ONE PARAMETER REGULARIZATIONS OF PRODUCTS OF RESIDUE CURRENTS

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ABSTRACT. We show that Coleff-Herrera type products of residue currents can be defined by analytic continuation of natural functions depending on one complex variable.

Dedicated to the memory of Mikael Passare

1. Introduction

Let f be a holomorphic function defined on a domain in $\mathbb{C}^n$. It is proved in [15] using Hironaka's desingularization theorem that if φ is a test form then

$$\lim_{\epsilon\to 0^+}\int_{|f|^2>\epsilon}\varphi/f$$

exists and defines the action of a current, denoted $1/f$. The $\bar\partial$-image, $\bar\partial(1/f)$, is the residue current of f and it has the useful property that it is annihilated by a holomorphic function g if and only if g is in the ideal generated by f. If $f_1,\dots,f_q$ are holomorphic functions then the *Coleff-Herrera product* of the currents $\bar\partial(1/f_j)$ is defined as follows. For a test form φ of bidegree $(n,n-q)$ consider the residue integral

$$I_f^\varphi(\epsilon)=\int_{T(\epsilon)}\frac{\varphi}{f_1\cdots f_q},$$

where $T(\epsilon)=\cap_1^q\{|f_j|^2=\epsilon_j\}$. It is proved in [12] that the limit of $\epsilon\mapsto I_f^\varphi(\epsilon)$ exists if $\epsilon=(\epsilon_1,\dots,\epsilon_q)\to 0$ along a path in $\mathbb{R}_+^q$ such that $\epsilon_j/\epsilon_{j+1}^k\to 0$ for all $k\in\mathbb{N}$ and $j=1,\dots,q-1$; such a path is said to be *admissible*. Moreover, the limit defines the action of a current, the Coleff-Herrera product

$$\bar\partial\frac{1}{f_q}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}.\varphi:=\text{“}\lim_{\epsilon\to 0}\text{”}I_f^\varphi(\epsilon), \tag{1.1}$$

where “ lim ” means the limit along an admissible path as above. Following Passare [19], let χ be a smooth approximation of the characteristic function $\mathbf{1}_{[1,\infty)}$ and consider the smooth form

$$\frac{\bar\partial\chi(|f_q|^2/\epsilon_q)}{f_q}\wedge\cdots\wedge\frac{\bar\partial\chi(|f_1|^2/\epsilon_1)}{f_1}. \tag{1.2}$$

It follows from [16, Theorem 2] or the proof of [19, Proposition 2] that the limit in the sense of currents of (1.2) as $\epsilon\to 0$ along an admissible path equals the Coleff-Herrera product, and moreover, that one gets the same result if one first lets $\epsilon_1\to 0$, then

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lets $\epsilon_2 \to 0$ and so on. The Coleff-Herrera product is thus indeed the result of an iterative procedure. In general there are no obvious commutation properties, e.g., $\bar\partial(1/zw) \wedge \bar\partial(1/z) = 0$ whereas $\bar\partial(1/z) \wedge \bar\partial(1/zw) = \bar\partial(1/z^2) \wedge \bar\partial(1/w)$, where the last product is simply a tensor product. However, if $f = (f_1, \dots, f_q)$ defines a complete intersection, i.e., codim $\{f = 0\} = q$, then the Coleff-Herrera product depends in an anticommutative way of the ordering of the tuple f; in fact by [11] the smooth form (1.2) then converges unconditionally. Moreover, also in the complete intersection case, a holomorphic function annihilates the Coleff-Herrera product if and only if it is in the ideal $\langle f_1, \dots, f_q\rangle$; this last property is called the *duality property* and it was proved independently by Dickenstein-Sessa, [13], and Passare, [18].

In this paper we consider another approach to Coleff-Herrera type products; it is based on analytic continuation and has been studied in, e.g., [6, 7, 10, 20, 27]. For $\lambda_j \in \mathbb{C}$ with $\mathfrak{Re}\,\lambda_j \gg 0$, let

$$\Gamma_f^\varphi(\lambda_1, \dots, \lambda_q) = \int \frac{\bar\partial |f_q|^{2\lambda_q} \wedge \cdots \wedge \bar\partial |f_1|^{2\lambda_1}}{f_1 \cdots f_q} \wedge \varphi,$$

where φ is a test form. It is standard to see that $\lambda_1 \mapsto \Gamma_f^\varphi(\lambda_1, \dots, \lambda_q)$ has an analytic continuation to a neighborhood of 0 and that $\Gamma_f^\varphi(0, \lambda_2, \dots, \lambda_q)$ equals

$$\frac{\bar\partial |f_q|^{2\lambda_q}}{f_q} \wedge \cdots \wedge \frac{\bar\partial |f_2|^{2\lambda_2}}{f_2} \wedge \bar\partial \frac{1}{f_1} . \varphi.$$

From [5, Proposition 2.1] it follows that $\lambda_2 \mapsto \Gamma_f^\varphi(0, \lambda_2, \dots, \lambda_q)$ is analytic at 0, that $\lambda_3 \mapsto \Gamma_f^\varphi(0, 0, \lambda_3, \dots, \lambda_q)$ is too, and so on. Once one knows that the Coleff-Herrera product is obtained by letting $\epsilon_j \to 0$ successively in (1.2) it is not that hard to see that

$$\bar\partial \frac{1}{f_q} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} . \varphi = \Gamma_f^\varphi(\lambda_1, \dots, \lambda_q)|_{\lambda_1 = 0} \cdots |_{\lambda_q = 0},$$

where the expression on the right hand side means that we first let $\lambda_1 \to 0$, then let $\lambda_2 \to 0$ etc; see, e.g., [16, Theorem 2]. However, from an algebraic point of view, cf. [8, Theorem 3.2], it is often desirable to have a current given as the value at 0 of a single one-variable analytic function; this is the motivation for this paper. From Theorem 1.2 below it follows that if $\mu_1 > \cdots > \mu_q > 0$ are integers, then $\lambda \mapsto \Gamma_f^\varphi(\lambda^{\mu_1}, \dots, \lambda^{\mu_q})$, a priori defined for $\mathfrak{Re}\,\lambda \gg 0$, has an analytic continuation to a neighborhood of $[0, \infty) \subset \mathbb{C}$ and that the value at $\lambda = 0$ equals the Coleff-Herrera product (1.1). Notice that this way of letting $(\lambda_1, \dots, \lambda_q) \to 0$ is analogous to limits along admissible paths in the sense that λ_j goes to zero much faster than λ_{j+1}, $j = 1, \dots, q-1$.

We remark that if f defines a complete intersection then it is showed in [23] that $\Gamma_f^\varphi(\lambda)$ is analytic in a neighborhood of the half-space $\{\mathfrak{Re}\,\lambda_j \geq 0, j = 1, \dots, q\}$.

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Let us now consider a more general setting. Let f be a section of a Hermitian vector bundle E of rank m over a reduced complex space X of pure dimension n. In [22] and [1] were introduced currents U and R, generalizing the currents $1/f$ and $\bar\partial(1/f)$, respectively. These currents are based on Bochner-Martinelli type expressions. To be precise, let $f = f_1 e_1 + \cdots + f_m e_m$, where $\{e_k\}_k$ is a local holomorphic frame for E with dual frame $\{e_k^*\}_k$, and let $s = s_1 e_1^* + \cdots + s_m e_m^*$ be the section of the dual bundle E^* with pointwise minimal norm such that $f \cdot s = |f|_E^2$. For $\lambda \in \mathbb{C}$, $\mathfrak{Re}\,\lambda \gg 0$, we let

$$U^\lambda := \sum_{k=1}^{m} |f|_E^{2\lambda} \frac{s \wedge (\bar\partial s)^{k-1}}{|f|_E^{2k}}, \tag{1.3}$$

where $(0,1)$-forms anticommute with the e_k^*. It turns out, [1], [22], that $\lambda \mapsto U^\lambda$, considered as a current-valued map, has an analytic continuation to a neighborhood of 0. The value at $\lambda = 0$ is a current U on X that takes values in ΛE^*; U is the standard extension of $\sum_k s \wedge (\bar\partial s)^{k-1}/|f|_E^{2k}$ across $\{f = 0\}$. If E has rank 1, then $U = (1/f)e^*$ for any choice of metric. Let

$$R^\lambda := 1 - |f|_E^{2\lambda} + \sum_{k=1}^{m} \bar\partial |f|_E^{2\lambda} \wedge \frac{s \wedge (\bar\partial s)^{k-1}}{|f|_E^{2k}}. \tag{1.4}$$

Letting $\nabla_f := \delta_f - \bar\partial$, where δ_f denotes interior multiplication with f, one can check that $R^\lambda = 1 - \nabla_f U^\lambda$, see [1] for details. It follows that $\lambda \mapsto R^\lambda$ has an analytic continuation to a neighborhood of 0 and the value at $\lambda = 0$ is the current R; it is straightforward to check that R has support on $\{f = 0\}$. If E has rank 1 then $R = \bar\partial(1/f) \wedge e^*$ and more generally, if f defines a complete intersection then $R = \bar\partial(1/f_m) \wedge \cdots \wedge \bar\partial(1/f_1) \wedge e_1^* \wedge \cdots \wedge e_m^*$ for any choice of metric, see [1] and [22].

The value at $\lambda = 0$ of the term $1 - |f|_E^{2\lambda}$ of R^λ is the restriction $\mathbf{1}_{\{f=0\}}$ to the zero set of f, see [5]. In itself it is zero unless f vanishes identically on some components of X in which case it simply is 1 there. However, when forming products of R's the role of $\mathbf{1}_{\{f=0\}}$ is much more significant, cf. [3] and Example 1.4.

Remark 1.1. For future reference we notice that if $\pi\colon X' \to X$ is a modification such that the pullback of the ideal sheaf defined by f is principal, then one can write $\pi^* f = f^0 f'$, where f^0 is a section of the line bundle $L \to X'$ corresponding to the exceptional divisor and f' is a non-vanishing section of $L^{-1} \otimes \pi^* E$. Equipping L with some Hermitian metric, for instance by setting $|f^0|_L := |\pi^* f|_{\pi^* E}$, we can thus write $\pi^*|f|_E = |f^0|_L |f'|_{L^{-1} \otimes \pi^* E}$. Locally on X' we can identify f^0 and f' by a holomorphic function and a non-vanishing holomorphic tuple, respectively, still denoted f^0 and f'. Hence, locally on X' we have $\pi^*|f|_E = |f^0| u$ for some smooth positive function u.

Let f_j be a section of a Hermitian vector bundle E_j of rank m_j, let U^j and R^j be the associated currents, and let $U^{j,\lambda}$ and $R^{j,\lambda}$ be the corresponding λ-regularizations. Following, e.g., [3] and [16] we define products of the currents R^j recursively as follows. Having defined $R^{k-1} \wedge \cdots \wedge R^1$, consider the current-valued function

$$\lambda \mapsto R^{k,\lambda} \wedge R^{k-1} \wedge \cdots \wedge R^1,$$

a priori defined for $\mathfrak{Re}\,\lambda \gg 0$. It turns out, see, e.g., [5] or [16], that it can be analytically continued to a neighborhood of 0, and we define $R^k \wedge \cdots \wedge R^1$ to be the value at $\lambda = 0$.

Theorem 1.2. *Let $\mu_1 > \cdots > \mu_q$ be positive integers. Then the current-valued function*

$$\lambda \mapsto R^{q,\lambda^{\mu_q}} \wedge \cdots \wedge R^{1,\lambda^{\mu_1}},$$

a priori defined for $\mathfrak{Re}\,\lambda \gg 0$, has an analytic continuation to a neighborhood of the half-axis $[0,\infty) \subset \mathbb{C}$ and the value at $\lambda = 0$ is $R^q \wedge \cdots \wedge R^1$.

To connect with Coleff-Herrera type products, let χ be the characteristic function $\mathbf{1}_{[1,\infty)}$ or a smooth regularization thereof and let

$$R^{j,\epsilon_j} := 1 - \chi(|f_j|^2_{E_j}/\epsilon_j) + \sum_{k=1}^{m_j} \bar\partial\chi(|f_j|^2_{E_j}/\epsilon_j) \wedge \frac{s_j \wedge (\bar\partial s_j)^{k-1}}{|f_j|^{2k}_{E_j}}.$$

If φ is a test form on X, then the limit of

$$\int_X R^{q,\epsilon_q} \wedge \cdots \wedge R^{1,\epsilon_1} \wedge \varphi \tag{1.5}$$

as $\epsilon \to 0$ along an admissible path exists and equals the action of $R^q \wedge \cdots \wedge R^1$ on φ, see [16].

Let us mention a version of Theorem 1.2 with connection to intersection theory. Let f be a section of E and let

$$M^\lambda := 1 - |f|^{2\lambda}_E + \sum_{k\geq 1} \bar\partial|f|^{2\lambda}_E \wedge \frac{\partial \log|f|^2_E}{2\pi i} \wedge (dd^c \log|f|^2_E)^{k-1},$$

where $dd^c = \bar\partial\partial/2\pi i$. It is showed in [3] that $\lambda \mapsto M^\lambda$ has an analytic continuation to a neighborhood of 0 and that the value at $\lambda = 0$ is a positive closed current, which we denote by M. One can give a meaning to the product $(dd^c \log|f|^2_E)^k$ for arbitrary k that extends the classical one for $k \leq \operatorname{codim}\{f = 0\}$, and from [3] it follows that

$$M = \mathbf{1}_Z + \sum_{k\geq 1} \mathbf{1}_Z (dd^c \log|f|^2_E)^k,$$

where $\mathbf{1}_Z$ is the restriction to the zero set Z of f. The current M is closely connected to R. For instance, if X is smooth and D is the Chern connection on E then it follows from [2] that

$$M_k = R_k \cdot (Df/2\pi i)^k/k!,$$

where the subscript k means the component of bidegree $(*, k)$.

Let $f_1, \ldots, f_q$ be sections of Hermitian vector bundles E_j and let $M^1, \ldots, M^q$ be the associated currents. One can define products of the M^j recursively as for the R^j and we have the following analogue of Theorem 1.2.

Theorem 1.3. *Let $\mu_1 > \cdots > \mu_q$ be positive integers. Then the current-valued function*

$$\lambda \mapsto M^{q,\lambda^{\mu_q}} \wedge \cdots \wedge M^{1,\lambda^{\mu_1}},$$

a priori defined for $\mathfrak{Re}\,\lambda \gg 0$, has an analytic continuation to a neighborhood of the half-axis $[0,\infty) \subset \mathbb{C}$ and the value at $\lambda = 0$ is $M^q \wedge \cdots \wedge M^1$.

Example 1.4 (Example 5.6 in [3])**.** Let $\mathcal{J}_x \subset \mathscr{O}_{X,x}$ be an ideal and let $h_1, \ldots, h_n \in \mathcal{J}_x$ be a generic Vogel sequence of $\mathcal{J}_x$; see, e.g., [3] for the definition. By the Stückrad-Vogel procedure, [24], adapted to the local situation, [17], [25], one gets an associated Vogel cycle V^h; the multiplicities of the components of various dimensions of V^h are the Segre numbers, [14], used in excess intersection theory. By Theorem 1.3 we have that

$$\lambda \mapsto \bigwedge_{k=1}^n \big(1 - |h_k|^{2\lambda^{\mu_k}} + \bar\partial|h_k|^{2\lambda^{\mu_k}} \wedge \partial\log|h_k|^2/2\pi i\big)$$

is analytic at 0 and by [3] the value there is the Lelong current associated with V^h; see [3] for more details.

Remark 1.5. Assume that codim $\cap_j \{f_j = 0\} = m_1 + \cdots + m_q$. Then $M^j = (dd^c \log |f_j|^2_{E_j})^{m_j} = [f_j = 0]$, where $[f_j = 0]$ is the Lelong current of the fundamental cycle of f_j, and more generally,

$$M^q \wedge \cdots \wedge M^1 = [f_q = 0] \wedge \cdots \wedge [f_1 = 0],$$

i.e., the current representing the proper intersection of the cycles $[f_j = 0]$.

In this case the current-valued function

$$(\lambda_1, \dots, \lambda_q) \mapsto R^{q,\lambda_q} \wedge \cdots \wedge R^{1,\lambda_1}$$

has an analytic continuation to a neighborhood of the origin in $\mathbb{C}^q$, [16], and the value at $\lambda = 0$ is the R-current associated to $\oplus_j f_j$, [26]. Moreover, by [16], (1.5) depends Hölder continuously on $\epsilon \in [0, \infty)^q$ if χ is smooth. The smoothness of χ is necessary in view of the example in [21, Section 1].

2. Proof of Theorems 1.2 and 1.3

We will actually prove a slightly more general result than Theorem 1.2; we will allow mixed products of U^j and R^k. Let P^j denote either U^j or R^j and let P^{j,λ_j} be the corresponding λ-regularization, (1.3) or (1.4). One defines products of the P^j recursively as above.

Theorem 1.2'. *Let $\mu_1 > \cdots > \mu_q$ be positive integers. Then the current-valued function*

$$\lambda \mapsto P^{q,\lambda^{\mu_q}} \wedge \cdots \wedge P^{1,\lambda^{\mu_1}},$$

a priori defined for $\mathfrak{Re}\, \lambda \gg 0$, has an analytic continuation to a neighborhood of the half-axis $[0, \infty) \subset \mathbb{C}$ and the value at 0 is $P^q \wedge \cdots \wedge P^1$.

Let $\pi \colon X' \to X$ be a smooth modification of X such that $\{\pi^* f_j = 0\}$, $j = 1, \dots, q$, and $\cup_j \{\pi^* f_j = 0\}$ are normal crossings divisors. Then locally in X' we can write $\pi^* f_j = f_j^0 f_j'$, where f_j^0 is a monomial in local coordinates and f_j' is a non-vanishing holomorphic tuple. It follows that $s_j = \bar{f}_j^0 s_j'$, where s_j' is a smooth section. A straightforward computation shows that

$$\pi^* R^{j,\lambda_j} = 1 - |f_j^0|^{2\lambda_j} u_j^{2\lambda_j} + \sum_{k=1}^{m_j} \frac{\bar{\partial}(|f_j^0|^{2\lambda_j} u_j^{2\lambda_j})}{(f_j^0)^k} \wedge \vartheta_{jk},$$

$$\pi^* U^{j,\lambda_j} = \sum_{k=1}^{m_j} \frac{|f_j^0|^{2\lambda_j} u_j^{2\lambda_j}}{(f_j^0)^k} \wedge \vartheta_{jk},$$

where u_j is a smooth non-vanishing function and $\vartheta_{jk} = s_j' \wedge (\bar{\partial} s_j')^{k-1} / u_j^{2k}$ is a smooth form, cf., Remark1.1. In the same way,

$$\pi^* M^{f_j,\lambda_j} = 1 - |f_j^0|^{2\lambda_j} u_j^{2\lambda_j} + \sum_{k \geq 1} \bar{\partial}\big(|f_j^0|^{2\lambda_j} u_j^{2\lambda_j}\big) \wedge \partial \log(|f_j^0|^2 u_j^2) \wedge \omega_{jk},$$

where ω_{jk} is smooth, cf. [3, Section 4]. Taking the identity $\partial \log(|f_j^0|^2 u_j^2) = df_j^0 / f_j^0 + 2\partial u_j / u_j$ into account, Theorems 1.2' and 1.3 are consequences of the following quite technical lemma.

Lemma 2.1. *Let $u_1, \dots, u_r$ be smooth non-vanishing functions defined in some neighborhood of the origin in $\mathbb{C}^n$, with coordinates $(x_1, \dots, x_n)$. For $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{C}^r$, $\mathfrak{Re}\,\lambda_j \gg 0$, $\alpha_1, \dots, \alpha_r \in \mathbb{N}^n$, and $k_1, \dots, k_r \in \mathbb{N}$, let*

$$\Gamma(\lambda) := \frac{|u_r x^{\alpha_r}|^{2\lambda_r} \cdots |u_{p+1} x^{\alpha_{p+1}}|^{2\lambda_{p+1}} \bar\partial |u_p x^{\alpha_p}|^{2\lambda_p} \wedge \cdots \wedge \bar\partial |u_1 x^{\alpha_1}|^{2\lambda_1}}{x^{k_r\alpha_r} \cdots x^{k_1\alpha_1}};$$

here $x^{k_\ell \alpha_\ell} = x_1^{k_\ell \alpha_{\ell,1}} \cdots x_n^{k_\ell \alpha_{\ell,n}}$ if $\alpha_\ell = (\alpha_{\ell,1}, \dots, \alpha_{\ell,n})$. If σ is a permutation of $\{1, \dots, r\}$, write $\Gamma^\sigma(\lambda_1, \dots, \lambda_r) := \Gamma(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(r)})$.

Let $\mu_1, \dots, \mu_r$ be positive integers. Then $\Gamma^\sigma(\kappa^{\mu_1}, \dots, \kappa^{\mu_r})$ has an analytic continuation to a connected neighborhood of the half-axis $[0, \infty)$ in $\mathbb{C}$, and if $\mu_1 > \dots > \mu_r$, then

$$\Gamma^\sigma(\kappa^{\mu_1}, \dots, \kappa^{\mu_r}) \,|_{\kappa=0} = \Gamma^\sigma(\lambda_1, \dots, \lambda_r) \,|_{\lambda_1=0} \cdots |_{\lambda_r=0}\,. \tag{2.1}$$

The reason for the permutation σ is that we have mixed products of U's and R's in Theorem 1.2'.

Proof. To begin with let us assume that all $u_j = 1$. A straightforward computation shows that

$$\Gamma(\lambda) = \lambda_1 \cdots \lambda_p \frac{\prod_{j=1}^r |x^{\alpha_j}|^{2\lambda_j}}{x^{\sum_{j=1}^r k_j \alpha_j}} \sum_I{}' A_I \frac{d\bar x_{i_1} \wedge \cdots \wedge d\bar x_{i_p}}{\bar x_{i_1} \cdots \bar x_{i_p}} =: \lambda_1 \cdots \lambda_p \sum_I{}' \Gamma_I,$$

where the sum is over all increasing multi-indices $I = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$ and A_I is the determinant of the matrix $(\alpha_{\ell, i_j})_{1 \le \ell \le p, 1 \le j \le p}$.

Pick a non-vanishing summand Γ_I; without loss of generality, assume that $I = \{1, \dots, p\}$ and $A_I = 1$. With the notation $b_k(\lambda) := \sum_{\ell=1}^r \lambda_\ell \alpha_{\ell,k}$ for $1 \le k \le n$,

$$\Gamma_I = \frac{\prod_{k=1}^n |x_k|^{2b_k(\lambda)}}{x^{\sum_{j=1}^r k_j \alpha_j}} \frac{d\bar x_1 \wedge \cdots \wedge d\bar x_p}{\bar x_1 \cdots \bar x_p} = \frac{1}{b_1(\lambda) \cdots b_p(\lambda)} \frac{\bigwedge_{k=1}^p \bar\partial |x_k|^{2b_k(\lambda)} \prod_{k=p+1}^n |x_k|^{2b_k(\lambda)}}{x^{\sum_{j=1}^r k_j \alpha_j}}.$$

Now the current-valued function

$$\widetilde\Gamma_I : (\lambda_1, \dots, \lambda_r) \mapsto \frac{\bigwedge_{j=1}^p \bar\partial |x_j|^{2b_j(\lambda)} \prod_{j=p+1}^n |x_j|^{2b_j(\lambda)}}{x^{\sum k_j \alpha_j}}$$

has an analytic continuation to a neighborhood of the origin in $\mathbb{C}^r$; in fact, it is a tensor product of one-variable currents. In particular, $\widetilde\Gamma_I(\kappa^{\mu_1}, \dots, \kappa^{\mu_r}) \,|_{\kappa=0} = \widetilde\Gamma_I(\lambda) \,|_{\lambda_1=0} \cdots |_{\lambda_r=0}$. Let

$$\gamma(\lambda) = \frac{\lambda_1 \cdots \lambda_p}{b_1(\lambda) \cdots b_p(\lambda)}$$

and $\gamma^\sigma = \gamma(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(r)})$. We claim that if $\mu_1 > \dots > \mu_r$, then

$$\gamma^\sigma(\lambda) \,|_{\lambda_1=0} \cdots |_{\lambda_r=0} = \gamma^\sigma(\kappa^{\mu_1}, \dots, \kappa^{\mu_r})|_{\kappa=0},$$

where it is a part of the claim that both sides make sense.

Let us prove the claim. Since $A_I = 1$, reordering the factors $b_1, \dots, b_p$ and multiplying $\gamma(\lambda)$ by a non-zero constant, we may assume that $\alpha_{kk} = 1$, $k = 1, \dots, p$, so that

$$\gamma(\lambda) = \frac{\lambda_1}{\lambda_1 + \alpha_{21}\lambda_2 + \cdots + \alpha_{r1}\lambda_r} \cdots \frac{\lambda_p}{\alpha_{p1}\lambda_1 + \cdots + \lambda_p + \cdots + \alpha_{rp}\lambda_r}.$$

For $j < r$ set $\tau_j := \lambda_j/\lambda_{j+1}$ and $\widetilde{\gamma}^\sigma(\tau_1, \dots, \tau_{r-1}) := \gamma^\sigma(\lambda)$; notice that γ^σ is 0-homogeneous, so that $\widetilde{\gamma}^\sigma$ is well-defined. Then $\lambda_j = \tau_j \cdots \tau_{r-1}\lambda_r$, and therefore $\widetilde{\gamma}^\sigma$ consists of p factors of the form

$$\frac{\tau_k \cdots \tau_{r-1}}{\alpha_{k1}\tau_1 \cdots \tau_{r-1} + \cdots + \tau_k \cdots \tau_{r-1} + \cdots + \alpha_{k,r-1}\tau_{r-1} + \alpha_{kr}}. \tag{2.2}$$

Observe that (2.2) is holomorphic in τ in some neighborhood of the origin. Indeed, if $\alpha_{kr} \neq 0$, then (2.2) is clearly holomorphic, whereas if $\alpha_{kr} = 0$ we can factor out τ_{r-1} from the denominator and numerator. In the latter case (2.2) is clearly holomorphic if $\alpha_{k,r-1} \neq 0$ etc; since $\alpha_{kk} = 1$ this procedure eventually stops. Hence, $\widetilde{\gamma}^\sigma(\tau)$ is holomorphic in a neighborhood of 0. It follows that $\gamma^\sigma(\kappa^{\mu_1}, \dots, \kappa^{\mu_r}) = \widetilde{\gamma}^\sigma(\kappa^{\mu_1-\mu_2}, \dots, \kappa^{\mu_{r-1}-\mu_r})$ is holomorphic in a neighborhood of 0 and since the denominator of $\gamma^\sigma(\kappa^{\mu_1}, \dots, \kappa^{\mu_r})$ is a polynomial in κ with non-negative coefficients it is in fact holomorphic in a neighborhood of $[0, \infty)$. Moreover, $\gamma^\sigma(\lambda_1, \dots, \lambda_r)$ is holomorphic in $\Delta = \{|\lambda_1/\lambda_2| < \epsilon, \dots, |\lambda_{r-1}/\lambda_r| < \epsilon\}$. Let us now fix $\lambda_2 \neq 0, \dots, \lambda_r \neq 0$ in Δ. Then $\gamma^\sigma(\lambda)$ is holomorphic in λ_1 in a neighborhood of the origin. Next, for $\lambda_3 \neq 0, \dots, \lambda_r \neq 0$ fixed in Δ, $\gamma^\sigma(\lambda)|_{\lambda_1=0}$ is holomorphic in λ_2 in a neighborhood of the origin, etc. It follows that

$$\gamma^\sigma(\lambda)|_{\lambda_1} \cdots |_{\lambda_r=0} = \widetilde{\gamma}^\sigma(\tau)|_{\tau=0} = \gamma^\sigma(\kappa^{\mu_1}, \dots, \kappa^{\mu_r})|_{\kappa=0},$$

which proves the claim. Thus (2.1) follows in the case $u_j = 1$, $j = 1, \dots, r$.

Now, consider the general case. Replace each $|u_j|^{2\lambda_j}$ in $\Gamma(\lambda)$ by $|u_j|^{2\omega_j}$, where $\omega_j \in \mathbb{C}$. Then Γ is a sum of terms of the following representative form:

$$\prod_{j=p+1}^{r} |u_j|^{2\omega_j} \prod_{j=1}^{p'} |u_j|^{2\omega_j} \bigwedge_{p'+1}^{p} \bar{\partial}|u_j|^{2\omega_j} \wedge \frac{\prod_{j=p'+1}^{r} |x^{\alpha_j}|^{2\lambda_j} \bigwedge_{j=1}^{p'} \bar{\partial}|x^{\alpha_j}|^{2\lambda_j}}{x^{k_r\alpha_r} \cdots x^{k_1\alpha_1}} \tag{2.3}$$

Fixing all λ_j and ω_j except for $\lambda_{\sigma(1)}$ and $\omega_{\sigma(1)}$, (2.3) becomes an analytic (current-valued) function $g(\lambda_{\sigma(1)}, \omega_{\sigma(1)})$ in a neighborhood of $0 \in \mathbb{C}^2$. Thus, the value at 0 of $g(\lambda_{\sigma(1)}, \lambda_{\sigma(1)})$ is the same as first letting $\omega_{\sigma(1)} = 0$ (which corresponds to setting $u_{\sigma(1)} = 1$) and then letting $\lambda_{\sigma(1)} = 0$ in $g(\lambda_{\sigma(1)}, \omega_{\sigma(1)})$. Continuing analogously for $(\lambda_{\sigma(2)}, \omega_{\sigma(2)})$ and so on, it follows that the right hand side of (2.1) is independent of the u_j.

To see that the left hand side of (2.1) is independent of u_j, replace each λ_j in (2.3) by $\kappa^{\mu_{\sigma(j)}}$ and denote the resulting expression by $\tilde{g}(\kappa, \omega_1, \dots, \omega_r)$. Then $\tilde{g}$ is clearly analytic in the ω_j and by the first part of the proof it is also analytic in a neighborhood of $[0, \infty) \subset \mathbb{C}_\kappa$. Hence, $\tilde{g}$ is analytic in a neighborhood of $0 \in \mathbb{C}^{r+1}$. The left hand side of (2.1) is obtained by evaluating $\kappa \mapsto \tilde{g}(\kappa, \kappa^{\mu_{\sigma(1)}}, \dots, \kappa^{\mu_{\sigma(r)}})$ at $\kappa = 0$; this is thus the same as evaluating $\tilde{g}(\kappa, 0)$ (which corresponds to setting all $u_j = 1$) at $\kappa = 0$. Hence also the left hand side of (2.1) is independent of the u_j and the lemma follows.

□

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