text{Reg } V$. The zero set of $\mathcal{J}(V)$ is contained in $\text{Sing } V$.

We will call the sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$, that now plays the role of $\mathcal{O}(V)$, the **adjunction sheaf** of V in M . In general the adjunction sheaf is not locally free.

As a consequence of Theorem 1.1, we can deduce:

¹We denote by $\mathcal{O}(V)$ the invertible sheaf of germs of meromorphic functions which are holomorphic outside V and are allowed to have a single pole along the divisor V (see e.g. [GR1], VII.1.3). $[V]$ can then be understood either as the line bundle associated with the invertible sheaf $\mathcal{O}(V)$ or as the line bundle associated with the divisor V . There is then a natural identification between $\mathcal{O}(V)$ and the sheaf of holomorphic sections in $[V]$, i.e., $\mathcal{O}([V])$.

Theorem 1.2. *The natural inclusion $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \hookrightarrow \mathcal{K}_M \otimes \mathcal{O}(V)$ induces a natural isomorphism*

$$\mathcal{K}_V \cong \omega_V \otimes \mathcal{J}(V)|_V.$$

In particular, there is a natural inclusion of the Grauert-Riemenschneider canonical sheaf into Grothendieck's dualizing sheaf, $\mathcal{K}_V \subset \omega_V$.

If V is normal, the inclusion $\mathcal{K}_V \subset \omega_V$ is due to Grauert-Riemenschneider, see [GR2]. In general this inclusion follows from Henkin-Passare, [HP, Section 2]. Also, when V is normal we have the following characterization of when this inclusion is an equality, which is probably well-known to experts in birational geometry.

Theorem 1.3. *Let V be a normal hypersurface in M . Then $\mathcal{K}_V = \omega_V$, i.e. $\mathcal{J}(V) = \mathcal{O}_M$, exactly if V has at worst canonical singularities.*

Taking the long exact sequence of (1.4) yields extension of cohomology classes of \mathcal{K}_V , cf. Remark 3.1. To obtain our L^2 -extension results we need to be able to talk about metrics “on” the adjunction sheaf. We believe that the following concept is a natural generalization of smooth metrics on the line bundle $[V]$ associated with $\mathcal{O}(V)$. If D is an effective divisor in a complex manifold, there is an associated canonical singular metric $e^{-2\varphi_D}$ on $[D]$; if D is locally defined by f_α , then φ_D is locally given as $\log |f_\alpha|$.

Definition 1.4. *Let $e^{-2\psi}$ be a singular metric on the normal bundle $[V]$ of the hypersurface V in M . We say that $e^{-2\psi}$ is a **smooth metric** on the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$ if there exists an embedded resolution of singularities with only normal crossings $\pi : (V', M') \rightarrow (V, M)$ such that $e^{-2\pi^*\psi + 2\varphi_\Delta}$ is a smooth metric on the normal bundle $[V']$ of V' in M' , where $\Delta = \pi^*V - V'$ is the difference of the total transform π^*V and the strict transform V' of V .*

In fact, if $\pi : (V', M') \rightarrow (V, M)$ is any embedded resolution of V in M , any smooth metric on $[V']$ induces a smooth metric on $\mathcal{O}(V) \otimes \mathcal{J}(V)$, see Theorem 4.1.

For any Hermitian complex space X and any Hermitian line bundle $F \rightarrow X$ (with a possibly singular metric), we denote by $\mathcal{C}_X^{p,q}(F)$ the sheaf of germs of (p, q) -forms g on the regular part of X with values in F , which are square-integrable up to the singular set and which are in the domain of the $\bar{\partial}$ -operator in the sense of distributions, $\bar{\partial}_w$, meaning that $\bar{\partial}_w g$ is again square-integrable up to the singular set, see Section 2. For an open set $U \subset X$, we define the L^2 -cohomology

$$H_{(2)}^{p,q}(U^*, F) := H^q(\Gamma(U, (\mathcal{C}_X^{p,*}(F), \bar{\partial}_w))).$$

Here and in the following, let $U^* = \text{Reg } U$ denote the regular part of U .

We can now give an alternative definition of $\mathcal{O}(V) \otimes \mathcal{J}(V)$ as the sheaf of holomorphic sections that are square integrable with respect to a smooth metric on $\mathcal{O}(V) \otimes \mathcal{J}(V)$.

Theorem 1.5. *Let M be a complex manifold of dimension n and let V a hypersurface in M . Moreover let $e^{-2\psi}$ be a singular metric on $[V]$, which is smooth on the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$. Then*

$$\mathcal{O}(V) \otimes \mathcal{J}(V) \cong \ker \bar{\partial}_w \subset \mathcal{C}_M^{0,0}([V]^{sing}), \quad (1.5)$$

where $[V]^{sing} := ([V], e^{-2\psi})$.

If $F \rightarrow M$ is any Hermitian line bundle (with smooth metric), then the complex $(\mathcal{C}_M^{n,}(F \otimes [V]^{sing}), \bar{\partial}_w)$ is a fine resolution for $\mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$, where $\mathcal{K}_M(F)$ denotes the sheaf of holomorphic n -forms with values in F .*

In particular, it follows from Theorem 1.5 that the (flabby) cohomology of $\mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$ can be expressed in terms of the L^2 -cohomology of forms with values in the line bundle $F \otimes [V]^{sing}$:

$$H^q(M, \mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)) \cong H_{(2)}^{n,q}(M, F \otimes [V]^{sing}). \quad (1.6)$$

As \mathcal{K}_M is invertible, Theorem 1.5 also gives an L^2 -resolution for the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$.

The proof of Theorem 1.5 relies on the fact that the metric $e^{-2\psi}$ behaves locally like $e^{-2\varphi}$ where φ is a plurisubharmonic defining function for the multiplier ideal sheaf $\mathcal{J}(V)$. Using Theorem 1.5, we will deduce that the adjunction map Ψ from (1.4) in Theorem 1.1 induces a natural map Ψ_V on the level of L^2 -cohomology,

$$\Psi_V : H_{(2)}^{n,q}(M, F \otimes [V]^{sing}) \rightarrow H_{(2)}^{n-1,q}(V^*, F),$$

and thus the extension problem for L^2 -cohomology classes amounts to studying the question whether Ψ_V is surjective.

Our second main result shows that this L^2 -extension problem is invariant under bimeromorphic modifications, which allows us to reduce the problem to the case of a smooth divisor where other methods can be applied.

Theorem 1.6. *Let M be a compact Hermitian manifold of dimension n , let $V \subset M$ be a singular hypersurface, and let $F \rightarrow M$ be a Hermitian holomorphic line bundle. Let $e^{-2\psi}$ be a singular Hermitian metric on $[V]$ that is smooth on the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$. Moreover, let*

$$\pi : (V', M') \rightarrow (V, M)$$

be an embedded resolution of singularities of V in M .

Then there exists a commutative diagram

$$\begin{array}{ccc} H_{(2)}^{n,q}(M, F \otimes [V]^{sing}) & \xrightarrow{\Psi_V} & H_{(2)}^{n-1,q}(V^*, F) \\ \cong \downarrow \pi^* & & \cong \downarrow \pi^* \\ H^{n,q}(M', \pi^* F \otimes [V']) & \xrightarrow{\Psi_{V'}} & H_{(2)}^{n-1,q}(V', \pi^* F), \end{array} \quad (1.7)$$

where $\Psi_{V'}$ is the adjunction map for the smooth divisor V' in M' , Ψ_V is induced by the adjunction map for the non-smooth divisor V in M as defined in (3.20). The vertical maps π^ are induced by pull-back of forms under π and both are isomorphisms.*

Using Theorem 1.6 together with a recent result by Berndtsson [B] we get the following L^2 -extension result under a quite weak positivity assumption on $F \rightarrow M$.

Theorem 1.7. *Let M be a compact Kähler manifold with Kähler metric ω . Furthermore, let V , $e^{-2\psi}$, and F be as in Theorem 1.6, and let $e^{-2\phi}$ be the smooth metric of the Hermitian line bundle $F \rightarrow M$.*

Let $0 \leq q \leq n-1$. Assume that there exists an $\epsilon > 0$ such that

$$i\partial\bar{\partial}\phi \wedge \omega^q \geq \epsilon i\partial\bar{\partial}\psi \wedge \omega^q \quad (1.8)$$

and

$$i\partial\bar{\partial}\phi \wedge \omega^q \geq 0. \quad (1.9)$$

Then the adjunction map Ψ_V in (1.7) is surjective.

Another condition to ensure surjectivity of the adjunction map Ψ_V in (1.7) is $H^{n,q+1}(M, F) = 0$. That can be seen by applying the long exact cohomology sequence to the adjunction formula (1.4) in Theorem 1.1, cf. Remark 3.4. Thus Ψ_V is surjective e.g. if $F \rightarrow M$ is a positive line bundle by Kodaira's vanishing theorem.

In fact, each L^2 -cohomology class in $H_{(2)}^{n,q}(M, F \otimes [V]^{sing})$ has a smooth representative in $\Gamma(M, \mathcal{C}_{n,q}^\infty(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))$, see Lemma 4.3.

Corollary 1.8. *Let M , V , and F be as in Theorem 1.7. Assume that the map Ψ_V in (1.7) is surjective. Let $u \in \Gamma(V, \mathcal{C}_V^{n-1,q}(F))$ be a $\bar{\partial}$ -closed L^2 -form of degree $(n-1, q)$*

If $q \geq 1$, then there exists an L^2 -form $g \in \Gamma(V, \mathcal{C}_V^{n-1,q-1}(F))$ and a $\bar{\partial}$ -closed section $U \in \Gamma(M, \mathcal{C}_{n,q}^\infty(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))$, i.e., a smooth (n, q) -form with values in $F \otimes [V]$ and some extra vanishing according to $\mathcal{J}(V)$, such that

$$U = \frac{df}{f} \wedge (u - \bar{\partial}g)$$

where f is any local defining function for the hypersurface V .

If $q = 0$ the statement holds without g .

We remark that one can deduce some L^2 -estimates for our extension by use of Berndtsson's theorem, but forgo this topic here as these estimates would depend on the resolution of singularities that is used. Anyway, it is easy to see (and a nice feature) that one can tensor F in such statements with another semi-positive line bundle $F' \rightarrow M$ without changing the estimate.

This paper is organized as follows. In Section 2 we give some background on L^2 -cohomology on a singular space and the Grauert–Riemenschneider sheaf. In Section 3 we discuss adjunction and prove Theorem 1.1 in Section 3.2 and Theorems 1.2 and 1.3 in Section 3.3. The relation to L^2 -extension is then discussed in Section 4; the proofs of Theorems 1.5, 1.6, and 1.7 are given in Sections 4.1, 4.2, and 4.3, respectively. Finally in Section 5 we illustrate our results by some examples.

2. SOME PRELIMINARIES

2.1. L^2 -cohomology for the $\bar{\partial}$ -operator on singular complex spaces. We need to recall briefly some of the concepts from [R]; we refer to that paper for details. Let (X, h) always be a (singular) Hermitian complex space of pure dimension n , $F \rightarrow X \setminus \text{Sing } X$ a Hermitian holomorphic line bundle, and $U \subset X$ an open subset. Moreover, let $\mathcal{L}^{*,*}(F)$ be the sheaf of forms with values in F that are square-integrable up to the singular set of X , i.e.,

$$\mathcal{L}^{p,q}(U, F) = \{f \in L_{p,q}^{2,loc}(U \setminus \text{Sing } X, F) : f|_K \in L_{p,q}^2(K \setminus \text{Sing } X, F) \ \forall K \subset\subset U\}.$$

We denote by

$$\bar{\partial}_w(U) : \mathcal{L}^{p,q}(U, F) \rightarrow \mathcal{L}^{p,q+1}(U, F)$$

the $\bar{\partial}$ -operator in the sense of distributions on $U \setminus \text{Sing } X$ which is closed and densely defined. When there is no danger of confusion, we will simply write $\bar{\partial}$ or $\bar{\partial}_w$ for $\bar{\partial}_w(U)$. Since $\bar{\partial}_w$ is a local operator, i.e. $\bar{\partial}_w(U)|_V = \bar{\partial}_w(V)$ for open sets $V \subset U$, we get sheaves

$$\mathcal{C}_X^{p,q}(F) := \mathcal{L}^{p,q}(F) \cap \bar{\partial}_w^{-1} \mathcal{L}^{p,q+1}(F),$$

given by

$$\mathcal{C}_X^{p,q}(U, F) = \mathcal{L}^{p,q}(U, F) \cap \text{Dom } \bar{\partial}_w(U).$$

The sheaves $\mathcal{C}_X^{p,q}(F)$ admit partitions of unity, and so we obtain fine sequences

$$\mathcal{C}_X^{p,0}(F) \xrightarrow{\bar{\partial}_w} \mathcal{C}_X^{p,1}(F) \xrightarrow{\bar{\partial}_w} \mathcal{C}_X^{p,2}(F) \xrightarrow{\bar{\partial}_w} \dots \quad (2.1)$$

We will use simply $\mathcal{C}_X^{p,q}$ to denote the sheaves of forms with values in the trivial line bundle.

The $L^{2,loc}$ -Dolbeault cohomology for forms with values in F with respect to the $\bar{\partial}_w$ -operator on an open set $U \subset X$ is by definition the cohomology of the complex (2.1) after taking global sections over U ; this is denoted by

$$H_{(2)}^{p,q}(U^*, F) := H^q(\Gamma(U, \mathcal{C}_X^{p,*}(F))).$$

To be precise, this is the cohomology of forms which are square-integrable on compact subsets of U , and it becomes the L^2 -cohomology of X^* when $U = X$ is compact.

2.2. The Grauert–Riemenschneider canonical sheaf. Let $\pi : M \rightarrow X$ be a resolution of singularities and let \mathcal{K}_M be the canonical sheaf of the smooth manifold M . Then the **Grauert–Riemenschneider canonical sheaf** \mathcal{K}_X of X is the direct image sheaf

$$\mathcal{K}_X := \pi_* \mathcal{K}_M. \quad (2.2)$$

The definition is independent of $\pi : M \rightarrow X$, see [GR2].

Since the L^2 -property of $(n, 0)$ -forms remains invariant under modifications of the metric and thus also under resolution of singularities, \mathcal{K}_X can alternatively be defined as

$$\mathcal{K}_X = \ker \bar{\partial}_w \subset \mathcal{C}_X^{n,0}.$$

Let us mention briefly a few important facts which will be used later. If $F \rightarrow M$ is a semi-positive line bundle and $\mathcal{K}_M(F)$ is the sheaf of holomorphic n -forms with values in F , then Takegoshi's vanishing theorem, [T][Remark 2], states that the higher direct images of \mathcal{K}_M vanish,

$$R^q \pi_* \mathcal{K}_M(F) = 0, \quad q > 0. \quad (2.3)$$

Let both X and M carry Hermitian metrics. Then the pull-back of forms under π maps square-integrable (n, q) -forms on X^* to square-integrable (n, q) -forms on M . By trivial extension over the exceptional set, we get well-defined mappings

$$\pi^* : \mathcal{C}_X^{n,q}(G) \rightarrow \mathcal{C}_M^{n,q}(\pi^* G)$$

if $G \rightarrow X$ is any Hermitian holomorphic line bundle. If the line bundle $\pi^* G \rightarrow M$ is locally semi-positive with respect to the base space X , then this map induces an isomorphism of L^2 -cohomology, see [R, Theorem 1.5]:

$$\pi^* : H_{(2)}^{n,q}(X^*, G) \xrightarrow{\cong} H_{(2)}^{n,q}(M, \pi^* G). \quad (2.4)$$

3. THE ADJUNCTION FORMULA

3.1. The adjunction formula for a smooth divisor. Let M be a complex manifold of dimension n , and V a smooth hypersurface in M . Let $[V]$ be the holomorphic line bundle whose holomorphic sections correspond to sections in $\mathcal{O}(V)$; by a slight abuse of notation, we call $[V]$ the normal bundle of V in M . The well-known adjunction formula states that

$$\iota_* \mathcal{K}_V \cong \mathcal{K}_M \otimes \mathcal{O}(V) / \mathcal{K}_M = \mathcal{K}_M([V]) / \mathcal{K}_M. \quad (3.1)$$

Here and throughout $\iota : V \hookrightarrow M$ denotes the natural inclusion, i.e., $\iota_*\mathcal{K}_V$ is the trivial extension of the canonical sheaf \mathcal{K}_V to M . If we denote the canonical bundles on V and M by K_V and K_M , respectively, then the adjunction formula can be expressed as

$$K_V \cong (K_M \otimes [V])|_V.$$

We shall explain how the isomorphism in (3.1) can be realized explicitly. For further use, we take a slightly more general point of view. Let $\{f_j\}_j$ be a system of holomorphic functions defining V and let ω be a smooth (n, q) -form with values in $[V]$; we identify ω with a semi-meromorphic (n, q) -form with at most a single pole along V . In local coordinates z_1, \dots, z_n , we can write

$$\omega = \frac{g_j}{f_j} \wedge dz_1 \wedge \dots \wedge dz_n,$$

where the g_j are smooth $(0, q)$ -forms which transform as $g_j = (f_j/f_k)g_k$. We can now define the adjunction morphism Ψ locally as follows. For each point $p \in V$, there exists an f_j with $df_j \neq 0$ in a neighborhood of p , i.e. $\partial f_j / \partial z_\mu \neq 0$ for some $1 \leq \mu \leq n$. In this neighborhood,

$$\Psi : \omega \mapsto \omega' := (-1)^{q+\mu-1} g_j \wedge \frac{dz_1 \wedge \dots \wedge \widehat{dz_\mu} \wedge \dots \wedge dz_n}{\partial f_j / \partial z_\mu} \Big|_V. \quad (3.2)$$

It is not hard to see that this assignment depends neither on the choice of μ nor on the choice of f_j or the local coordinates z_1, \dots, z_n . Thus, (3.2) gives a well-defined mapping

$$\Psi : C_{n,q}^\infty(M, [V]) \rightarrow C_{n-1,q}^\infty(V).$$

Note that Ψ maps holomorphic n -forms with values in $[V]$ on M to holomorphic $(n-1)$ -forms on V . This proves the adjunction formula (3.1). Note that on V , we have locally:

$$\omega = \frac{df_j}{f_j} \wedge \omega' = \frac{df_j}{f_j} \wedge \Psi(\omega), \quad (3.3)$$

which can be actually used to define Ψ . It follows directly from the definition that $\Psi \circ \bar{\partial} = \bar{\partial} \circ \Psi$ so that the adjunction morphism defines an adjunction map also on the level of cohomology classes

$$\Psi_V : H^{n,q}(M, [V]) \rightarrow H^{n-1,q}(V).$$

Remark 3.1. The question whether $\bar{\partial}$ -cohomology classes on V extend to M or not has a nice cohomological realization. From the considerations above, we obtain the short exact sequence (1.2). By use of the long exact cohomology sequence, it follows that the induced map

$$H^q(M, \mathcal{K}_M \otimes \mathcal{O}(V)) \longrightarrow H^q(M, \iota_*\mathcal{K}_V) \cong H^q(V, \mathcal{K}_V)$$

is surjective if $H^{q+1}(M, \mathcal{K}_M) \cong H^{n,q+1}(M) = 0$.

3.2. The adjunction formula for a singular divisor. Again, let M be a complex manifold of dimension n , but V a hypersurface in M which is not necessarily smooth. Two problems occur. First, it is not clear what we mean by a canonical sheaf on V . Second, the adjunction morphism cannot be defined in a 'smooth' way as above because it will happen that $df_j = 0$ in singular points of V .

Let

$$\pi : (V', M') \rightarrow (V, M)$$

be an embedded resolution of V in M , i.e. $\pi : M' \rightarrow M$ is a surjective proper holomorphic map such that $\pi|_{M' \setminus E} : M' \setminus E \rightarrow M \setminus \text{Sing } V$ is a biholomorphism, where E is the exceptional divisor which consists of normal crossings only, and the regular hypersurface V' is the strict transform of V , see e.g. [BM], Theorem 13.2. Hence, $\pi|_{V'} : V' \rightarrow V$ is a resolution of singularities. For ease of notation, throughout this paper we will denote $\pi|_{V'}$ again by π .

The adjunction formula (1.2) for the pair (V', M') tells us that

$$0 \longrightarrow \mathcal{K}_{M'} \hookrightarrow \mathcal{K}_{M'} \otimes \mathcal{O}(V') \longrightarrow \iota_* \mathcal{K}_{V'} \longrightarrow 0$$

is exact. By Takegoshi's vanishing theorem (2.3) we get the short exact sequence

$$0 \longrightarrow \pi_* \mathcal{K}_{M'} \hookrightarrow \pi_* (\mathcal{K}_{M'} \otimes \mathcal{O}(V')) \longrightarrow \pi_* \iota_* \mathcal{K}_{V'} \longrightarrow 0$$

on M . In light of (2.2)

$$\pi_* \iota_* \mathcal{K}_{V'} = \iota_* \pi_* \mathcal{K}_{V'} \cong \iota_* \mathcal{K}_V,$$

where ι denotes also the embedding $\iota : V \hookrightarrow M$.

We thus get the short exact sequence

$$0 \longrightarrow \mathcal{K}_M \hookrightarrow \pi_* (\mathcal{K}_{M'} \otimes \mathcal{O}(V')) \longrightarrow \iota_* \mathcal{K}_V \longrightarrow 0 \quad (3.4)$$

on M , and obtain the following adjunction formula for the Grauert-Riemenschneider canonical sheaf on a singular hypersurface:

$$\iota_* \mathcal{K}_V \cong \pi_* (\mathcal{K}_{M'} \otimes \mathcal{O}(V')) / \mathcal{K}_M. \quad (3.5)$$

Next, we will show that there exists a multiplier ideal sheaf $\mathcal{J}_\pi(V)$ such that

$$\pi_* (\mathcal{K}_{M'} \otimes \mathcal{O}(V')) \cong \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}_\pi(V). \quad (3.6)$$

Since $\pi_* \mathcal{K}_{M'} \cong \mathcal{K}_M$, it is clear that (3.6) holds with $\mathcal{J}_\pi(V)_x = \mathcal{O}_{M,x}$ for points $x \notin \text{Sing } V$. So, consider a point $x \in \text{Sing } V$. There exists a holomorphic function f in a neighborhood U_x of the point x defining the hypersurface V , i.e. $V = (f)$. Consider the pullback $\pi^* f = f \circ \pi$ on $\pi^{-1}(U_x)$. Then $\pi^* f$ is vanishing precisely of order 1 on the strict transform V' of V because f is vanishing precisely of order 1 on the regular part of V . Let

$$(\pi^* f) = V' + E_f,$$

where E_f is a divisor with support on the exceptional set of the embedded resolution $\pi : M' \rightarrow M$. E_f is a normal crossing divisor since the exceptional set E has only normal crossings. $\mathcal{O}(-E_f)$ is the sheaf of germs of holomorphic functions which vanish at least to the order of $\pi^* f$ on E . For the direct image sheaf, we have that

$$\pi_* \mathcal{O}(-E_f) \subset \pi_* \mathcal{O}_{M'} \cong \mathcal{O}_M$$

on U_x , i.e. $\pi_* \mathcal{O}(-E_f)$ can be considered as a coherent sheaf of ideals in \mathcal{O}_M . So, there exist holomorphic functions $g_1, \dots, g_k \in \mathcal{O}(U_x)$ that generate the direct image sheaf in a neighborhood of the point x . By restricting U_x , we can assume that this is the case in U_x . Then $\pi^* g_j \in \mathcal{O}(-E_f)(\pi^{-1}(U_x))$ for $j = 1, \dots, k$, i.e., all the $\pi^* g_j$ vanish at least to the order of $\pi^* f$ on the exceptional set E , and the common zero set of the $\pi^* g_j$ is contained in E . On the other hand, f is in the direct image sheaf, and so $f = h_1 g_1 + \dots + h_k g_k$ for holomorphic functions $h_1, \dots, h_k \in \mathcal{O}(U_x)$. Hence,

$$\pi^* f = \pi^* h_1 \pi^* g_1 + \dots + \pi^* h_k \pi^* g_k,$$

meaning that not all the π^*g_j can vanish to an order strictly higher than π^*f on any irreducible component of E . Thus, we conclude that

$$E_f = (\pi^*g_1, \dots, \pi^*g_k), \quad (3.7)$$

i.e. the $\pi^*g_1, \dots, \pi^*g_k$ generate $\mathcal{O}(-E_f)$. Let $\varphi := \log(|g_1| + \dots + |g_k|)$ and let $\mathcal{J}(\varphi)$ be the multiplier ideal sheaf associated with the plurisubharmonic weight φ on U_x , i.e. the sheaf of germs of holomorphic functions F such that $|F|^2 e^{-2\varphi}$ is locally integrable. On the other hand, $\mathcal{J}(\pi^*\varphi)$ is the sheaf of germs of holomorphic functions H such that $H/(|\pi^*g_1| + \dots + |\pi^*g_k|)$ is locally square-integrable. But E has only normal crossings, and so (3.7) implies that

$$\mathcal{J}(\pi^*\varphi) = \mathcal{O}(-E_f) \quad (3.8)$$

on $\pi^{-1}(U_x)$. We are now in the position to prove:

Proposition 3.2. *On U_x , the pull-back of forms under π induces the isomorphism*

$$\pi^* : \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(\varphi) \xrightarrow{\cong} \pi_*(\mathcal{K}_{M'} \otimes \mathcal{O}(V')).$$

Proof. It is well-known that

$$\pi^* : \mathcal{K}_M \otimes \mathcal{J}(\varphi) \xrightarrow{\cong} \pi_*(\mathcal{K}_{M'} \otimes \mathcal{J}(\pi^*\varphi)),$$

see e.g. [D2], Proposition 15.5. By use of (3.8), we obtain

$$\pi^* : \mathcal{K}_M \otimes \mathcal{J}(\varphi) \xrightarrow{\cong} \pi_*(\mathcal{K}_{M'} \otimes \mathcal{O}(-E_f)). \quad (3.9)$$

Let W be an open set in U_x . Then $\mathcal{K}_M \otimes \mathcal{J}(\varphi)(W)$ are the holomorphic n -forms on W with coefficients in $\mathcal{J}(\varphi)$, and $\pi_*(\mathcal{K}_{M'} \otimes \mathcal{O}(-E_f))(W)$ are the holomorphic n -forms on $\pi^{-1}(W)$ with coefficients in $\mathcal{O}(-E_f)$. The isomorphism in (3.9) is given by pull-back of n -forms under π . So, the isomorphism (3.9) is preserved if we multiply the germs on the left-hand side by $1/f$, and the germs on the right-hand side by $1/\pi^*f$. Recalling that $(f) = V$ and $(\pi^*f) = V' + E_f$, it follows that

$$\pi^* : \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(\varphi) \xrightarrow{\cong} \pi_*(\mathcal{K}_{M'} \otimes \mathcal{O}(V' + E_f) \otimes \mathcal{O}(-E_f)),$$

meaning nothing else but

$$\pi^* : \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(\varphi) \xrightarrow{\cong} \pi_*(\mathcal{K}_{M'} \otimes \mathcal{O}(V')).$$

□

Note that (3.9) shows that $\mathcal{J}(\varphi)$ does not depend on the specific choice of φ . Since \mathcal{K}_M and $\mathcal{O}(V)$ are invertible sheaves, we conclude the following result which together with (3.4) proves Theorem 1.1.

Theorem 3.3. *Let*

$$\mathcal{J}_\pi(V) := \mathcal{O}(V)^{-1} \otimes \mathcal{K}_M^{-1} \otimes \pi_*(\mathcal{K}_{M'} \otimes \mathcal{O}(V')).$$

Then $\mathcal{J}_\pi(V)$ is a multiplier ideal sheaf such that pull-back of forms under π induces the isomorphism

$$\pi^* : \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}_\pi(V) \xrightarrow{\cong} \pi_*(\mathcal{K}_{M'} \otimes \mathcal{O}(V')).$$

Locally, $\mathcal{J}_\pi(V) = \mathcal{J}(\varphi)$ for a plurisubharmonic function $\varphi = \log(|g_1| + \dots + |g_k|)$, where the g_1, \dots, g_k are holomorphic functions such that

$$(\pi^*f)|_E = E_f = (\pi^*g_1, \dots, \pi^*g_k).$$

By (3.4) and (3.5), we obtain the short exact sequence

$$0 \longrightarrow \mathcal{K}_M \hookrightarrow \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}_\pi(V) \longrightarrow \iota_* \mathcal{K}_V \longrightarrow 0 \quad (3.10)$$

and the **adjunction formula for the Grauert-Riemenschneider canonical sheaf on a singular hypersurface**:

$$\iota_* \mathcal{K}_V \cong \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}_\pi(V) / \mathcal{K}_M. \quad (3.11)$$

Since \mathcal{K}_M and $\mathcal{O}(V)$ are invertible, it follows from (3.10) that $\mathcal{J}_\pi(V)$ does not depend on the resolution π . So, we write $\mathcal{J}(V) = \mathcal{J}_\pi(V)$ and call $\mathcal{O}(V) \otimes \mathcal{J}(V)$ the **adjunction sheaf of V in M** .

Note that the natural injection in (3.10) makes sense since $\mathcal{O}_M \subset \mathcal{O}(V) \otimes \mathcal{J}(V)$, $\mathcal{J}(V)$ is coherent, and the zero set of $\mathcal{J}(V)$ is contained in $\text{Sing } V$.

Remark 3.4. As in Remark 3.1 we obtain from (3.10) by use of the long exact cohomology sequence the adjunction map for the flabby cohomology

$$H^q(M, \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)) \longrightarrow H^q(M, \iota_* \mathcal{K}_V) \cong H^q(V, \mathcal{K}_V), \quad (3.12)$$

which is surjective e.g. if $H^{q+1}(M, \mathcal{K}_M) \cong H^{n,q+1}(M) = 0$.

3.3. Relation to the Grothendieck dualizing sheaf. As mentioned in the introduction, one can also consider the adjunction formula for Grothendieck's dualizing sheaf ω_V . As V is a complete intersection,

$$\omega_V \cong (\mathcal{K}_M \otimes \mathcal{O}(V))|_V,$$

see, e.g., [PR, §5.3]. Thus we obtain the natural short exact sequence

$$0 \rightarrow \mathcal{K}_M \hookrightarrow \mathcal{K}_M \otimes \mathcal{O}(V) \rightarrow \iota_* \omega_V \rightarrow 0.$$

Combining with Theorem 3.3 we get the exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}_M & \longrightarrow & \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) & \longrightarrow & \iota_* \mathcal{K}_V \longrightarrow 0 \\ & & \downarrow = & & \downarrow j & & \downarrow \text{dotted} \\ 0 & \longrightarrow & \mathcal{K}_M & \longrightarrow & \mathcal{K}_M \otimes \mathcal{O}(V) & \longrightarrow & \iota_* \omega_V \longrightarrow 0 \end{array} \quad (3.13)$$

where the map j is the natural inclusion. The diagram (3.13) induces a natural isomorphism

$$\iota_* \mathcal{K}_V \cong (\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)) / \mathcal{K}_M \cong \mathcal{J}(V) \otimes (\mathcal{K}_M \otimes \mathcal{O}(V)) / \mathcal{K}_M \cong \mathcal{J}(V) \otimes \iota_* \omega_V.$$

As $\mathcal{J}(V)|_V \subset \mathcal{O}_V$, this implies particularly that there is a natural inclusion of the Grauert-Riemenschneider canonical sheaf into Grothendieck's dualizing sheaf, $\mathcal{K}_V \subset \omega_V$. This proves Theorem 1.2.

Let us now prove Theorem 1.3; assume therefore that V is a normal. Then $\omega_V \cong (\mathcal{K}_M \otimes \mathcal{O}(V))|_V$ is an invertible sheaf corresponding to a canonical Cartier divisor K_V , in particular V is Gorenstein, see e.g. [PR, §5.4]. If $\pi : N \rightarrow V$ is any resolution of singularities and K_N is the canonical divisor of N , we can write

$$K_N = \pi^* K_V + \sum a_j E_j, \quad (3.14)$$

where the E_j are the irreducible components of the exceptional divisor and the a_j are rational coefficients. Then V has (at worst) **canonical singularities** if and only if $a_j \geq 0$ for all indices j . As (3.14) is equivalent to $\mathcal{K}_N = \pi^* \omega_V \otimes \mathcal{O}(\sum a_j E_j)$, V has canonical singularities precisely if $\pi^* \omega_V \subset \mathcal{K}_N$.

Proof of Theorem 1.3. Let $\pi : N \rightarrow V$ be any resolution of singularities.

First assume that $\mathcal{K}_V = \omega_V$. Then, in light of (2.2),

$$\pi^* \omega_V = \pi^* \mathcal{K}_V = \pi^* \pi_* \mathcal{K}_N \subset \mathcal{K}_N,$$

and so V has canonical singularities.

Conversely, assume that V has canonical singularities. Then

$$\omega_V \subset \pi_* \pi^* \omega_V \subset \pi_* \mathcal{K}_N = \mathcal{K}_V$$

and thus $\mathcal{K}_V = \omega_V$, since the inverse inclusion follows from Theorem 1.2. \square

3.4. The commutative adjunction diagram. We will now give an explicit realization of the adjunction map (3.12) on the level of (n, q) -forms. We can achieve that easily by tensoring (3.10) with the sheaf of germs of smooth $(0, q)$ -forms $\mathcal{C}_{0,q}^\infty$.

Let M be a compact Hermitian manifold, V a singular hypersurface in M , and $\pi : (V', M') \rightarrow (V, M)$ an embedded resolution of V in M . Give M' any positive definite metric. As in Section 3.1, let $\{f'_j\}_j$ be a finite defining system for V' in M' . We are now in the position to describe an adjunction morphism on M similar to the procedure in Section 3.1. On M , we have to replace the normal bundle $[V]$ by the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$.

Consider the adjunction map

$$\Psi' : \mathcal{K}_{M'} \otimes \mathcal{O}(V') \longrightarrow \iota_* \mathcal{K}_{V'}$$

for the non-singular hypersurface V' in M' defined as in Section 3.1. Using Proposition 3.2 and $\pi^* : \mathcal{K}_V \xrightarrow{\cong} \pi_* \mathcal{K}_{V'}$, we obtain the first commutative adjunction diagram

$$\begin{array}{ccc} \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) & \xrightarrow{\Psi} & \iota_* \mathcal{K}_V \\ \cong \downarrow \pi^* & & \cong \downarrow \pi^* \\ \pi_* (\mathcal{K}_{M'} \otimes \mathcal{O}(V')) & \xrightarrow{\Psi'} & \pi_* \iota_* \mathcal{K}_{V'} \end{array} \quad (3.15)$$

defining the adjunction map Ψ on M . Here the vertical maps are induced by pull-back of forms under π . Note that $\pi_* \iota_* \mathcal{K}_{V'} = \iota_* \pi_* \mathcal{K}_{V'}$ and that the kernel of Ψ is just the natural inclusion of \mathcal{K}_M in $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$. So, Ψ realizes the isomorphism (3.11), i.e.

$$\Psi : \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) / \mathcal{K}_M \xrightarrow{\cong} \iota_* \mathcal{K}_V.$$

Considering the maps on global sections, (3.15) yields the commutative diagram

$$\begin{array}{ccc} \Gamma(M, \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)) & \xrightarrow{\Psi} & \Gamma(V, \mathcal{K}_V) \\ \cong \downarrow \pi^* & & \cong \downarrow \pi^* \\ \Gamma(M', \mathcal{K}_{M'} \otimes \mathcal{O}(V')) & \xrightarrow{\Psi'} & \Gamma(V', \mathcal{K}_{V'}) \end{array}$$

We can now describe Ψ explicitly. Let $\omega \in \Gamma(M, \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))$. By use of (3.3), we get that

$$\pi^* \omega = \frac{df'_j}{f'_j} \wedge \Psi'(\pi^* \omega) = \frac{df'_j}{f'_j} \wedge \pi^* \Psi(\omega) \quad (3.16)$$

locally on V' .

Let f be a local defining function for V in M . Then π^*f is vanishing precisely to order 1 on $V' \setminus E$. Hence

$$\pi^* \left(\frac{df}{f} \right) = \frac{\pi^* df}{\pi^* f} = \frac{df'_j}{f'_j}$$

locally as $(1,0)$ -forms in $\mathcal{O}(V')|_{V' \setminus E}$. Since π is a biholomorphism on $M' \setminus E$, it follows from (3.16) that

$$\omega = \frac{df}{f} \wedge \Psi(\omega) \quad (3.17)$$

on V^* . This shows that the adjunction map Ψ does not depend on the resolution $\pi : M' \rightarrow M$ since df does not vanish on V^* .

Let us now define the adjunction map for (germs of) (n,q) -forms. That can be done simply by tensoring the diagram (3.15) by the sheaves of germs of smooth $(0,q)$ -forms, so that we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{C}_{0,q}^\infty \otimes (\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)) & \xrightarrow{(1,\Psi)} & \mathcal{C}_{0,q}^\infty \otimes \iota_* \mathcal{K}_V \\ \downarrow (\pi^*, \pi^*) & & \downarrow (\pi^*, \pi^*) \\ \pi_* \mathcal{C}_{0,q}^\infty \otimes \pi_* (\mathcal{K}_{M'} \otimes \mathcal{O}(V')) & \xrightarrow{(1,\Psi')} & \pi_* \mathcal{C}_{0,q}^\infty \otimes \pi_* \iota_* \mathcal{K}_{V'} \end{array} \quad (3.18)$$

Now the vertical arrows are not isomorphisms any more.

It is easy to see that we can complement the diagram on the right hand side by natural mappings to the sheaves of germs of L^2 -forms in the domain of $\bar{\partial}$ on V and V' , respectively:

$$\begin{array}{ccc} \mathcal{C}_{0,q}^\infty \otimes \iota_* \mathcal{K}_V & \longrightarrow & \iota_* \mathcal{C}_V^{n-1,q} \\ \downarrow (\pi^*, \pi^*) & & \downarrow \pi^* \\ \pi_* \mathcal{C}_{0,q}^\infty \otimes \pi_* \iota_* \mathcal{K}_{V'} & \longrightarrow & \pi_* \iota_* \mathcal{C}_{V'}^{n-1,q} \end{array} \quad (3.19)$$

Merging (3.18), (3.19) and adopting the notation, we obtain the commutative adjunction diagram for (germs of) (n,q) -forms:

$$\begin{array}{ccc} \mathcal{C}_{n,q}^\infty \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) & \xrightarrow{\Psi} & \iota_* \mathcal{C}_V^{n-1,q} \\ \downarrow \pi^* & & \downarrow \pi^* \\ \pi_* (\mathcal{C}_{n,q}^\infty \otimes \mathcal{O}(V')) & \xrightarrow{\Psi'} & \pi_* \iota_* \mathcal{C}_{V'}^{n-1,q} \end{array}$$

For global sections, we have the commutative diagram

$$\begin{array}{ccc} \Gamma(M, \mathcal{C}_{n,q}^\infty \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)) & \xrightarrow{\Psi} & \Gamma(V, \mathcal{C}_V^{n-1,q}) \\ \downarrow \pi^* & & \downarrow \pi^* \\ \Gamma(M', \mathcal{C}_{n,q}^\infty \otimes \mathcal{O}(V')) & \xrightarrow{\Psi'} & \Gamma(V', \mathcal{C}_{V'}^{n-1,q}) \end{array} \quad (3.20)$$

Let $\omega \in \Gamma(M, \mathcal{C}_{n,q}^\infty \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))$. It follows from (3.17) that we still have

$$\omega = \frac{df}{f} \wedge \Psi(\omega)$$

locally on V^* .

Since $\bar{\partial}$ commutes with π^* and Ψ' , it must also commute with Ψ , and so we deduce from (3.20) the commutative diagram on the level of $\bar{\partial}$ -cohomology:

$$\begin{array}{ccc} H^q(\Gamma(M, \mathcal{C}_{n,*}^\infty \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))) & \xrightarrow{\Psi_V} & H^q(\Gamma(V, \mathcal{C}_V^{n-1,*})) \\ \downarrow \pi^* & & \downarrow \pi^* \\ H^q(\Gamma(M', \mathcal{C}_{n,*}^\infty \otimes \mathcal{O}(V'))) & \xrightarrow{\Psi_{V'}} & H^q(\Gamma(V', \mathcal{C}_{V'}^{n-1,*})) \end{array} \quad (3.21)$$

Note that the groups on the right hand-side are by definition the L^2 -cohomology groups for the $\bar{\partial}$ -operator in the sense of distributions.

It does not cause any additional difficulty to define the adjunction morphisms as above also for (n, q) -forms with values in a Hermitian line bundle $F \rightarrow M$ and in $\pi^*F \rightarrow M'$, respectively. Using (2.4) we get:

Theorem 3.5. *Let M be a compact Hermitian manifold of dimension n , $V \subset M$ a singular hypersurface in M , and $F \rightarrow M$ a Hermitian line bundle. Let $\pi : (V', M') \rightarrow (V, M)$ be an embedded resolution of singularities, and let $\mathcal{J}(V)$ be the multiplier ideal sheaf as defined in Theorem 3.3. Then there exists a commutative diagram*

$$\begin{array}{ccc} H^q(\Gamma(M, \mathcal{C}_{n,*}^\infty(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))) & \xrightarrow{\Psi_V} & H_{(2)}^{n-1,q}(V^*, F) \\ \downarrow \pi^* & & \cong \downarrow \pi^* \\ H^q(\Gamma(M', \mathcal{C}_{n,*}^\infty(\pi^*F) \otimes \mathcal{O}(V'))) & \xrightarrow{\Psi_{V'}} & H_{(2)}^{n-1,q}(V', \pi^*F), \end{array} \quad (3.22)$$

where $\Psi_{V'}$ is the usual adjunction map for the smooth divisor V' in M' , Ψ_V is the adjunction map for the non-smooth divisor V in M as defined in (3.21), and the vertical maps π^* in (3.22) are induced by pull-back of forms under π .

Note that the lower line in (3.22) can be understood as

$$\Psi_{V'} : H^{n,q}(M', \pi^*F \otimes [V']) \longrightarrow H_{(2)}^{n-1,q}(V', \pi^*F),$$

where $[V'] \rightarrow M'$ is the normal bundle of V' in M' , see Section 3.1.

Our purpose is to determine conditions under which the adjunction map Ψ_V in the upper line of the commutative diagram (3.22) is surjective. For this, it would be interesting to know whether the vertical map on the left-hand side of the diagram (3.22) is also an isomorphism. We will see later in Section 4 that this is actually true if we replace the upper left corner of the diagram by a certain L^2 -cohomology group. In, particular it follows that the vertical map on the left-hand side is surjective, see Corollary 4.4.

Before studying L^2 -cohomology in the next section, let us first investigate the case of C^∞ -cohomology on the left hand side of (3.22) a bit closer.

It is well known (just solve the $\bar{\partial}$ -equation locally in the C^∞ -category for forms with values in a line bundle) that the complex

$$(\mathcal{C}_{n,*}^\infty(\pi^*F \otimes [V']), \bar{\partial})$$

is a fine resolution of $\mathcal{K}_{M'}(\pi^*F \otimes [V'])$. On the other hand, we have already seen that pull-back of forms under π induces the isomorphism

$$\pi^* : \mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \xrightarrow{\cong} \pi_*(\mathcal{K}_{M'}(\pi^*F) \otimes \mathcal{O}(V')), \quad (3.23)$$

since the line-bundle $F \rightarrow M$ is added easily to the statement of Proposition 3.2.

Lemma 3.6. *If $U \subset M$ is sufficiently small, then $(\pi^*F \otimes [V'])|_{\pi^{-1}(U)}$ is semi-positive.*

Proof. The factor π^*F is irrelevant since the statement is local with respect to M and semi-positivity is stable under pullback by holomorphic mappings. Let $f \in \mathcal{O}(U)$ be a defining function for $V|_U$ so that

$$(\pi^*f) = V'|_{\pi^{-1}(U)} + E_f|_{\pi^{-1}(U)},$$

where E_f is a divisor with support on the exceptional set of the resolution. Then

$$[V']|_{\pi^{-1}(U)} = \pi^*[V] \otimes [-E_f]|_{\pi^{-1}(U)}$$

and so, by the argument above, it is sufficient to see that $[-E_f]|_{\pi^{-1}(U)}$ is semi-positive.

Take a covering $\{U_\alpha\}$ of $\pi^{-1}(U)$ such that $\pi^*f = f_\alpha^0 \cdot f'_\alpha$ in U_α , where $(f_\alpha^0) = V'|_{U_\alpha}$ and $(f'_\alpha) = E_f|_{U_\alpha}$. Recall from Section 3.2 that (possibly after shrinking U) there are $g_1, \dots, g_k \in \mathcal{O}(U)$ that generate the direct image of $\mathcal{O}(-E_f)$. It follows that there is a non-vanishing tuple $h_\alpha = (h_{1\alpha}, \dots, h_{k\alpha}) \in \mathcal{O}(U_\alpha)^k$ such that

$$\pi^*g = (\pi^*g_1, \dots, \pi^*g_k) = f'_\alpha h_\alpha \quad \text{in } U_\alpha.$$

Letting $|h_\alpha| = (|h_{1\alpha}|^2 + \dots + |h_{k\alpha}|^2)^{1/2}$, it is straight forward to check that the local functions $e^{-2\log|h_\alpha|}$ transform as a metric on $[-E_f]$. Since $\log|h_\alpha|$ is plurisubharmonic, $[-E_f]|_{\pi^{-1}(U)}$ is semi-positive. \square

Thus, $\pi^*F \otimes [V']$ is locally semi-positive with respect to M . Since $\mathcal{K}_{M'}(\pi^*F \otimes [V']) \cong \mathcal{K}_{M'}(\pi^*F) \otimes \mathcal{O}(V')$ we conclude by Takegoshi's vanishing theorem (2.3):

$$R^q\pi_*(\mathcal{K}_{M'}(\pi^*F) \otimes \mathcal{O}(V')) = 0 \quad \text{for } q > 0. \quad (3.24)$$

So, the direct image complex

$$(\pi_*(\mathcal{C}_{n,*}^\infty(\pi^*F) \otimes \mathcal{O}(V')), \pi_*\bar{\partial})$$

is a fine resolution of $\pi_*(\mathcal{K}_{M'}(\pi^*F) \otimes \mathcal{O}(V'))$. Combining with (3.23) and taking global cohomology we would get that the vertical map on the left-hand side of (3.22) is an isomorphism if we knew that the complex

$$(\mathcal{C}_{n,*}^\infty(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V), \bar{\partial})$$

were exact. But this is a delicate problem since it involves the multiplier ideal sheaf $\mathcal{J}(V)$. We can prove the required local exactness only in the L^2 -category, see Section 4.

4. EXTENSION OF COHOMOLOGY CLASSES

4.1. Smooth metrics on the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$. Given a line bundle $F \rightarrow M$ we will use the notation $e^{-2\varphi}$ for the Hermitian metric on F with weight φ . When $e^{-2\varphi}$ is a singular Hermitian metric, i.e. the weight φ is in $L^{1,loc}$ rather than smooth, cf. [D2, Definition 11.20], we will sometimes write F^{sing} . The following result asserts that smooth metrics on adjunction sheaves, as defined in Definition 1.4 do actually exist.

Theorem 4.1. *Let*

$$\pi : (V', M') \rightarrow (V, M)$$

be any embedded resolution of singularities of V in M with only normal crossings, and $e^{-2\psi'}$ a smooth Hermitian metric on the normal bundle $[V']$ of V' in M' . Then $e^{-2\psi'}$ induces a singular metric $e^{-2\psi}$ on $[V]$ that is smooth on the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$. Moreover, if $[V']|_{\pi^{-1}(U)}$ is semi-positive, then $[V]^{sing}|_U$ is too.

Proof. It is sufficient to prove the theorem locally in M so let $U \subset M$ such that $V|_U$ is defined by $f \in \mathcal{O}(U)$. Then $\pi^*V = (\pi^*f) = V'|_{\pi^{-1}(U)} + E_f|_{\pi^{-1}(U)}$, where V' is the strict transform of V , and E_f is an effective divisor with support on the exceptional set E ; cf. the proof of Lemma 3.6. Take a covering $\{U_\alpha\}$ of $\pi^{-1}(U)$ such that $\pi^*f = f_\alpha^0 f'_\alpha$ in U_α where f_α^0 defines $V'|_{U_\alpha}$ and f'_α defines $E_f|_{U_\alpha}$. Now, if s is any section of $[V]$ over U then π^*s/f'_α transforms as the f_α^0 and hence defines a semi-meromorphic section of $[V']$ over U_α . We define the singular metric on $[V]$ by letting

$$|s|_V^2 := |\pi^*s|^2 / |f'_\alpha|^2 e^{-2\psi'_\alpha} = |\pi^*s|^2 e^{-2(\psi'_\alpha + \varphi_{E_f})}, \quad (4.1)$$

where $e^{-2\psi'_\alpha}$ are the local functions for the metric on $[V']$. Clearly, this singular metric on $[V]$ induces the original metric $e^{-2\psi'}$ on $[V']$ and hence, the singular metric on $[V]$ is smooth on the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$.

Assume now that $[V']|_{\pi^{-1}(U)}$ is semi-positive, i.e., that $dd^c\psi'_\alpha \geq 0$; notice that by Lemma 3.6 this can always be achieved if U is small enough. By (4.1), the singular metric $e^{-2\psi}$ on $[V]$ has the property that $\pi^*\psi = \psi'_\alpha + \log|f'_\alpha|$ and so, in the sense of currents,

$$dd^c\psi = \pi_*(dd^c\psi'_\alpha + T) = \pi_*dd^c\psi'_\alpha,$$

where T is the current of integration on $|E_f|$; the last equality follows since π_*T is a normal $(1,1)$ -current with support on $\text{Sing } V$, which has codimension ≥ 2 . Hence, $[V]^{sing}|_U$ is semi-positive. \square

Next, we will prove Theorem 1.5. Let $e^{-2\psi}$ be the induced metric on $[V]$ from Theorem 4.1. With the notation from the proof above, we then have that $\pi^*e^{-2\psi} \sim |f'_\alpha|^{-2}$. Recall from Section 3.2 that we can choose holomorphic functions $g_1, \dots, g_k \in \mathcal{O}(U)$ such that g_1, \dots, g_k generate the direct image of $\mathcal{O}(-E_f)$ over U , where U is a small open set in M , i.e. E_f is precisely the common zero set of $\pi^*g_1, \dots, \pi^*g_k$ (counted with multiplicities). Hence, $|f'_\alpha| \sim |\pi^*g_1| + \dots + |\pi^*g_k|$, and so

$$e^{-2\psi} \sim (|g_1| + \dots + |g_k|)^{-2} = e^{-2\varphi}, \quad (4.2)$$

where φ is a local defining function for the multiplier ideal sheaf $\mathcal{J}(V)$.

Remark 4.2. Let $e^{-2\psi}$ and $e^{-2\psi'}$ be smooth metrics on $\mathcal{O}(V) \otimes \mathcal{J}(V)$. Then note, in light of (4.2), that being L^2 with respect to $e^{-2\psi}$ is equivalent to being L^2 with respect to $e^{-2\psi'}$.

Proof of Theorem 1.5. By Definition 1.4, there exists an embedded resolution $\pi : (V', M') \rightarrow (V, M)$ of V in M such that $\pi^*e^{-2\psi}$ induces a smooth metric on the normal bundle $[V']$ of V' in M' . But then, by (4.2), $\psi \sim \varphi$ where φ is a local defining function for the multiplier ideal sheaf $\mathcal{J}(V)$. Hence, a holomorphic section h of $[V]$ is in $\mathcal{C}_M^{0,0}([V]^{sing})$ precisely if $|h|^2 e^{-2\varphi}$ is locally integrable (in a trivialization of $[V]$). This proves (1.5).

It follows from (1.5) that sections of $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$ can be identified with square-integrable holomorphic sections of $[V]^{sing}$, i.e.

$$\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \cong \ker \bar{\partial}_w \subset \mathcal{C}_M^{n,0}([V]^{sing}).$$

By Remark 4.2 we may assume that $e^{-2\psi}$ is locally semi-positive. Then exactness of the complex $(\mathcal{C}_M^{n,*}([V]^{sing}), \bar{\partial}_w)$ is equivalent to local L^2 -exactness of the $\bar{\partial}$ -equation for (n, q) -forms with values in a holomorphic line bundle with a singular Hermitian metric which is positive semi-definite. But this is well-known, see, e.g., [D2, Corollary 14.3], and the proof of the Nadel vanishing theorem, [D2, Theorem 15.8]. It is

furthermore clear that the sheaves $\mathcal{C}_M^{n,q}([V]^{sing})$ admit a smooth partition of unity, so that $(\mathcal{C}_M^{n,*}([V]^{sing}), \bar{\partial}_w)$ is in fact a fine resolution of $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$. \square

4.2. The adjunction diagram for L^2 -cohomology classes. Let $[V]^{sing}$ be the normal bundle of a hypersurface V with a singular Hermitian metric, smooth on $\mathcal{O}(V) \otimes \mathcal{J}(V)$. In order to define the adjunction map for L^2 -cohomology classes with values in $[V]^{sing}$ we need the following lemma. This is necessary as L^2 -forms do not behave well under restriction to lower-dimensional subspaces.

Lemma 4.3. *Each L^2 -cohomology class $[\phi] \in H_{(2)}^{n,q}(M, [V]^{sing})$ has a smooth representative $\phi \in \Gamma(M, \mathcal{C}_{n,q}^\infty \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))$.*

Proof. We will use the DeRham-Weil-Dolbeault isomorphism (see e.g. Demailly, [D3, IV.6], or adopt the procedure from [H, Chapter 7.4]). Let $\mathcal{U} = \{U_\alpha\}$ be a Stein cover for M and $\{\chi_\alpha\}$ a smooth partition of unity subordinate to \mathcal{U} . Recall that the DeRham-Weil-Dolbeault map on Čech cohomology

$$[\Lambda_q] : \check{H}^q(\mathcal{U}, \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)) \longrightarrow H_{(2)}^{n,q}(M, [V]^{sing}) \quad (4.3)$$

is defined as follows: given a Čech cocycle $c \in C^q(\mathcal{U}, \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))$, set

$$\Lambda_q c := \sum_{\nu_0, \dots, \nu_q} c_{\nu_0 \dots \nu_q} \chi_{\nu_q} \bar{\partial} \chi_{\nu_0} \wedge \dots \wedge \bar{\partial} \chi_{\nu_{q-1}}. \quad (4.4)$$

As $(\mathcal{C}_M^{n,*}([V]^{sing}), \bar{\partial}_w)$ is a fine resolution for $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$ (see Theorem 1.5 and (1.6)), $[\Lambda_q]$ is an isomorphism. So, each class $[\phi] \in H_{(2)}^{n,q}(M, [V]^{sing})$ has a representative

$$\phi = \Lambda_q c \in \Gamma(M, \mathcal{C}_{n,q}^\infty \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)).$$

\square

We are now ready to prove Theorem 1.6, replacing the upper left cohomology group in (3.22) by $H_{(2)}^{n,q}(M, [V]^{sing})$. Note that it does not cause any difficulty to include a Hermitian holomorphic line bundle $F \rightarrow M$ in the statement of Theorem 1.5 and Lemma 4.3.

Proof of Theorem 1.6. Starting from Proposition 3.5, we replace the cohomology group

$$H^q(\Gamma(M, \mathcal{C}_{n,*}^\infty(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)))$$

in the upper left corner of the commutative diagram (3.22) by the L^2 -cohomology

$$H_{(2)}^{n,q}(M, F \otimes [V]^{sing}) = H^q(\Gamma(M, \mathcal{C}_M^{n,*}(F \otimes [V]^{sing}))).$$

We do that by adding the map

$$\Lambda_q \circ [\Lambda_q]^{-1} : H_{(2)}^{n,q}(M, F \otimes [V]^{sing}) \longrightarrow H^q(\Gamma(M, \mathcal{C}_{n,*}^\infty(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)))$$

to the diagram (3.22), where Λ_q is the DeRham-Weil-Dolbeault map as defined in (4.3), (4.4). The application of $\Lambda_q \circ [\Lambda_q]^{-1}$ means to choose smooth representatives in $\Gamma(M, \mathcal{C}_{n,q}^\infty(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))$ for cohomology classes in $H_{(2)}^{n,q}(M, F \otimes [V]^{sing})$. Note that $\Lambda_q \circ [\Lambda_q]^{-1}$ does not depend on the choices made in Lemma 4.3 as we consider the map on cohomology classes.

By use of Proposition 3.5, it only remains to show that

$$\pi^* \circ \Lambda_q \circ [\Lambda_q]^{-1} : H_{(2)}^{n,q}(M, F \otimes [V]^{sing}) \longrightarrow H^{n,q}(M', \pi^* F \otimes [V'])$$

is an isomorphism. This is equivalent to showing that

$$\pi^* \circ \Lambda_q : \check{H}^q(\mathcal{U}, \mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)) \rightarrow H^{n,q}(M', \pi^* F \otimes [V'])$$

is an isomorphism, where $\mathcal{U} = \{U_\alpha\}$ is a Stein cover for M and Λ_q is the DeRham-Weil-Dolbeault map with respect to a suitable partition of unity $\{\chi_\alpha\}$ subordinate to \mathcal{U} . But $\pi^* \circ \Lambda_q = \Lambda'_q \circ \pi^*$, where we let Λ'_q denote the DeRham-Weil-Dolbeault map with respect to the covering $\pi^*\mathcal{U} = \{\pi^{-1}(U_\alpha)\}$ and the partition of unity $\{\pi^*\chi_\alpha\}$ on M' . Using (3.23) we get

$$\pi^* : \check{H}^q(\mathcal{U}, \mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)) \xrightarrow{\cong} \check{H}^q(\pi^*\mathcal{U}, \mathcal{K}_{M'}(\pi^*F) \otimes \mathcal{O}(V'))$$

and thus by (3.24),

$$H^q(\pi^{-1}(U), \mathcal{K}_{M'}(\pi^*F) \otimes \mathcal{O}(V')) = H^q(U, \mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))$$

on open sets $U \subset M$. Hence, $\pi^*\mathcal{U}$ is a Leray cover for $\mathcal{K}_{M'}(\pi^*F) \otimes \mathcal{O}(V')$ on M' , meaning that the DeRham-Weil-Dolbeault map Λ'_q on M' is also an isomorphism. Hence, $\pi^* \circ \Lambda_q = \Lambda'_q \circ \pi^*$ is an isomorphism. \square

The proof of Theorem 1.6 yields immediately:

Corollary 4.4. *The vertical map π^* on the left-hand side of the commutative diagram (3.22) in Proposition 3.5 is surjective.*

4.3. Extension of L^2 -cohomology classes.

Proof of Theorem 1.7. By Theorem 1.6, we can consider instead the extension problem for the smooth hypersurface V' in M' . So, we have to discuss the question whether

$$\Psi_{V'} : H^{n,q}(M', \pi^*F \otimes [V']) \longrightarrow H_{(2)}^{n-1,q}(V', \pi^*F)$$

is surjective, where π^*F carries the smooth Hermitian metric $\pi^*e^{-2\phi}$ and $[V']$ carries the smooth Hermitian metric $e^{-2\psi'}$ where $\psi' = \pi^*\psi - \varphi_\Delta$ with $\Delta = \pi^*V - V'$. Recall that locally $\varphi_\Delta = \log |f'_\alpha|$ (see the proof of Theorem 4.1), so that

$$i\partial\bar{\partial}\psi' = i\partial\bar{\partial}\pi^*\psi - i\partial\bar{\partial}\log |f'_\alpha| \leq i\partial\bar{\partial}\pi^*\psi. \quad (4.5)$$

As $\pi : M' \rightarrow M$ is holomorphic, (1.8), (1.9) and (4.5) give

$$(i\partial\bar{\partial}\pi^*\phi - \epsilon i\partial\bar{\partial}\psi') \wedge (\pi^*\omega)^q \geq (i\partial\bar{\partial}\pi^*\phi - \epsilon i\partial\bar{\partial}\pi^*\psi) \wedge (\pi^*\omega)^q \geq 0, \quad (4.6)$$

$$i\partial\bar{\partial}\pi^*\phi \wedge (\pi^*\omega)^q \geq 0. \quad (4.7)$$

We may assume that the embedded resolution of V in M is obtained by finitely many blow-ups (i.e. monoidal transformations) along smooth centers, see [BM, Theorem 13.4]. So, M' can be interpreted as a submanifold in a finite product of Kähler manifolds and it inherits a Kähler metric ω' .

As ω' is strictly positive definite and $\pi^*\omega$ is only positive semi-definite, there exists a constant $C > 0$ such that $\omega' \geq C\pi^*\omega$. Thus, (4.6) and (4.7) imply

$$(i\partial\bar{\partial}\pi^*\phi - \epsilon i\partial\bar{\partial}\psi') \wedge (\omega')^q \geq C^q (i\partial\bar{\partial}\pi^*\phi - \epsilon i\partial\bar{\partial}\psi') \wedge (\pi^*\omega)^q \geq 0,$$

$$i\partial\bar{\partial}\pi^*\phi \wedge (\omega')^q \geq C^q i\partial\bar{\partial}\pi^*\phi \wedge (\pi^*\omega)^q \geq 0.$$

So, the smooth metrics $\pi^*e^{-2\phi}$ and $e^{-2\psi'}$ satisfy the assumptions of Berndtsson's extension Theorem 3.1 in [B] for the smooth divisor V' in the Kähler manifold (M', ω') , and this gives the required surjectivity of $\Psi_{V'}$. \square

Combining Theorem 1.7 with Lemma 4.3, we obtain also Corollary 1.8.

5. EXAMPLES FOR THE MULTIPLIER IDEAL SHEAF $\mathcal{J}(V)$

We shall illustrate the role of the multiplier ideal sheaf $\mathcal{J}(V)$ and of our adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$, respectively, in three simple examples.

Example 5.1. Let us discuss briefly what would happen if we blew up a regular hypersurface. So, let V be the regular hypersurface in \mathbb{C}^2 (with coordinates z_1, z_2) given as the zero set of $f(z) = z_1$. Let $\pi : M' \rightarrow \mathbb{C}^2$ be the blow up of the origin, i.e. M' is given by the equation $z_1 w_2 = z_2 w_1$ in $\mathbb{C}^2 \times \mathbb{CP}^1$ with coordinates $((z_1, z_2); [w_1 : w_2])$ and π is the projection $\mathbb{C}^2 \times \mathbb{CP}^1 \rightarrow \mathbb{C}^2, (z, w) \mapsto z$.

We cover M' by two charts. The first is given by $w_1 = 1$ (coordinates z_1, w_2). Here, $\pi^* f = z_1$ and the exceptional divisor E appears as $\{z_1 = 0\}$.

The second chart is given by $w_2 = 1$ (coordinates w_1, z_2). Here, $\pi^* f = z_2 w_1$, the exceptional divisor E appears as $\{z_2 = 0\}$, and the strict transform V' of V is just $\{w_1 = 0\}$.

Thus, in the notation of Section 3.2, we have $E_f = E$ so that E_f is generated by the two holomorphic functions $\pi^* g_1$ and $\pi^* g_2$ where $g_1 = z_1$ and $g_2 = z_2$. So, $\mathcal{J}(V) = \mathcal{J}(\varphi)$ with $\varphi = \log(|z_1| + |z_2|)$. Let $h \in (\mathcal{O}_{\mathbb{C}^2})_0$ be a germ of a holomorphic function at the origin of \mathbb{C}^2 . Then $h e^{-\varphi} = h/(|z_1| + |z_2|)$ is locally square-integrable at the origin. We conclude that $\mathcal{J}(V) = \mathcal{J}(\varphi) = \mathcal{O}_{\mathbb{C}^2}$, which is expected since V is smooth.

Example 5.2. Let V be the cusp in \mathbb{C}^2 (with coordinates z_1, z_2) given as the zero set of $f(z) = z_1^3 - z_2^2$. We obtain an embedded resolution $\pi : (V', M') \rightarrow (V, \mathbb{C}^2)$ with only normal crossings by a sequence of three blow-ups. M' can be realized as follows. We consider $\mathbb{C}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ with coordinates $((z_1, z_2); [w_1 : w_2]; [x_1 : x_2]; [y_1 : y_2])$ and define M' by the three equations

$$z_1 w_2 = z_2 w_1, \quad z_1 x_2 = w_2 x_1, \quad x_1 y_2 = w_2 y_1.$$

The resolution π is given by the projection on the first factor \mathbb{C}^2 . The exceptional set consists of three copies E_1, E_2, E_3 of \mathbb{CP}^1 coming from the three blow-ups. It is not hard to check that $(\pi^* f) = V' + 2E_1 + 3E_2 + 6E_3$. The whole resolution M' can be covered by eight charts, but we can get a good picture by just considering two of them.

The first is given by $w_1 = x_2 = y_1 = 1$ (coordinates x_1, y_2). Then $z_1 = x_1^2 y_2$ and $z_2 = x_1^3 y_2^2$ so that $\pi^* f = x_1^6 y_2^3 (1 - y_2)$. Here, $E_2 = (y_2)$, $E_3 = (x_1)$ and $V' = \{1 - y_2 = 0\}$.

The second interesting chart is given by $w_1 = x_2 = y_2 = 1$ (coordinates y_1, w_2). Then $z_1 = y_1 w_2^2$ and $z_2 = y_1 w_2^3$ so that $\pi^* f = w_2^6 y_1^2 (y_1 - 1)$. Here, $E_1 = (y_1)$, $E_3 = (w_2)$ and $V' = \{y_1 - 1 = 0\}$.

One can check that $E_f = 2E_1 + 3E_2 + 6E_3$ is generated by the two holomorphic functions $\pi^* g_1$ and $\pi^* g_2$ where $g_1 = z_1^3$ and $g_2 = z_2^2$. So, we obtain here that $\mathcal{J}(V) = \mathcal{J}(\varphi)$ with $\varphi = \log(|z_1|^3 + |z_2|^2)$. It is a standard exercise to compute the multiplier ideal sheaf $\mathcal{J}(\varphi)$ (see e.g. [D2], Exercise 15.7):

$$\mathcal{J}(V) = \mathcal{J}(\varphi) = (z_1, z_2),$$

i.e. we obtain the ideal sheaf of the origin (with multiplicity one).

It is now interesting to check that this makes sense in view of the adjunction mapping

$$\Psi : \mathcal{K}_{\mathbb{C}^2} \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \rightarrow \mathcal{K}_V.$$

A germ ω of $\mathcal{K}_{\mathbb{C}^2} \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$ can be written as

$$\omega = g \frac{dz_1 \wedge dz_2}{f} = g \frac{dz_1 \wedge dz_2}{z_1^3 - z_2^2},$$

where g is a germ of a holomorphic function in $\mathcal{J}(V)$. As a cusp, V has a well-defined tangential space at the origin, that is $T_0V = \{z_2 = 0\}$. Thus, $|dz_1|_V \sim 1$ in a neighborhood of the origin on V (measured in the metric on V induced by the Euclidean metric of \mathbb{C}^2). Recall that

$$\Psi(\omega) = -g \frac{dz_1}{\partial f / \partial z_2} = g \frac{dz_1}{2z_2}.$$

Recall that by definition $\Psi(\omega)$ is in the Grauert-Riemenschneider canonical sheaf \mathcal{K}_V if and only if it is square-integrable on V . But

$$|\Psi(\omega)|_V = \left| g \frac{dz_1}{2z_2} \right|_V \sim |g| \frac{1}{|z_1|^{3/2}},$$

which would not be square-integrable on V at the origin if g were just a holomorphic function. But g is of the form $g = z_1 h_1$ or $g = z_2 h_2$ so that $\Psi(\omega)$ is in fact square-integrable on V . This illustrates the role of the multiplier ideal sheaf $\mathcal{J}(V)$ in the adjunction formula.

Example 5.3. Consider the hypersurface V generated by the polynomial $f(x, y, z) = z^2 - xy$ in \mathbb{C}^3 (with coordinates x, y, z). The embedded resolution $\pi : (V', M') \rightarrow (V, M)$ is obtained by a single blow-up of the origin. The exceptional set E is a single copy of \mathbb{CP}^2 and it easy to check that $E_f = 2E$ as f vanishes to order 2 in the origin.

One can check that $E_f = 2E$ is generated by the three holomorphic functions π^*g_1 , π^*g_2 and π^*g_3 where $g_1 = x^2$, $g_2 = y^2$ and $g_3 = z^2$. It follows that $\mathcal{J}(V) = \mathcal{J}(\varphi)$ with $\varphi = \log(|x|^2 + |y|^2 + |z|^2)$. But if h is a holomorphic function, then $he^{-\varphi} = h/(|x|^2 + |y|^2 + |z|^2)$ is square-integrable in \mathbb{C}^3 . Thus, $\mathcal{J}(V) = \mathcal{J}(\varphi) = \mathcal{O}_{\mathbb{C}^3}$, i.e. V has a canonical singularity (see Theorem 1.3).

We shall check that the adjunction map

$$\Psi : \mathcal{K}_{\mathbb{C}^3} \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) = \mathcal{K}_{\mathbb{C}^3} \otimes \mathcal{O}(V) \longrightarrow \mathcal{K}_V$$

makes sense. A section ω of $\mathcal{K}_{\mathbb{C}^3} \otimes \mathcal{O}(V)$ can be written as

$$\omega = g \frac{dx \wedge dy \wedge dz}{z^2 - xy},$$

where g is just a holomorphic function. To check that $\Psi(\omega)$ is square-integrable on V , we cover the variety V with two charts. For that purpose we cover V by the two parts where either $|x| \geq |y|$ or $|y| \geq |x|$, respectively.

Let $|x| \geq |y|$. Note that on V , this is equivalent to $|z| \leq |x|$. So, we can represent V as a graph with bounded gradient over \mathbb{C}^2 with coordinates x, z under the map $y = G(x, z) = z^2/x$. For the gradient, we get $\nabla G = (-z^2/x^2, 2z/x)$, which is in fact bounded as $|z| \leq |x|$. In the coordinates x, z we have

$$\Psi(\omega) = -g \frac{dx \wedge dz}{\partial f / \partial y} = g \frac{dx \wedge dz}{x}$$

so that

$$|\Psi(\omega)|_V \lesssim \frac{1}{|x|}.$$

But this is in fact square-integrable in \mathbb{C}^2 with coordinates x, z over the region $|z| \leq |x|$ where we have to integrate (the pole of $1/x$ is only met in $0 \in \mathbb{C}^2$).

The second chart is completely analogous by symmetry. Let $|y| \geq |x|$. On V , this is equivalent to $|xy| \leq |y|^2$ or $|z| \leq |y|$. So, we can represent V as a graph with bounded slope over \mathbb{C}^2 with coordinates y, z under the map $x = G(y, z) = z^2/y$. In the coordinates y, z we have

$$\Psi(\omega) = g \frac{dy \wedge dz}{\partial f / \partial x} = -g \frac{dy \wedge dz}{y}$$

so that

$$|\Psi(\omega)|_V \sim \frac{1}{|y|}$$

is square-integrable over the region $|z| \leq |y|$ in \mathbb{C}^2 .

That shows that $\Psi(\omega)$ is in fact square-integrable over V , thus a section in \mathcal{K}_V .

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