CHERN FORMS OF HERMITIAN METRICS WITH ANALYTIC SINGULARITIES ON VECTOR BUNDLES

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Abstract. We define Chern and Segre forms, or rather currents, associated with a Griffiths positive singular hermitian metric h with analytic singularities on a holomorphic vector bundle E. The currents are constructed as pushforwards of generalized Monge-Ampère products on the projectivization of E. The Chern and Segre currents represent the Chern and Segre classes of E, respectively, and coincide with the Chern and Segre forms of E and h where h is smooth. Moreover, our currents coincide with the Chern and Segre forms constructed by the first three authors and Ruppenthal in the cases when these are defined.

1. Introduction

Singular metrics on line bundles were introduced by Demailly in [De4], and have since developed to be an influential analytic tool in complex algebraic geometry. In [BP] Berndtsson and Păun introduced singular metrics on vector-bundles in order to prove results about pseudo-effectivity of relative canonical bundles. These have been further studied in a series of papers including, e.g., [H, HPS, R]. In order to develop a theory for singular metrics on vector bundles it seems crucial to have Chern forms. In the line bundle case the (first) Chern form is a well-defined current, whereas any attempt to construct Chern forms of singular metrics on higher rank bundles seems to involve multiplication of currents. In [LRRS] the first three authors together with Ruppenthal defined Chern forms for positive singular metrics on vector bundles under a certain natural condition on the dimension of the degeneracy locus. In this paper we define Chern forms without this assumption but for metrics with so-called analytic singularities. To do this we develop a new formalism for generalized (mixed) Monge-Ampère operators for plurisubharmonic functions with analytic singularities extending the construction in [AW].

Let E be a holomorphic vector bundle of rank r over a complex manifold X of dimension n and let h be a smooth hermitian metric on E. Let $\pi : \mathbf{P}(E) \to X$ be the projective bundle of lines in E^*. Then h^* induces a metric on the tautological line bundle $\mathcal{O}_{\mathbf{P}(E)}(-1) \subset \pi^* E^*$; let $e^{-\varphi}$ be the dual metric on $\mathcal{O}_{\mathbf{P}(E)}(1)$. If h is Griffiths positive, then $e^{-\varphi}$ is a positive metric, i.e., the local weights φ are plurisubharmonic (psh), and the first Chern form of $e^{-\varphi}$ is given as $dd^c\varphi$, where $d^c = (1/4\pi i)(\partial - \bar\partial)$. Note that this is a well-defined global positive $(1,1)$-form. Following the ideas in, e.g., [F] one can define the associated *kth Segre form* as

$$s_k(E,h) := (-1)^k \pi_*(dd^c\varphi)^{k+r-1}, \tag{1.1}$$

cf. [M, Section 7.1]. Since π is a submersion this is a smooth form of bidegree (k,k). It was proved in [M, Proposition 6], see also [Di, Proposition 1.1] and [G, Proposition 3.1], that (1.1) coincides with the classical definition of Segre forms, which means that the total Segre form $s(E,h) = 1 + s_1(E,h) + s_2(E,h) + \cdots$ is the multiplicative inverse of the total Chern form $c(E,h) = 1 + c_1(E,h) + c_2(E,h) + \cdots$. Identifying components of the same bidegree, this can be expressed as

$$s_k(E,h) + s_{k-1}(E,h) \wedge c_1(E,h) + \cdots + c_k(E,h) = 0,\ k = 1,2,\ldots . \tag{1.2}$$

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In particular, (1.2) holds on cohomology level, i.e., the total Segre class $s(E) = 1+s_1(E)+s_2(E)+\cdots$ is the multiplicative inverse of the total Chern class $c(E) = 1 + c_1(E) + c_2(E) + \cdots$; here $c_k(E)$ and $s_k(E)$ are the *kth Chern* and *Segre classes* of E, defined as the de Rham cohomology classes of $c_k(E, h)$ and $s_k(E, h)$, respectively.

The aim of this paper is to construct Chern and Segre forms, or rather currents, associated with singular metrics. Therefore let h be a Griffiths positive singular metric on E in the sense of Berndtsson-Păun, [BP], see Section 5; then the induced singular metric $e^{-\varphi}$ on $\mathcal{O}_{\mathbf{P}(E)}$ is positive, cf. Proposition 5.2. Our strategy is to mimic the construction (1.1) of Segre forms and use them to construct Chern and Segre currents. However, in general one cannot take products of currents and in particular $(dd^c\varphi)^k$ is not always well-defined.

Recall that a psh function u has *analytic singularities* if it is locally of the form

$$u = c \log |F|^2 + v, \tag{1.3}$$

where $c > 0$, F is a tuple of holomorphic functions f_j, $|F|^2 = \sum |f_j|^2$, and v is bounded. We say that h has *analytic singularities* if the weights φ are psh with analytic singularities; for a direct definition in terms of h, see Proposition 5.4. In [H] Hosono constructed a class of examples of singular hermitian metrics on vector bundles, that in fact have analytic singularities, see Example 5.5.

Given a psh function u with analytic singularities, in [AW] Andersson and the last author defined *generalized Monge-Ampère products* $(dd^cu)^m$ recursively as

$$(dd^cu)^k := dd^c(u\mathbf{1}_{X\setminus Z}(dd^cu)^{k-1}), \tag{1.4}$$

where Z is the unbounded locus of u, i.e., locally defined as $\{F = 0\}$ where u is given by (1.3); for u of the form $u = \log |F|^2$ the currents (1.4) were defined in [A]. The current $(dd^cu)^m$ is positive and closed and of bidegree (m, m). For $m \leq \operatorname{codim} Z$, it coincides with Bedford-Taylor-Demailly's classically defined $(dd^cu)^m$, cf. Section 2.1. If α is a closed smooth $(1, 1)$-form, inspired by [ABW, Theorem 1.2], cf. Remark 3.6, we let

$$[dd^cu]^m_\alpha := (dd^cu)^m + \sum_{\ell=0}^{m-1} \alpha^{m-\ell}\mathbf{1}_Z(dd^cu)^\ell \tag{1.5}$$

for $m \geq 1$ and $[dd^cu]^0_\alpha = 1$; see (3.11) for a recursive description. Note that if $m \leq \operatorname{codim} Z$, then $\mathbf{1}_Z(dd^cu)^\ell = 0$ for $\ell < m$ and thus $[dd^cu]^m_\alpha = (dd^cu)^m$.

Now assume that h has analytic singularities and let θ be the first Chern form of a smooth metric $e^{-\psi}$ on $\mathcal{O}_{\mathbf{P}(E)}(1)$; e.g., $e^{-\psi}$ can be chosen as the metric on $\mathcal{O}_{\mathbf{P}(E)}(1)$ induced by a smooth metric on E. Since the difference of two local weights φ is of the form $\log |f|^2$, where f is a nonvanishing holomorphic function, $[dd^c\varphi]^m_\theta$ is a globally defined current on $\mathbf{P}(E)$, see Section 4. Inspired by (1.1) we define

$$s_k(E, h, \theta) := (-1)^k\pi_*[dd^c\varphi]^{k+r-1}_\theta. \tag{1.6}$$

If the φ are smooth, then clearly $s_k(E, h, \theta)$ coincides with $s_k(E, h)$ defined by (1.1).

To construct Chern currents we need to define products of this kind of currents. Let $E_1, \dots, E_t$ be disjoint copies of E and let $\pi : Y \to X$ be the fiber product $Y = \mathbf{P}(E_t) \times_X \cdots \times_X \mathbf{P}(E_1)$. Let φ_j and θ_j denote the pullbacks to Y of the metric and the form on $\mathbf{P}(E_j)$ corresponding to φ and θ, respectively. By extending ideas in [AW] and [ASWY] we give meaning to products

$$[dd^c\varphi_t]^{m_t}_{\theta_t} \wedge \cdots \wedge [dd^c\varphi_1]^{m_1}_{\theta_1}$$

on Y, see Sections 3 and 4. Next for $k_j \geq 1$ we define

$$s_{k_t}(E, h, \theta) \wedge \cdots \wedge s_{k_1}(E, h, \theta) := (-1)^k\pi_*\big([dd^c\varphi_t]^{k_t+r-1}_{\theta_t} \wedge \cdots \wedge [dd^c\varphi_1]^{k_1+r-1}_{\theta_1}\big), \tag{1.7}$$

see Section 6.1; here and throughout $k := k_1 + \cdots + k_t$. If h, and thus φ, is smooth, then (1.7) just coincides with the product $s_{k_t}(E, h) \wedge \cdots \wedge s_{k_1}(E, h)$ of smooth Segre forms, cf. Section 6.2.

The currents (1.7) are in general not commutative in the factors $s_{k_j}(E,h,\theta)$, see Example 8.4. Now we can recursively define Chern currents $c_k(E,h,\theta)$ using the identities (1.2), i.e.,

$$\begin{aligned} c_1(E,h,\theta) &:= -s_1(E,h,\theta), \\ c_2(E,h,\theta) &:= s_1(E,h,\theta)^2 - s_2(E,h,\theta), \\ &\vdots \\ c_k(E,h,\theta) &:= \sum_{k_1+\cdots+k_t=k} (-1)^t s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta). \end{aligned} \tag{1.8}$$

Theorem 1.1. *Let h be a Griffiths positive hermitian metric with analytic singularities on the holomorphic vector bundle $E\to X$ over a complex manifold X of dimension n and let θ be the first Chern form of a smooth metric on $\mathcal{O}_{\mathbf{P}(E)}(1)$. Then for $k=1,2,\dots$ $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ defined by* (1.6) *and* (1.8)*, respectively, are closed normal (k,k)-currents; more precisely they are locally differences of closed positive currents. Moreover*

1. *$c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ represent the kth Chern and Segre classes $c_k(E)$ and $s_k(E)$ of E, respectively, as de Rham cohomology classes of currents,*
2. *$c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ coincide with the Chern and Segre forms $c_k(E,h)$ and $s_k(E,h)$, respectively, where h is smooth,*
3. *the Lelong numbers of $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ at each $x\in X$ are independent of θ.*

Note that Lelong numbers of $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ are well-defined since the currents are locally differences of closed positive currents, cf. Section 6.4.

Assume that the unbounded locus of $\log\det h^*$ is contained in a variety V of pure codimension p. Then for $k\le p$, $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ are independent of θ, see Section 8. In general, however, $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ do depend on θ, cf. Examples 8.1, 8.2 and 8.4.

In [LRRS] the first three authors together with Ruppenthal showed that if h is a singular hermitian metric (not necessarily with analytic singularities) such that the unbounded locus of $\log\det h^*$ is contained in a variety V of pure codimension p, then for $k_1+\cdots+k_t\le p$ one can give meaning to currents $s_{k_t}(E,h)\wedge\cdots\wedge s_{k_1}(E,h)$ as limits of $s_{k_t}(E,h_{\varepsilon_t})\wedge\cdots\wedge s_{k_1}(E,h_{\varepsilon_1})$, where h_{ε_j} are smooth metrics approximating h, see Section 7. Analogously to (1.8) one can then define Chern currents $c_k(E,h)$ for $k\le p$. We should remark that this construction cannot be extended to general k, see Example 8.1.

Theorem 1.2. *Let h be a Griffiths positive hermitian metric with analytic singularities on the holomorphic vector bundle $E\to X$ over a complex manifold X and let θ be the first Chern form of a smooth metric on $\mathcal{O}_{\mathbf{P}(E)}(1)$. Assume that the unbounded locus of $\log\det h^*$ is contained in a variety $V\subset X$. Then for $k_1+\cdots+k_t\le \operatorname{codim}V$,*

$$s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta) = s_{k_t}(E,h)\wedge\cdots\wedge s_{k_1}(E,h). \tag{1.9}$$

In particular $c_k(E,h,\theta)=c_k(E,h)$ for $k\le\operatorname{codim}V$.

The paper is organized as follows. In Section 2 we give some background on currents and classical Monge-Ampère products. In Section 3 we introduce mixed Monge-Ampère products of psh functions with analytic singularities generalizing (1.4). Next in Sections 4 and 5 we study metrics with analytic singularities on line bundles and vector bundles, respectively. The proofs of Theorems 1.1 and 1.2 occupy Sections 6 and 7, respectively. Finally, in Section 8 we conclude with some examples and remarks.

2. Preliminaries

Let us first recall some results on (closed positive) currents. If $\pi:\widetilde{X}\to X$ is a proper map, μ is a current on $\widetilde{X}$, and α is a smooth form on X, then we have the projection formula

$$\alpha\wedge\pi_*\mu = \pi_*(\pi^*\alpha\wedge\mu). \tag{2.1}$$

Moreover, if p is a proper submersion, μ is a current on X, and α is a smooth form on $\widetilde{X}$, then

$$p_*\alpha \wedge \mu = p_*(\alpha \wedge p^*\mu). \tag{2.2}$$

The *Poincaré-Lelong formula* asserts that if f is a holomorphic function defining a divisor D, then

$$dd^c \log |f|^2 = [D],$$

where $[D]$ is the current of integration along the divisor of f.

Given a subset $A \subset X$, let $\mathbf{1}_A$ denote the characteristic function of A. If $Z \subset X$ is a subvariety and T is a closed positive current on X, then the Skoda-El Mir theorem asserts that $\mathbf{1}_{X\setminus Z}T$ is again positive and closed. It follows that if $U \subset X$ is any constructible set[1], i.e., a set in the Boolean algebra generated by Zariski open sets in X, then also $\mathbf{1}_U T$ is positive and closed. Note that if U_1 and U_2 are two constructible sets in X, then

$$\mathbf{1}_{U_1\cap U_2}T = \mathbf{1}_{U_1}\mathbf{1}_{U_2}T. \tag{2.3}$$

Also, note that if χ_ϵ is any sequence of bounded functions such that $\chi_\varepsilon \to \mathbf{1}_U$ pointwise, then by dominated convergence, $\mathbf{1}_U T = \lim_\varepsilon \chi_\epsilon T$. It follows that if π is as above, then

$$\mathbf{1}_U \pi_* T = \pi_*(\mathbf{1}_{\pi^{-1}U}T). \tag{2.4}$$

Moreover, if $Z \subset X$ is a subvariety (locally) defined by a holomorphic tuple F, then $\mathbf{1}_{X\setminus Z}T$ equals the limit of $|F|^{2\lambda}T$ as $\lambda \to 0^+$.

Finally recall that a closed positive (or normal) current of bidegree (k,k) on X that has support on a subvariety of X of codimension $> k$ vanishes. We refer to this as the *dimension principle.* In particular, if $W, Z \subset X$ are subvarieties such that W is of pure codimension p and $\operatorname{codim}(Z \cap W) > p$, then

$$\mathbf{1}_Z[W] = 0. \tag{2.5}$$

2.1. **Classical Bedford-Taylor-Demailly Monge-Ampère products.** Let $u_1, \dots, u_m$ be locally bounded psh functions on a complex manifold X and let T be a closed positive current on X. The classical Bedford-Taylor theory asserts that one can define a closed positive current

$$dd^c u_m \wedge \cdots \wedge dd^c u_1 \wedge T$$

recursively as

$$dd^c u_k \wedge \cdots \wedge dd^c u_1 \wedge T := dd^c\big(u_k dd^c u_{k-1} \wedge \cdots \wedge dd^c u_1 \wedge T\big), \tag{2.6}$$

for $k = 1, \dots, m$. This current satisfies the following version of the Chern-Levine-Nirenberg inequalities, see [De3, Chapter III, Propositions 3.11 and 4.6].

Proposition 2.1. *Given compacts $L \subset\subset K$ there is a constant $C_{K,L}$ such that for all closed positive currents T and psh functions $v, u_1, \dots, u_m$, where $u_1, \dots, u_m$ are locally bounded,*

$$||v dd^c u_m \wedge \cdots \wedge dd^c u_1 \wedge T||_L \le C_{K,L}||v||_{L^1(K)}||u_1||_{L^\infty(K)} \cdots ||u_m||_{L^\infty(K)}||T||_K.$$

Here $\|S\|_K$ is the *mass semi-norm* of the order zero current S with respect to the compact set K, see [De3, Chapter III.3].

Recall that the *unbounded locus* $L(u)$ of a psh function u is the set of points $x \in X$ such that u is unbounded in every neighborhood of x. Note that if u has analytic singularities, then $L(u)$ is an analytic set, locally defined by $F = 0$ where u is given by (1.3). Demailly [De1] extended the definition (2.6) to the case when the intersection of the unbounded loci of the u_j is small in a certain sense. The following is a simple corollary of [De3, Chapter III, Theorem 4.5].

Proposition 2.2. *Let $u_1, \dots, u_m$ be psh functions on a complex manifold X such that the unbounded locus $L(u_j)$ is contained in analytic set $Z_j \subset X$ for each j. Moreover, let T be a closed positive current of bidegree (p,p) with support contained in an analytic set $W \subset X$. Assume that*

$$\operatorname{codim}(Z_{i_1} \cap \cdots \cap Z_{i_\ell} \cap W) \ge \ell + p$$

[1]Simple examples of constructible sets are $V \setminus W$ or $(X \setminus V) \cup W$ for analytic sets V and W in X.

for all choices of $1 \le i_1 < \cdots < i_\ell \le m$*. Then* $u_m dd^c u_{m-1} \wedge \cdots \wedge dd^c u_1 \wedge T$ *and* $dd^c u_m \wedge \cdots \wedge dd^c u_1 \wedge T$ *are well-defined and have locally finite mass. The latter is a closed positive current.*

These products satisfy the following continuity properties, see, e.g., [De3, Chapter III, Proposition 4.9].

Proposition 2.3. *Let* u_j *and* T *be as in Proposition 2.2. If* $u_j^{(\iota)}$ *are sequences of psh functions decreasing to* u_j*, then*

$$u_m^{(\iota)} dd^c u_{m-1}^{(\iota)} \wedge \cdots \wedge dd^c u_1^{(\iota)} \wedge T \to u_m dd^c u_{m-1} \wedge \cdots \wedge dd^c u_1 \wedge T$$
$$dd^c u_m^{(\iota)} \wedge \cdots \wedge dd^c u_1^{(\iota)} \wedge T \to dd^c u_m \wedge \cdots \wedge dd^c u_1 \wedge T.$$

We have the following generalization of (2.5).

Lemma 2.4. *Assume that* $W \subset X$ *is a subvariety of pure codimension* p*, and assume that* $Z \subset X$ *is a subvariety, such that* $\operatorname{codim}_X(Z \cap W) > p$*. Moreover, assume that* $b_1, \ldots, b_\ell$ *are locally bounded psh functions. Then*

$$\mathbf{1}_Z dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] = 0. \tag{2.7}$$

Remark 2.5. Assume that $W \subset X$ is a subvariety and $U \subset X$ is a constructible set. Then it follows from Lemma 2.4 that

$$\mathbf{1}_U dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] = dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge \mathbf{1}_U [W]. \tag{2.8}$$

To prove (2.8) we may assume that U is a subset of W and that W is irreducible. We then need to prove that

$$\mathbf{1}_U dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] = dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W]$$

if U is dense in W and $\mathbf{1}_U dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] = 0$ otherwise.

To see this, note on the one hand, that if U is not dense in W, then, since W is irreducible, $\overline{U}$ is a subvariety of W of positive codimension. Thus

$$\mathbf{1}_U dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] = \mathbf{1}_U \mathbf{1}_{\overline{U}} dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] = 0$$

in view of (2.3) and Lemma 2.4. On the other hand, if U is dense in W, i.e., $\overline{U} = W$, since $\overline{\overline{U} \setminus U}$ is a subvariety of W of positive codimension, then again using Lemma 2.4, note that

$$\mathbf{1}_U dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] = \mathbf{1}_W dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] -$$
$$\mathbf{1}_{\overline{U} \setminus U} \mathbf{1}_{\overline{\overline{U} \setminus U}} dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] = dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W].$$

□

Proof of Lemma 2.4. We may assume that X is connected. Let us first assume that $W = X$ so that $[W] = 1$ and $Z \subset X$ is a subvariety of positive codimension. Then the lemma follows by a small modification of the proof of Corollary 3.3 in [AW]: It is enough to consider the case when Z is smooth. The general case then follows by stratification. Since it is a local statement, we may choose coordinates z so that $Z = \{z_1 = \cdots = z_q = 0\}$, where $q = \operatorname{codim} Z$. In view of (2.3) it is enough to prove that $\mathbf{1}_{\{z_1=0\}} dd^c b_\ell \wedge \cdots \wedge dd^c b_1 = 0$. Notice that in a set $|z_1| \le r, |(z_2, \ldots, z_n)| \le r'$, we have that $\mathbf{1}_{\{z_1=0\}}(dd^c b_\ell \wedge \cdots \wedge dd^c b_1)$ is the limit of

$$-(|z_1|^{2\lambda} - 1)(dd^c b_\ell \wedge \cdots \wedge dd^c b_1)$$

as $\lambda \to 0^+$, cf. the beginning of this section. Since $|z_1|^{2\lambda} - 1$ is psh, (2.7) follows from Proposition 2.1 since the total mass of $|z_1|^{2\lambda} - 1$ tends to 0 when $\lambda \to 0^+$.

Next, let us assume that W is smooth. We claim that then

$$dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] = i_*(dd^c i^* b_\ell \wedge \cdots \wedge dd^c i^* b_1), \tag{2.9}$$

where i is the inclusion $i : W \to X$. Taking this for granted, since $\dim(Z \cap W) > p$ it follows that $i^{-1}Z$ is a proper subvariety of W and thus $\mathbf{1}_{i^{-1}Z} dd^c i^* b_\ell \wedge \cdots \wedge dd^c i^* b_1$ vanishes by the argument above, and thus in view of (2.4),

$$\mathbf{1}_Z dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] = i_*\big(\mathbf{1}_{i^{-1}Z} dd^c i^* b_\ell \wedge \cdots \wedge dd^c i^* b_1\big) = 0.$$

It is clear that (2.9) holds if the b_j are smooth. For general locally bounded psh b_j let $b_j^{(\iota)}$ be sequences of smooth psh functions decreasing to b_j. Then (2.9) follows from the smooth case and Proposition 2.3.

For the general case, let $\pi : \widetilde{X} \to X$ be an embedded resolution of W in X. In particular, $\widetilde{W} := \overline{\pi^{-1}W_{\text{reg}}}$ is a smooth manifold of pure codimension p in $\widetilde{X}$ and π is a biholomorphism outside a hypersurface $H \subset \widetilde{X}$, such $\pi(H \cap \widetilde{W})$ has codimension $> p$ in X. It follows that outside $\pi(H \cap \widetilde{W})$, $\pi_*[\widetilde{W}] = [W]$, and by the dimension principle this holds everywhere on X.

If b_j are smooth, then in view of (2.1)

$$dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] = \pi_*\big(dd^c \pi^* b_\ell \wedge \cdots \wedge dd^c \pi^* b_1 \wedge [\widetilde{W}]\big). \tag{2.10}$$

For general b_j (2.10) follows by approximating by sequences of smooth psh functions converging to b_j using Proposition 2.3.

Next let $\widetilde{Z} = \pi^{-1}Z$. Then $\widetilde{Z} \cap \widetilde{W} \subset \widetilde{X}$ is a subvariety of codimension $> p$. Indeed, for each connected component $\widetilde{W}_j$ of $\widetilde{W}$, $\widetilde{Z} \cap \widetilde{W}_j$ is a subvariety of $\widetilde{W}_j$. Assume that $\widetilde{Z} \cap \widetilde{W}_j = \widetilde{W}_j$ for some j. Then

$$Z \cap W \supset \pi(\widetilde{Z} \cap \widetilde{W}) \supset \pi(\widetilde{W}_j);$$

however $\pi(\widetilde{W}_j)$ has codimension p, which contradicts that $Z \cap W$ has codimension $> p$. Thus $\widetilde{Z} \cap \widetilde{W}_j$ is a proper subvariety of $\widetilde{W}_j$ for each j. Thus, as proved above,

$$\mathbf{1}_{\widetilde{Z}} dd^c \pi^* b_\ell \wedge \cdots \wedge dd^c \pi^* b_1 \wedge [\widetilde{W}] = 0$$

and, in view of (2.4), we conclude that

$$\mathbf{1}_Z dd^c b_\ell \wedge \cdots \wedge dd^c b_1 \wedge [W] = \pi_*\big(\mathbf{1}_{\widetilde{Z}} dd^c \pi^* b_\ell \wedge \cdots \wedge dd^c \pi^* b_1 \wedge [\widetilde{W}]\big) = 0.$$

□

3. Generalized mixed Monge-Ampère products

Assume that $u_1, \ldots, u_m$ are psh functions with analytic singularities on a complex manifold X with unbounded loci $Z_1, \ldots, Z_m$, respectively. Moreover assume that $U_1, \ldots, U_m \subset X$ are constructible sets contained in $X \setminus Z_1, \ldots, X \setminus Z_m$, respectively. Inspired by [AW, Section 4] we consider currents

$$dd^c u_m \mathbf{1}_{U_m} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1} \tag{3.1}$$

defined recursively as

$$dd^c u_k \mathbf{1}_{U_k} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1} := dd^c\big(u_k \mathbf{1}_{U_k} dd^c u_{k-1} \mathbf{1}_{U_{k-1}} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1}\big) \tag{3.2}$$

for $k = 1, \ldots, m$. In particular, if $u_j = u$ and $U_j = X \setminus Z_j$ for all j, then (3.2) coincides with (1.4). For aesthetic reasons, and to emphasize that U_j is associated with u_j, we choose to write (3.1) rather than

$$dd^c u_m \wedge \mathbf{1}_{U_m} dd^c u_{m-1} \cdots dd^c u_2 \wedge \mathbf{1}_{U_2} dd^c u_1 \mathbf{1}_{U_1}.$$

We say that a current of the form $\mathbf{1}_U dd^c u_m \mathbf{1}_{U_m} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1}$, where U is a constructible set, is a *(closed positive) current with analytic singularities.* We also include currents (3.1) with no factor $dd^c u_j \mathbf{1}_{U_j}$; in other words $\mathbf{1}_U$ is also a current with analytic singularities. For u_j of the form $\log |F_j|^2$ currents like (3.1) were defined in [ASWY, Section 5]. Note that $dd^c u_m \mathbf{1}_{U_m} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1}$ vanishes unless U_1 is dense in (at least one connected component of) X.

Proposition 3.2 below asserts that this definition makes sense and that the currents $dd^c u_k \mathbf{1}_{U_k} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1}$ are positive and closed. Moreover, Proposition 3.4 asserts that

$$dd^c u_m \mathbf{1}_{X\setminus Z_m} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{X\setminus Z_1} \tag{3.3}$$

coincides with $dd^c u_m \wedge \cdots \wedge dd^c u_1$, when this current is well-defined, cf. Proposition 2.2. It is therefore tempting to think of (3.3) as a generalized mixed Monge-Ampère product, and just denote it by $dd^c u_m \wedge \cdots \wedge dd^c u_1$. The following example shows, however, that this "product" lacks some properties one would naturally ask for of a product. In particular, it is not additive in any factor except the right-most one nor commutative.

Example 3.1. Let $u_1 = \log|z_1|^2$ and $u_2 = \log|z_1 z_2|^2$ in $X = \mathbf{C}^2$. Then u_1 and u_2 are psh with analytic singularities with unbounded loci $Z_1 = \{z_1 = 0\}$ and $Z_2 = \{z_1 = 0\} \cup \{z_2 = 0\}$, respectively. In view of the Poincaré-Lelong formula it follows that

$$dd^c u_2 \mathbf{1}_{X\setminus Z_2} \wedge dd^c u_1 \mathbf{1}_{X\setminus Z_1} = dd^c\big(u_2 \mathbf{1}_{X\setminus Z_2}[z_1 = 0]\big) = 0$$

but

$$dd^c u_1 \mathbf{1}_{X\setminus Z_1} \wedge dd^c u_2 \mathbf{1}_{X\setminus Z_2} = dd^c\big(u_1 \mathbf{1}_{X\setminus Z_1}([z_1 = 0] + [z_2 = 0])\big) = [z_1 = 0] \wedge [z_2 = 0] = [0],$$

so that $dd^c u_2 \mathbf{1}_{X\setminus Z_2} \wedge dd^c u_1 \mathbf{1}_{X\setminus Z_1}$ is not commutative in the factors $dd^c u_j \mathbf{1}_{X\setminus Z_j}$.

Moreover, let $u_3 = \log|z_2|^2$. Then u_3 is psh with analytic singularities with unbounded locus $Z_3 = \{z_2 = 0\}$. Note that $u_2 = u_1 + u_3$. Now

$$\begin{aligned} &dd^c u_1 \mathbf{1}_{X\setminus Z_1} \wedge dd^c u_1 \mathbf{1}_{X\setminus Z_1} + dd^c u_3 \mathbf{1}_{X\setminus Z_3} \wedge dd^c u_1 \mathbf{1}_{X\setminus Z_1} = \\ &\qquad\qquad 0 + [z_2 = 0] \wedge [z_1 = 0] = [0] \neq dd^c u_2 \mathbf{1}_{X\setminus Z_2} \wedge dd^c u_1 \mathbf{1}_{X\setminus Z_1}. \end{aligned}$$

□

Proposition 3.2. *Let u_j and U_j be as above. Assume that $dd^c u_k \mathbf{1}_{U_k} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1}$ is inductively defined via* (3.2)*. Let $u_{k+1}^{(\iota)}$ be a sequence of smooth psh functions in X decreasing to u_{k+1}. Then*

$$u_{k+1} \mathbf{1}_{U_{k+1}} dd^c u_k \mathbf{1}_{U_k} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1} := \lim_\iota u_{k+1}^{(\iota)} \mathbf{1}_{U_{k+1}} dd^c u_k \mathbf{1}_{U_k} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1}$$

has locally finite mass and does not depend on the choice of sequence $u_{k+1}^{(\iota)}$. Moreover $dd^c u_{k+1} \mathbf{1}_{U_{k+1}} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1}$, defined by (3.2)*, is positive and closed.*

Remark 3.3. Note that Proposition 3.2 asserts that if T is a current with analytic singularities, u is a psh function with analytic singularities with unbounded locus Z, and U is a constructible set contained in $X \setminus Z$, then

$$dd^c u \wedge \mathbf{1}_U T := dd^c(u \mathbf{1}_U T)$$

is a well-defined current with analytic singularities. □

The proof is a generalization of the proof of Proposition 4.1 in [AW].

Proof. Since the statement is local we may assume that $u_j = \log|F_j|^2 + v_j$. Moreover without loss of generality we may assume that X is connected and that U_1 is dense in X, and thus $\mathbf{1}_{U_1} = 1$ as a distribution. Indeed, otherwise $dd^c u_k \mathbf{1}_{U_k} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1} = 0$.

Let $\pi\colon \widetilde{X} \to X$ be a smooth modification such that locally on $\widetilde{X}$, $\pi^* F_j = f_j f_j'$, where f_j is a holomorphic function and f_j' is a nonvanishing tuple of holomorphic functions, for each j. It follows that $\pi^* u_j = \log|f_j|^2 + b_j$, where $b_j := \log|f_j'|^2 + \pi^* v_j$ is psh and locally bounded, cf. the proof of Proposition 4.1 in [AW]. Note that for two different local representations, the f_j differ by a nonvanishing holomorphic factor, and thus the b_j differ by a pluriharmonic term. Therefore the local divisors $\{f_j = 0\}$ define a global divisor D_j on $\widetilde{X}$, such that $\pi^{-1} Z_j = |D_j|$ and moreover the currents $dd^c b_j$ define a global positive current on $\widetilde{X}$. By the Poincaré-Lelong formula

$$dd^c \pi^* u_j = [D_j] + dd^c b_j.$$

Let $u_1^{(\iota)}$ be a sequence of smooth psh functions decreasing to u_1. Since π is a modification

$$dd^c u_1^{(\iota)} = \pi_*(dd^c\pi^* u_1^{(\iota)}) \to \pi_*\big(dd^c\pi^* u_1\big) = \pi_*\big([D_1] + dd^c b_1\big)$$

and it follows that

$$dd^c u_1 = \pi_*\big([D_1] + dd^c b_1\big).$$

Let us now assume that we have proved that $u_k\mathbf{1}_{U_k} dd^c u_{k-1}\mathbf{1}_{U_{k-1}}\wedge\cdots\wedge dd^c u_1\mathbf{1}_{U_1}$ is well-defined with the desired properties and that $dd^c u_k\mathbf{1}_{U_k}\wedge\cdots\wedge dd^c u_1\mathbf{1}_{U_1}$ is the pushforward of

$$\sum_{I=\{i_1,\dots,i_s\}\subset\{1,\dots,k\}} dd^c b_I \wedge [V_I], \tag{3.4}$$

where $dd^c b_I = dd^c b_{i_s}\wedge\cdots\wedge dd^c b_{i_1}$ and V_I are analytic cycles[2] of pure codimension $k-s$ on $\widetilde{X}$. Since the b_j are determined up to addition by a pluriharmonic term, each $dd^c b_I\wedge[V_I]$ is a globally defined current on $\widetilde{X}$, cf. Section 2.1.

We will prove that:

(i) $u_{k+1}\mathbf{1}_{U_{k+1}} dd^c u_k\mathbf{1}_{U_k}\wedge\cdots\wedge dd^c u_1\mathbf{1}_{U_1} := \lim_\iota u_{k+1}^{(\iota)}\mathbf{1}_{U_{k+1}} dd^c u_k\mathbf{1}_{U_k}\wedge\cdots\wedge dd^c u_1\mathbf{1}_{U_1}$ *has locally finite mass and is independent of* $u_{k+1}^{(\iota)}$,

(ii) *the current*

$$dd^c u_{k+1}\mathbf{1}_{U_{k+1}}\wedge\cdots\wedge dd^c u_1\mathbf{1}_{U_1} := dd^c\big(u_{k+1}\mathbf{1}_{U_{k+1}} dd^c u_k\mathbf{1}_{U_k}\wedge\cdots\wedge dd^c u_1\mathbf{1}_{U_1}\big),$$

is the pushforward of a current of the form (3.4).

As soon as (i) and (ii) are verified, the proposition follows by induction.

Note that $\widetilde{U}_j := \pi^{-1}U_j$ is a constructible set in $\widetilde{X}$. Let us consider one summand $dd^c b_I\wedge[V_I]$ in (3.4). Let V_I' be the union of the irreducible components V_I^j of V_I such that $\widetilde{U}_{k+1}\cap V_I^j$ is dense in V_I^j. Then in view of Remark 2.5

$$\mathbf{1}_{\widetilde{U}_{k+1}} dd^c b_I\wedge[V_I] = dd^c b_I\wedge[V_I'].$$

We claim that $|D_{k+1}|\cap|V_I'|$ has positive codimension in $|V_I'|$. To see this let V_I^j be an irreducible component of $|V_I'|$. Then either $V_I^j\subset|D_{k+1}|$ or $|D_{k+1}|\cap V_I^j$ has positive codimension in V_I^j. However V_I^j cannot be contained in $|D_{k+1}|$ since $\widetilde{U}_{k+1}\cap V_I^j\subset(\widetilde{X}\setminus|D_{k+1}|)\cap V_I^j$ is dense in V_I^j; this proves the claim.

Since $\operatorname{codim}(|V_I'|\cap|D_{k+1}|) > \operatorname{codim}|V_I'|$, by Proposition 2.2, $\pi^*u_{k+1}dd^c b_I\wedge[V_I']$ has locally finite mass and by Proposition 2.3,

$$\pi^* u_{k+1}^{(\iota)} dd^c b_I\wedge[V_I'] \to \pi^* u_{k+1} dd^c b_I\wedge[V_I']$$

if $u_{k+1}^{(\iota)}$ is any sequence of psh functions decreasing to u_{k+1}. If $u_{k+1}^{(\iota)}$ are smooth, using (2.1), that $\pi^{-1}U_{k+1} = \widetilde{U}_{k+1}$, and (2.4), we get

$$u_{k+1}^{(\iota)}\mathbf{1}_{U_{k+1}} dd^c u_k\mathbf{1}_{U_k}\wedge\cdots\wedge dd^c u_1\mathbf{1}_{U_1} = \pi_*\big(\pi^* u_{k+1}^{(\iota)}\sum_I dd^c b_I\wedge[V_I']\big).$$

Proposition 2.3 then yields

$$u_{k+1}\mathbf{1}_{U_{k+1}} dd^c u_k\mathbf{1}_{U_k}\wedge\cdots\wedge dd^c u_1\mathbf{1}_{U_1} = \pi_*\big(\pi^* u_{k+1}\sum_I dd^c b_I\wedge[V_I']\big). \tag{3.5}$$

Clearly the right hand side has locally finite mass, and thus (i) is verified.

We now consider (ii). Since the local representation $\pi^*F_j = f_j f_j'$ is determined up to multiplication by a pluriharmonic factor, it follows, e.g., from Proposition 2.3 that

$$dd^c\pi^* u_{k+1}\wedge dd^c b_I\wedge[V_I'] = dd^c\log|f_{k+1}|^2\wedge dd^c b_I\wedge[V_I'] + dd^c b_{k+1}\wedge dd^c b_I\wedge[V_I'], \tag{3.6}$$

[2]formal linear combinations of irreducible analytic sets

cf. the proof of Proposition 4.1 in [AW]. The second term is clearly of the desired form. Since D_{k+1} and V_I' intersect properly, $[D_{k+1}] \wedge [V_I']$ is the current of integration of a cycle V of pure codimension $k-s+1$, cf. [De3, Chapter III, Proposition 4.12]. Thus

$$dd^c \log |f_{k+1}|^2 \wedge dd^c b_I \wedge [V_I'] = [D_{k+1}] \wedge dd^c b_I \wedge [V_I'] = dd^c b_I \wedge [V]. \tag{3.7}$$

This proves (ii). □

Proposition 3.4. *Let $u_1, \dots, u_m$ be psh functions with analytic singularities with corresponding unbounded loci $Z_1, \dots, Z_m$. Assume that*

$$\operatorname{codim}(Z_{i_1} \cap \cdots \cap Z_{i_\ell}) \geq \ell, \tag{3.8}$$

for each choice of $1 \leq i_1 < \cdots < i_\ell \leq m$. Then

$$dd^c u_m \mathbf{1}_{X \setminus Z_m} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{X\setminus Z_1} = dd^c u_m \wedge \cdots \wedge dd^c u_1, \tag{3.9}$$

where the right hand side is defined in the sense of Bedford-Taylor-Demailly.

Proof. The statement is local, so it is enough to assume that X is a fixed relatively compact coordinate neighborhood of any given point. We let $u_m^N := \max(u_m, -N)$, which is psh and decreases pointwise to u_m when $N \to \infty$. Since u_m has analytic singularities, $u_m^N \equiv -N$ in some neighborhood of Z_m. We let $u_m^{N,\varepsilon}$ be obtained from u_m^N through convolution with an approximate identity, so that $u_m^{N,\varepsilon}$ is smooth, psh and decreases pointwise to u_m^N when $\varepsilon \to 0$. Since $u_m^N \equiv -N$ in a neighborhood of Z_m, it follows that $u_m^{N,\varepsilon} \equiv -N$ in some smaller neighborhood of Z_m when ε is small enough. We can thus find a sequence $u_m^{(\iota)}$ of smooth psh functions decreasing pointwise to u_m such that each $u_m^{(\iota)}$ is constant in some neighborhood of Z_m.

We then get that

$$dd^c u_m \mathbf{1}_{X\setminus Z_m} \wedge (dd^c u_{m-1} \wedge \cdots \wedge dd^c u_1) = \lim_\iota dd^c u_m^{(\iota)} \wedge \mathbf{1}_{X\setminus Z_m}(dd^c u_{m-1} \wedge \cdots \wedge dd^c u_1) =$$
$$\lim_\iota dd^c u_m^{(\iota)} \wedge dd^c u_{m-1} \wedge \cdots \wedge dd^c u_1 = dd^c u_m \wedge dd^c u_{m-1} \wedge \cdots \wedge dd^c u_1,$$

where the first equality follows from the definition of the first current, the second equality follows since $u_m^{(\iota)}$ is smooth and constant in a neighborhood of Z_m, and the last equality follows from Proposition 2.3 because of (3.8). The proposition then follows by induction over m. □

Recall that a function q is *quasiplurisubharmonic (qpsh)* if it is of the form $q = u + a$, where u is psh and a is smooth. We say that the qpsh function q has *analytic singularities* if u has analytic singularities. The *unbounded locus* of q is defined as the unbounded locus of u.

Lemma 3.5. *Let T be a current with analytic singularities, let q be a qpsh function with analytic singularities with unbounded locus Z, and let $U \subset X \setminus Z$ be a constructible set. Then,*

$$q\mathbf{1}_U T := u\mathbf{1}_U T + a\mathbf{1}_U T$$

is independent of the decomposition $q = u + a$, where u is psh and a is smooth.

Proof. Let $q = u_1 + a_1 = u_2 + a_2$ be two decompositions of q such that u_j are psh and a_j are smooth. The statement is local. Therefore we may approximate q by convoluting with a sequence of regularizing kernels $\rho^{(\iota)}$ so that for $j = 1, 2$, $u_j^{(\iota)} := u_j * \rho^{(\iota)}$ is a sequence of smooth psh functions decreasing to u_j, and $a_j^{(\iota)} := a_j * \rho^{(\iota)}$ is a sequence of smooth functions converging to a_j, and for each ι, $u_1^{(\iota)} + a_1^{(\iota)} = u_2^{(\iota)} + a_2^{(\iota)}$. Thus in light of Proposition 3.2 we get

$$u_1\mathbf{1}_U T + a_1\mathbf{1}_U T = \lim_\iota \left(u_1^{(\iota)}\mathbf{1}_U T + a_1^{(\iota)}\mathbf{1}_U T\right) = \lim_\iota \left(u_2^{(\iota)}\mathbf{1}_U T + a_2^{(\iota)}\mathbf{1}_U T\right) = u_2\mathbf{1}_U T + a_2\mathbf{1}_U T.$$

□

Let u be a psh function with analytic singularities with unbounded locus Z, let α be a closed smooth $(1,1)$-form, and let T be a current (locally) of the form

$$T = \sum \beta_i \wedge T_i, \tag{3.10}$$

where the sum is finite, β_i are closed smooth forms, and T_i are currents with analytic singularities. We define the operator $T \mapsto [dd^c u]_\alpha \wedge T$ by

$$[dd^c u]_\alpha \wedge T := dd^c u \wedge \mathbf{1}_{X \setminus Z} T + \alpha \wedge \mathbf{1}_Z T := dd^c(u \mathbf{1}_{X\setminus Z} T) + \alpha \wedge \mathbf{1}_Z T.$$

By Remark 3.3 this is a well-defined current of the form (3.10). Using that $\mathbf{1}_{X\setminus Z}(\alpha^\ell \wedge \mathbf{1}_Z (dd^c u)^k) = 0$ for $k, \ell \geq 0$, we get that

$$[dd^c u]^m_\alpha = [dd^c u]_\alpha \wedge [dd^c u]^{m-1}_\alpha, \tag{3.11}$$

where $[dd^c u]^m_\alpha$ is defined by (1.5).

Next, for currents T of the form (3.10) we define operators $T \mapsto [dd^c u]^m_\alpha \wedge T$ recursively by

$$[dd^c u]^k_\alpha \wedge T := [dd^c u]_\alpha \wedge [dd^c u]^{k-1}_\alpha \wedge T. \tag{3.12}$$

Again by Remark 3.3 these are currents of the form (3.10). In particular, if $u_1, \dots, u_t$ are psh functions with analytic singularities, and $\alpha_1, \dots, \alpha_t$ are closed smooth $(1,1)$-forms, the current

$$[dd^c u_t]^{m_t}_{\alpha_t} \wedge \cdots \wedge [dd^c u_1]^{m_1}_{\alpha_1}$$

is a well-defined current of the form (3.10).

Note that if β is a closed smooth form, then

$$[dd^c u]^k_\alpha \wedge \beta \wedge T = \beta \wedge [dd^c u]^k_\alpha \wedge T.$$

Indeed, multiplication with $\mathbf{1}_U$ and $dd^c u$ commutes with multiplication with closed smooth forms.

Remark 3.6. Recall that if (X, ω) is a Kähler manifold, then a function $\phi \colon X \to \mathbf{R} \cup \{-\infty\}$ is called *ω-plurisubharmonic (ω-psh)* if locally the function $g + \phi$ is psh, where g is a local potential for ω, i.e., $\omega = dd^c g$. Moreover ϕ is said to have *analytic singularities* if the functions $g + \phi$ have analytic singularities. If ϕ is an ω-psh function with analytic singularities, we can define a global positive current $(\omega + dd^c\phi)^k$, by locally defining it as $(dd^c(g+\phi))^k$, see [ABW, Lemma 5.1]. Analogously to (1.5) we can define

$$[\omega + dd^c\phi]^m_\omega := (\omega + dd^c\phi)^m + \sum_{k=0}^{m-1} \omega^{m-k} \wedge \mathbf{1}_Z (\omega + dd^c\phi)^k,$$

where Z is the unbounded locus of ϕ, cf. [ABW]. With this notation, Theorem 1.2 in [ABW] can be formulated as:

Let ϕ be an ω-psh function with analytic singularities on a compact Kähler manifold (X, ω) of dimension n. Then

$$\int_X [\omega + dd^c\phi]^n_\omega = \int_X \omega^n.$$

□

4. Hermitian metrics with analytic singularities on line bundles

Let $L \to X$ be a holomorphic line bundle. A *singular hermitian metric* on L, as introduced by Demailly [De4], consists of (possibly infinite) seminorms $\|\cdot\|_{h(x)}$ on L_x for all $x \in X$ such that if $\vartheta : L|_{\mathcal{U}} \to \mathcal{U} \times \mathbf{C}$ is a local trivialization of L and ξ is a local section, then $\|\xi\|^2_h = |\vartheta(\xi)|^2 e^{-\varphi}$, where φ is a locally integrable function in $\mathcal{U}$ called the *(local) weight* of h with respect to ϑ, see, e.g., [L, Chapter 9.4.D]. The metric h is often denoted by $e^{-\varphi}$ or just φ. If $\vartheta' : L|_{\mathcal{U}'} \to \mathcal{U}' \times \mathbf{C}$ is another local trivialization, with transition function g, then in $\mathcal{U} \cap \mathcal{U}'$ the corresponding weight φ' satisfies

$$\varphi' = \varphi + \log|g|^2. \tag{4.1}$$

From (4.1) and the local integrability of the weights it follows that the curvature current $\Theta_h = \partial\bar\partial\varphi$ is a well-defined global current on X. The *Chern form* $c_1(L,h) = (i/2\pi)\Theta_h = dd^c\varphi$ represents the Chern class $c_1(L)$.

The metric $e^{-\varphi}$ is *positive* if the weights φ are psh. We say that a positive singular metric $e^{-\varphi}$ has *analytic singularities* if the weights φ have analytic singularities. In view of (4.1) there is a well-defined associated *unbounded locus* $Z \subset X$, that is a subvariety of X, locally defined as the unbounded loci of the φ. Since the local weights are integrable it follows that Z has positive codimension in X.

Example 4.1. Assume that $s_1, \dots, s_N$ are nontrivial holomorphic sections of a line bundle $L \to X$. Then $h = e^{-\varphi}$ with

$$\varphi = \log \sum_{j=1}^N |s_j|^2$$

is a positive metric with analytic singularities, cf. [De4, Example 2.4]. In other words, if $\vartheta : L|_{\mathcal{U}} \to \mathcal{U} \times \mathbf{C}$ is a local trivialization and ξ is a local section, then

$$\|\xi\|_h^2 = \frac{|\vartheta(\xi)|^2}{\sum_{j=1}^N |\vartheta(s_j)|^2}.$$

□

Lemma 4.2. *Assume that $\varphi_1, \dots, \varphi_t$ are positive metrics with analytic singularities on line bundles $L_1, \dots, L_t$ over X with unbounded loci $Z_1, \dots, Z_t$, respectively. If $U_1, \dots, U_t$ are constructible sets contained in $X \setminus Z_1, \dots, X \setminus Z_t$, respectively, and $\theta_1, \dots, \theta_t$ are closed $(1,1)$-forms, then the a priori locally defined currents*

$$dd^c\varphi_t \mathbf{1}_{U_t} \wedge \cdots \wedge dd^c\varphi_1 \mathbf{1}_{U_1}, \tag{4.2}$$

$$[dd^c\varphi_t]_{\theta_t}^{m_t} \wedge \cdots \wedge [dd^c\varphi_1]_{\theta_1}^{m_1} \tag{4.3}$$

are globally defined currents; (4.2) *has analytic singularities and* (4.3) *is of the form* (3.10).

Proof. Since the local weights of φ_j are psh functions with analytic singularities, (4.2) and (4.3) are locally well-defined and of the desired form in view of Section 3. Since two local weights differ by a pluriharmonic function, cf. (4.1), it follows using Lemma 3.5 that they are globally defined. □

If $\varphi_j = \varphi$, with singular set Z, and $U_j = X \setminus Z$ for all j, we write $(dd^c\varphi)^t$ for the generalized Monge-Ampère product (4.2), cf. (1.4).

For the proofs of Theorems 1.1 and 1.2 we will need the following results.

Proposition 4.3. *Let $\varphi_1, \dots, \varphi_t$ be hermitian metrics with analytic singularities on holomorphic line bundles $L_1, \dots, L_t$, respectively, over a complex manifold X. Moreover, let $\theta_1, \dots, \theta_t$ be first Chern forms of smooth metrics on $L_1, \dots, L_t$, respectively. Then*

$$[dd^c\varphi_t]_{\theta_t}^{m_t} \wedge \cdots \wedge [dd^c\varphi_1]_{\theta_1}^{m_1} = \theta_t^{m_t} \wedge \cdots \wedge \theta_1^{m_1} + dd^c S, \tag{4.4}$$

where S is a current on X.

Proof. In view of (3.12) it is enough to prove the result for $m_j = 1$, $j = 1, \dots, t$. Assume that θ_j is the first Chern form of the smooth metric ψ_j. Then note that $\varphi_j - \psi_j$ defines a global qpsh function on X for each j, cf. (4.1).

Let T be a current of the form (3.10). Then

$$\begin{aligned}(4.5)\quad [dd^c\varphi_j]_{\theta_j} \wedge T &= dd^c(\varphi_j \mathbf{1}_{X\setminus Z_j} T) + \theta_j \wedge \mathbf{1}_{Z_j} T = \\ &dd^c(\varphi_j \mathbf{1}_{X\setminus Z_j} T) - \theta_j \wedge \mathbf{1}_{X\setminus Z_j} T + \theta_j \wedge T = dd^c\big((\varphi_j - \psi_j)\mathbf{1}_{X\setminus Z_j} T\big) + \theta_j \wedge T,\end{aligned}$$

where we have used that $dd^c\psi_j = \theta_j$, Lemma 3.5, and that $\mathbf{1}_{X\setminus Z_j} T$ is closed for the last equality.

Assume that $t = 1$. Then it follows from (4.5) (applied to $T = 1$ and $j = 1$) that (4.4) holds with $S = (\varphi_1 - \psi_1)\mathbf{1}_{X\setminus Z_1}$. In fact, $S = \varphi_1 - \psi_1$ since Z_1 has positive codimension in X. Next assume that (4.4) holds for $t = \kappa$. Then (4.5) (applied to $T = [dd^c\varphi_\kappa]_{\theta_\kappa} \wedge \cdots \wedge [dd^c\varphi_1]_{\theta_1}$ and $j = \kappa + 1$) gives

$$[dd^c\varphi_{\kappa+1}]_{\theta_{\kappa+1}} \wedge \cdots \wedge [dd^c\varphi_1]_{\theta_1} = dd^cU + \theta_{\kappa+1} \wedge [dd^c\varphi_\kappa]_{\theta_\kappa} \wedge \cdots \wedge [dd^c\varphi_1]_{\boldsymbol{\theta_1}} =$$
$$dd^cU + \theta_{\kappa+1} \wedge \big(\theta_\kappa \wedge \cdots \wedge \theta_1 + dd^cS\big) = \theta_{\kappa+1} \wedge \cdots \wedge \theta_1 + dd^c(U + \theta_{\kappa+1} \wedge S),$$

where

$$U = (\varphi_{\kappa+1} - \psi_{\kappa+1})\mathbf{1}_{X\setminus Z_j}[dd^c\varphi_\kappa]_{\theta_\kappa} \wedge \cdots \wedge [dd^c\varphi_1]_{\boldsymbol{\theta_1}}.$$

Thus (4.4) holds for $t = \kappa + 1$ and the lemma follows by induction. □

Lemma 4.4. *Let φ be a positive metric with analytic singularities on a holomorphic line bundle L over a complex manifold X, and let θ be the first Chern form of a smooth metric on L. Let φ_ε be a sequence of smooth positive metrics decreasing to φ and let ω_ϵ be the corresponding first Chern forms. Moreover, let T be a current of the form* (3.10)*, and let β be a test form such that the support of $dd^c\beta$ does not intersect the unbounded locus Z of φ. Then*

$$\int_X [dd^c\varphi]^m_\theta \wedge T \wedge \beta = \lim_{\varepsilon\to 0} \int_X \omega_\varepsilon^m \wedge T \wedge \beta. \tag{4.6}$$

Proof. Assume that θ is the first Chern form of the smooth metric ψ. First note that for any current S of the form (3.10)

$$[dd^c\varphi]_\theta \wedge S - \omega_\varepsilon \wedge S = dd^c(\varphi\mathbf{1}_{X\setminus Z}S) + \theta \wedge \mathbf{1}_Z S - \omega_\varepsilon \wedge S =$$
$$dd^c\big((\varphi - \varphi_\varepsilon)\mathbf{1}_{X\setminus Z}S\big) + dd^c\big((\psi - \varphi_\varepsilon)\mathbf{1}_Z S\big). \tag{4.7}$$

Since $\varphi - \varphi_\varepsilon$ and $\psi - \varphi_\varepsilon$ are globally defined qpsh functions, cf. (4.1), in view of Lemma 3.5 the currents on the last line of (4.7) are globally defined currents of the form (3.10).

By applying (4.7) to $S = [dd^c\varphi]^{\ell-1}_\theta \wedge T$ for $\ell = 1, \ldots, m$, it follows that

$$[dd^c\varphi]^m_\theta \wedge T - \omega^m_\varepsilon \wedge T = \sum_{\ell=1}^m \omega_\varepsilon^{m-\ell} \wedge \big([dd^c\varphi]^\ell_\theta \wedge T - \omega_\varepsilon \wedge [dd^c\varphi]^{\ell-1}_\theta \wedge T\big) = \tag{4.8}$$
$$\sum_{\ell=1}^m \omega_\varepsilon^{m-\ell} \wedge dd^c\big((\varphi - \varphi_\varepsilon)\mathbf{1}_{X\setminus Z}[dd^c\varphi]^{\ell-1}_\theta \wedge T\big) + \sum_{\ell=1}^m \omega_\varepsilon^{m-\ell} \wedge dd^c\big((\psi - \varphi_\varepsilon)\mathbf{1}_Z[dd^c\varphi]^{\ell-1}_\theta \wedge T\big).$$

Again, the currents in the last line are well-defined global currents by Lemma 3.5.

Let us integrate one of the currents in the second sum against β. Then by Stokes' theorem

$$\int_X \omega_\varepsilon^{m-\ell} \wedge dd^c\big((\psi - \varphi_\varepsilon)\mathbf{1}_Z[dd^c\varphi]^{\ell-1}_\theta \wedge T\big) \wedge \beta =$$
$$\int_X \omega_\varepsilon^{m-\ell} \wedge (\psi - \varphi_\varepsilon)\mathbf{1}_Z[dd^c\varphi]^{\ell-1}_\theta \wedge T \wedge dd^c\beta = 0,$$

where the last equality follows since $\operatorname{supp}(\mathbf{1}_Z[dd^c\varphi]^{\ell-1}_\theta \wedge T) \subset Z$ is disjoint from $\operatorname{supp} dd^c\beta$. Outside Z, the current

$$\omega_\varepsilon^{m-\ell} \wedge (\varphi - \varphi_\varepsilon)\mathbf{1}_{X\setminus Z}[dd^c\varphi]^{\ell-1}_\theta \wedge T = (\varphi - \varphi_\varepsilon)(dd^c\varphi_\varepsilon)^{m-\ell} \wedge \mathbf{1}_{X\setminus Z}[dd^c\varphi]^{\ell-1}_\theta \wedge T$$

converges weakly to 0 by Proposition 2.3. Thus, integrating one of the terms in the first sum in the last line of (4.8) against β and taking the limit gives

$$\lim_{\varepsilon\to 0} \int_X \omega_\varepsilon^{m-\ell} \wedge dd^c\big((\varphi - \varphi_\varepsilon)\mathbf{1}_{X\setminus Z}[dd^c\varphi]^{\ell-1}_\theta \wedge T\big) \wedge \beta =$$
$$\lim_{\varepsilon\to 0} \int_X \omega_\varepsilon^{m-\ell} \wedge (\varphi - \varphi_\varepsilon)\mathbf{1}_{X\setminus Z}[dd^c\varphi]^{\ell-1}_\theta \wedge T \wedge dd^c\beta = 0.$$

Now (4.6) follows by integrating (4.8) against β and taking the limit. □

5. Hermitian metrics with analytic singularities on vector bundles

Let $E \to X$ be a holomorphic vector bundle over a complex manifold X. A *singular hermitian metric* h on E in the sense of Berndtsson-Păun, [BP], is a measurable function from X to the space of nonnegative hermitian forms on the fibers. The hermitian forms are allowed to take the value ∞ at some points in the base (i.e., the norm function $\|\xi\|_h$ is a measurable function with values in $[0,\infty]$), but for any fiber E_x the subset $E_0 := \{\xi \in E_x \,;\, \|\xi\|_{h(x)} < \infty\}$ has to be a linear subspace, and the restriction of the metric to this subspace must be an ordinary hermitian form.

Every singular hermitian metric h on E induces a canonical dual singular hermitian metric h^* on E^* such that $(h^*)^* = h$ under the natural isomorphism $(E^*)^* \cong E$, see, e.g., [LRRS, Lemma 3.1]. Following [BP, Section 3] we say that h is *Griffiths negative* if the function[3]

$$\chi_h(x,\xi) := \log \|\xi\|^2_{h(x)}$$

is psh on the total space of E. Moreover we say that h is *Griffiths positive* if the dual metric h^* on E^* is negative.

Proposition 5.1. *Let h be a singular hermitian metric on a holomorphic vector bundle E. Let 0_E denote the zero section of E. Then the following conditions are equivalent.*

(1) *h is Griffiths negative, i.e., χ_h is psh on the total space of E,*
(2) *χ_h is psh on $E \setminus 0_E$,*
(3) *the function $x \mapsto \log \|u(x)\|^2_{h(x)}$ is psh for each local section u of E,*
(4) *$\log \|u\|^2_h$ is psh for each local nonvanishing section u of E.*

Proof. We first prove that (3) is equivalent to (1) and that (4) is equivalent to (2). Note that if u is a local holomorphic section of E, then $\log \|u(x)\|^2_{h(x)} = \chi_h \circ u(x)$. Thus if χ_h is psh, then so is $\log \|u\|^2_h$ since u is a holomorphic map. Hence (1) implies (3). If $u \neq 0$, then it is enough that χ_h is psh on $E \setminus 0_E$ in order for u to be psh. Thus (2) implies (4).

For the converses, since plurisubharmonicity is a local property, we may assume that X is an open subset of $\mathbf{C}^n$ and that $E = X \times \mathbf{C}^r$. To prove that χ_h is psh it is then sufficient to prove that $\chi_h \circ \gamma(t)$ is subharmonic on (the restriction to E of) any complex line $\gamma(t)$ in $\mathbf{C}^n \times \mathbf{C}^r$. We let γ_0 and γ_1 denote the components of γ in $\mathbf{C}^n$ and $\mathbf{C}^r$, respectively. If $\gamma_0(t)$ is constant then

$$\chi_h \circ \gamma(t) = \log \|\gamma_1(t)\|^2_{h(\gamma_0(t))} = \log \|\gamma_1(t)\|^2_{h_0},$$

where h_0 is the constant metric $h(\gamma_0)$. If $\|u\|^2_h$ is psh for all u, then h_0 has to be finite, and thus since $\gamma_1(t)$ is a holomorphic curve, it follows that $\chi_h \circ \gamma$ is subharmonic. If $\gamma_0(t)$ is not constant, then note that $\gamma_1(t) = u \circ \gamma_0(t)$ for some (linear) holomorphic function u, that can be extended to a holomorphic section on X. Thus

$$\chi_h \circ \gamma(t) = \log \|\gamma_1(t)\|^2_{h(\gamma_0(t))} = \log \|u \circ \gamma_0(t)\|^2_{h(\gamma_0(t))} = \log \|u\|^2_h \circ \gamma_0(t).$$

Since γ_0 is holomorphic, $\chi_h \circ \gamma$ is subharmonic if $\|u\|^2_h$ is psh. Hence (3) implies (1). To show that χ_h is psh outside the zero section of E, u can be chosen nonvanishing, and thus (2) follows from (4).

Next we prove that (1) is equivalent to (2). Clearly (1) implies (2). To prove the converse assume that χ_h is psh on $E \setminus 0_E$. Then $\|\xi\|^2_{h(x)}$ is finite on $E \setminus 0_E$ and thus by homogeneity it must vanish on 0_E, which means that $\chi_h|_{0_E} \equiv -\infty$. It follows that χ_h trivially satisfies the sub-mean value property at each point of 0_E.

To prove that χ_h is upper semicontinuous at 0_E, choose $(x_0,\xi_0) \in 0_E$ and let (x_k,ξ_k) be a sequence of points converging to (x_0,ξ_0). We need to prove that $\lim_k \chi_h(x_k,\xi_k) = -\infty$. As above, we may assume that X is an open subset of $\mathbf{C}^n_x$ and $E = X \times \mathbf{C}^r_\xi$, and moreover that $0_E = \{\xi = 0\}$ and $(x_0,\xi_0) = (0,0)$. Also we may assume that (x_k,ξ_k) are contained in the set $\{|x| \leq 1, |\xi| \leq 1\}$. Let C be the compact "cylinder"

$$C = \{|x| \leq 1, |\xi| = 1\}.$$

[3]The function χ_h is sometimes called the *logarithmic indicatrix* of the (Finsler) metric h, see, e.g., [De2].

Since χ_h is psh and thus upper semi-continuous outside 0_E, $\chi_h|_C \leq M$ for some $M < \infty$. By homogeneity it follows that

$$\chi_h(x_k, \xi_k) \leq M + \log |\xi_k|^2 \to -\infty.$$

Thus $\chi_h(x, \xi)$ is upper semicontinuous at 0_E and hence it is psh in E. □

Let $\pi : \mathbf{P}(E) \to X$ denote the *projectivization* of E, i.e., the projective bundle of lines in the dual bundle E^* of E, i.e., $\mathbf{P}(E)_x = \mathbf{P}(E^*_x)$. The pullback bundle $\pi^* E^* \to \mathbf{P}(E)$ then carries a tautological line bundle

$$\mathcal{O}_{\mathbf{P}(E)}(-1) = \{(x, [\xi]; v), v \in \mathbf{C}\xi\} \subset \pi^* E^*.$$

Let e^{φ} denote the restriction of $\pi^* h^*$ to $\mathcal{O}_{\mathbf{P}(E)}(-1)$. Then $e^{-\varphi}$ is the dual metric on the dual line bundle $\mathcal{O}_{\mathbf{P}(E)}(1)$. If E is a line bundle, then $\mathcal{O}_{\mathbf{P}(E)}(1) \cong E$ and $e^{-\varphi} \cong h$.

Let us describe φ in a local trivialization. After possibly shrinking X we may assume that $E = X \times \mathbf{C}^r$; then $\mathbf{P}(E) = X \times \mathbf{P}^{r-1}$. For $i = 1, \dots, r$, let

$$\mathcal{U}_i = \{(x, [\xi]) \in \mathbf{P}(E), \xi_i \neq 0\}.$$

Then $\{\mathcal{U}_i\}$ is an open cover of $\mathbf{P}(E)$ and $\mathcal{O}_{\mathbf{P}(E)}(-1)$ is defined by the trivializations

$$\vartheta_i : \mathcal{O}_{\mathbf{P}(E)}(-1)|_{\mathcal{U}_i} \to \mathcal{U}_i \times \mathbf{C}, \ (x, [\xi]; v) \mapsto (x, [\xi]; v_i).$$

Now, on $\mathcal{U}_i$,

$$\|v\|^2_{\pi^* h^*(x,[\xi])} = |\vartheta_i(v)|^2 e^{\varphi_i(x,[\xi])} = |v_i|^2 e^{\varphi_i(x,[\xi])}. \tag{5.1}$$

Moreover, since $\pi^* h^*$ is a pullback metric

$$\|v\|_{\pi^* h^*(x,[\xi])} = \|v\|_{h^*(x)}. \tag{5.2}$$

By applying (5.1) and (5.2) to $v = \xi$ we get

$$\varphi_i(x, [\xi]) = \log \|\xi/\xi_i\|^2_{h^*(x)} = \chi_{h^*}(x, \xi/\xi_i). \tag{5.3}$$

Note that this is well-defined since the second and third expressions only depend on $[\xi]$.

Proposition 5.2. *Let h be a singular hermitian metric on a holomorphic vector bundle E. Then h is Griffiths positive if and only if the induced singular metric $e^{-\varphi}$ on $\mathcal{O}_{\mathbf{P}(E)}(1)$ is positive.*

Proof. Since this is a local statement we may assume that we are in the situation above. Then $e^{-\varphi}$ is positive if and only if φ_i is psh on $\mathcal{U}_i$ for all i. Moreover, by Proposition 5.1, h is Griffiths positive if and only if $\chi_{h^*}(x, \xi)$ is psh on E or equivalently on $E \setminus 0_E$.

Since $\xi_i \neq 0$ on $\mathcal{U}_i$, in view of (5.3), φ_i is psh there if χ_{h^*} is psh. Thus $e^{-\varphi}$ is positive if h is Griffiths positive. For the converse, if $(x, \xi) \in E \setminus 0_E$, then $\xi_i \neq 0$ for some i in some neighborhood $\mathcal{U}$ of (x, ξ). Then, by (5.3),

$$\chi_{h^*}(x, \xi) = \chi_{h^*}(x, \xi/\xi_i) + \log |\xi_i|^2 = \varphi_i(x, [\xi]) + \log |\xi_i|^2 \tag{5.4}$$

there. Since $\xi \to [\xi]$ is holomorphic and $\log |\xi_i|^2$ is pluriharmonic where $\xi_i \neq 0$, it follows that χ_{h^*} is psh in $\mathcal{U}$ if φ_i is psh. We conclude that h is Griffiths positive if $e^{-\varphi}$ is positive. □

Definition 5.3. We say that a Griffiths positive hermitian metric has *analytic singularities* if the induced positive metric $e^{-\varphi}$ on $\mathcal{O}_{\mathbf{P}(E)}(1)$ has analytic singularities.

Proposition 5.4. *Let h be a Griffiths positive hermitian metric on a holomorphic vector bundle E. Then h has analytic singularities if and only if χ_{h^*} is psh with analytic singularities on $E \setminus 0_E$.*

Proof. Let us assume that we are in the local situation above. Then h has analytic singularities if and only if φ_i are psh with analytic singularities for all i. In view of the proof of Proposition 5.2 this is in turn equivalent to that χ_{h^*} is psh with analytic singularities on $E \setminus 0_E$. □

We do not know whether it is possible to express analytic singularities of h in terms (of analytic singularities) of the functions $\log \|u\|^2_h$.

Example 5.5. In [H, Example 3.6] Hosono constructed a family of examples of singular hermitian metrics that generalize the metrics in Example 4.1: Assume that $E \to X$ is a holomorphic vector bundle with global holomorphic sections $s_1, \dots, s_N$. Let s be the morphism from the dual bundle E^* to the trivial bundle $X \times \mathbf{C}^N$ given by $(x,\xi) \mapsto (s_1(x,\xi), \dots, s_N(x,\xi))$ and let h^* be the pullback under s of the trivial metric on $X \times \mathbf{C}^N$, i.e.,

$$\langle \xi, \eta \rangle_{h^*(x)} := \langle s(x,\xi), s(x,\eta) \rangle.$$

Then

$$\|\xi\|^2_{h^*(x)} = |s(x,\xi)|^2 = \sum_j |s_j(x,\xi)|^2.$$

It follows that $\chi_{h^*}(\xi, x) = \log |s(x,\xi)|^2$ is psh with analytic singularities on E^*. Thus by Proposition 5.4 the dual metric h of h^* on $E = (E^*)^*$ is Griffiths positive with analytic singularities. □

Given a Griffiths positive singular metric h, $\log\det h^*$ is psh, see [R, Proposition 1.3], and we can define the *degeneracy locus* of h as the unbounded locus of $\log\det h^*$. The following lemma gives alternative definitions in terms of (the unbounded loci) of χ_{h^*} and φ.

Lemma 5.6. *Assume that h is a Griffiths positive singular metric on $E \to X$. Then, using the notation from above and denoting the projection $E^* \to X$ by p,*

$$L(\log\det h^*) = p\big(L(\chi_{h^*}) \setminus 0_E\big) = \pi\big(L(\varphi)\big). \tag{5.5}$$

In particular, it follows that if h has analytic singularities, then the degeneracy locus of h is a subvariety of X.

Proof. In view of (5.4), $(x,\xi) \in E \setminus 0_E$ is in $L(\chi_{h^*})$ if and only if $(x,[\xi]) \in \mathbf{P}(E)$ is in $L(\varphi)$, and thus the second equality in (5.5) follows. The inclusion $\pi\big(L(\varphi)\big) \subset L(\log\det h^*)$ is an immediate consequence of Lemma 3.7 in [LRRS].

Thus it remains to prove that

$$L(\log\det h^*) \subset p\big(L(\chi_{h^*}) \setminus 0_E\big). \tag{5.6}$$

Since the statement is local we may assume that $E = X \times \mathbf{C}^r$. Take $x \in L(\log\det h^*)$. Then, by definition there is a sequence $x_k \to x$ such that $\log\det h^*(x_k) \to -\infty$. This means that there is a sequence $\varepsilon_k \to 0$ such that $\det h^*(x_k) < \varepsilon_k^r$, which implies that $h^*(x_k)$ has at least one eigenvalue less than ε_k. Thus there are $\xi_k \in E^*_{x_k} = \mathbf{C}^r$ such that $\|\xi_k\|_{\mathbf{C}^r} = 1$ and $\|\xi_k\|_{h^*(x_k)} < \varepsilon_k$. Since $\|\xi_k\|_{\mathbf{C}^r} = 1$, $\{\xi_k\}$ has at least one accumulation point ξ in $\mathbf{C}^r$ and thus we can find a subsequence $(x_k, \xi_k) \to (x,\xi)$. Since $\|\xi_k\|_{h^*(x_k)} \to 0$, $(x,\xi) \in L(\chi_{h^*})$. Moreover, since $\|\xi_k\|_{\mathbf{C}^r} = 1$, $\|\xi\|_{\mathbf{C}^r} \neq 0$ and thus $x \in p\big(L(\chi_{h^*}) \setminus 0_E\big)$, which proves (5.6). □

6. Construction of Segre and Chern currents, proof of Theorem 1.1

6.1. **Construction, basic properties.** Assume that X is a complex manifold of dimension n, that $E \to X$ is a holomorphic vector bundle of rank r, and that h is a Griffiths positive hermitian metric with analytic singularities on E. Let $\pi : \mathbf{P}(E) \to X$ be the projectivization of E and let φ denote the metric on $L := \mathcal{O}_{\mathbf{P}(E)}(1) \to \mathbf{P}(E)$ induced by h. Then φ has analytic singularities, cf. Definition 5.3; let $Z \subset \mathbf{P}(E)$ denote the unbounded locus of φ. Moreover, assume that ψ is a smooth metric on L and let θ be the corresponding first Chern form.

Next, let $E_1, \dots, E_t$ be t disjoint copies of E, let π_i denote the projections $\mathbf{P}(E_i) \to X$, and p_i the identifications $\mathbf{P}(E_i) \to \mathbf{P}(E)$. Let $\tilde\varphi_i$ denote the metric $p_i^*\varphi$ on $\widetilde{L}_i := p_i^* L \to \mathbf{P}(E_i)$ induced by h with unbounded locus $\widetilde{Z}_i := p_i^{-1}(Z)$ and let $\tilde\theta_i = p_i^*\theta$ and $\tilde\psi_i = p_i^*\psi$. Moreover, let Y be the fiber product

$$Y = \mathbf{P}(E_t) \times_X \cdots \times_X \mathbf{P}(E_1),$$

with projections $\varpi_i : Y \to \mathbf{P}(E_i)$ and $\pi : Y \to X$. Let φ_i denote the pullback metric $\varpi_i^*\tilde\varphi_i$ on $L_i := \varpi_i^*\widetilde{L}_i$ with unbounded locus $Z_i := \varpi_i^{-1}(\widetilde{Z}_i)$ and let $\theta_i = \varpi_i^*\tilde\theta_i$ and $\psi_i = \varpi_i^*\tilde\psi_i$. Now, in view of Lemma 4.2, (1.7) is a well-defined (k,k)-current.

Remark 6.1. Let V be the degeneracy locus of h. Then in $Y \setminus \pi^{-1}V$, φ_j are locally bounded by Lemma 5.6, and thus

$$[dd^c\varphi_t]_{\theta_t}^{k_t+r-1} \wedge \cdots \wedge [dd^c\varphi_1]_{\theta_1}^{k_1+r-1} = (dd^c\varphi_t)^{k_t+r-1} \wedge \cdots \wedge (dd^c\varphi_1)^{k_1+r-1},$$

where the right hand side is locally defined in the sense of Bedford-Taylor. Hence outside V,

$$s_{k_t}(E,h,\theta) \wedge \cdots \wedge s_{k_1}(E,h,\theta) = (-1)^k \pi_*\big((dd^c\varphi_t)^{k_t+r-1} \wedge \cdots \wedge (dd^c\varphi_1)^{k_1+r-1}\big);$$

in particular it is independent of θ, and thus so are $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$. □

Lemma 6.2. *Let X, E, and h be as above. Given $x \in X$, there is a neighborhood $x \in \mathcal{U} \subset X$ such that in $\mathcal{U}$,*

$$s_{k_t}(E,h,\theta) \wedge \cdots \wedge s_{k_1}(E,h,\theta) = S_+ - S_-,$$

where S_+, S_- are closed positive currents.

In view of (1.8), it follows that $s_k(E,h,\theta)$ and $c_k(E,h,\theta)$ are differences of positive closed (k,k)-currents; this proves the first part of Theorem 1.1.

Proof. Let us use the notation from above. Since the statement is local we may assume that X is an open neighborhood of x in $\mathbf{C}^n$ and that $E = X \times \mathbf{C}^r$ is a trivial bundle. Then $\mathbf{P}(E_j) \cong X \times Y_j$, where $Y_j \cong \mathbf{P}^{r-1}$. Let ρ_j be the projection $\mathbf{P}(E_j) \to \mathbf{P}^{r-1}$. Moreover, let $\omega_{\rm FS}$ denote the Fubini-Study metric on $\mathbf{P}^{r-1}$, let ω_0 be the standard Euclidean metric on X, and let $\omega_j = \rho_j^*\omega_{\rm FS} + \pi_j^*\omega_0$. Then for some large enough $C > 0$, there is a neighborhood $x \in \mathcal{U} \subset X$ such that $\tilde\alpha_j := C\omega_j$ satisfies that

$$\tilde\beta_j := \tilde\alpha_j + \tilde\theta_j \geq 0$$

in $\pi_j^{-1}\mathcal{U} \subset \mathbf{P}(E_j)$ for each j. Let $\alpha_j = \varpi_j^*\tilde\alpha_j$ and $\beta_j = \varpi_j^*\tilde\beta_j$ be the corresponding closed $(1,1)$-forms on $\pi^{-1}\mathcal{U} \subset Y$, so that $\theta_j = \beta_j - \alpha_j$.

We claim that in $\pi^{-1}\mathcal{U}$, for $m_j \geq 1$,

$$[dd^c\varphi_t]_{\theta_t}^{m_t} \wedge \cdots \wedge [dd^c\varphi_1]_{\theta_1}^{m_1} = T_+ - T_-,$$

where $T_\pm$ are closed positive currents with analytic singularities, cf. the beginning of Section 3. Then in view of (1.7), $s_{k_t}(E,h,\theta) \wedge \cdots \wedge s_{k_1}(E,h,\theta)$ is of the desired form in $\mathcal{U}$.

To prove the claim, first note that if $m_1 = 0$, then,

$$[dd^c\varphi_1]_{\theta_1}^{m_1} = [dd^c\varphi_1]_{\theta_1}^{0} = 1 \geq 0.$$

Next, assume that

$$T := [dd^c\varphi_\kappa]_{\theta_\kappa}^{m_\kappa-1} \wedge \cdots \wedge [dd^c\varphi_1]_{\theta_1}^{m_1} = T_+ - T_-,$$

where $m_\kappa \geq 1$, and where $T_\pm$ are as above. Then

$$\begin{aligned}
[dd^c\varphi_\kappa]_{\theta_\kappa}^{m_\kappa} \wedge \cdots \wedge [dd^c\varphi_1]_{\theta_1}^{m_1} &= [dd^c\varphi_\kappa]_{\theta_\kappa} \wedge T = dd^c\varphi_\kappa \wedge \mathbf{1}_{Y\setminus Z_\kappa}T + \theta_\kappa \wedge \mathbf{1}_{Z_\kappa}T = \\
& dd^c\varphi_\kappa \wedge \mathbf{1}_{Y\setminus Z_\kappa}(T_+ - T_-) + (\beta_\kappa - \alpha_\kappa) \wedge \mathbf{1}_{Z_\kappa}(T_+ - T_-) = \\
& \big(dd^c\varphi_\kappa \wedge \mathbf{1}_{Y\setminus Z_\kappa}T_+ + \beta_\kappa \wedge \mathbf{1}_{Z_\kappa}T_+ + \alpha_\kappa \wedge \mathbf{1}_{Z_\kappa}T_-\big) - \\
& \big(dd^c\varphi_\kappa \wedge \mathbf{1}_{Y\setminus Z_\kappa}T_- + \beta_\kappa \wedge \mathbf{1}_{Z_\kappa}T_- + \alpha_\kappa \wedge \mathbf{1}_{Z_\kappa}T_+\big) =: T'_+ - T'_-.
\end{aligned}$$

Since $T_\pm$ are closed positive currents with analytic singularities and β_κ and α_κ are positive $(1,1)$-forms, $T'_\pm$ are well-defined closed positive currents with analytic singularities. The claim now follows by induction. □

6.2. **Comparison to the smooth case, proof of statement** (2) **in Theorem 1.1.** Note that to prove statement (2) in Theorem 1.1 it suffices to show that

$$s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta)=s_{k_t}(E,h)\wedge\cdots\wedge s_{k_1}(E,h) \tag{6.1}$$

when h is smooth.

To this end, assume that h, and thus φ, is smooth. Let α_j be the smooth form

$$\alpha_j:=(dd^c\tilde{\varphi}_j)^{k_j+r-1},$$

where we use the notation from Section 6.1. Then $s_{k_j}(E,h)=(-1)^{k_j}(\pi_j)_*\alpha_j$, cf. (1.1), and thus

$$s_{k_t}(E,h)\wedge\cdots\wedge s_{k_1}(E,h)=(-1)^k(\pi_t)_*\alpha_t\wedge\cdots\wedge(\pi_1)_*\alpha_1. \tag{6.2}$$

Note that in this case

$$\varpi_j^*\alpha_j=(dd^c\varpi_j^*\tilde{\varphi}_j)^{k_j+r-1}=(dd^c\varphi_j)^{k_j+r-1},$$

and thus in view of Remark 6.1

$$\varpi_t^*\alpha_t\wedge\cdots\wedge\varpi_1^*\alpha_1=[dd^c\varphi_t]_{\theta_t}^{k_t+r-1}\wedge\cdots\wedge[dd^c\varphi_1]_{\theta_1}^{k_1+r-1},$$

so that

$$s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta)=(-1)^k\pi_*\big(\varpi_t^*\alpha_t\wedge\cdots\wedge\varpi_1^*\alpha_1\big).$$

Now (6.1) follows from the following lemma (with $Y_j=\mathbf{P}(E_j)$).

Lemma 6.3. *Let X be a complex manifold, let $\pi_j\colon Y_j\to X$, $j=1,\dots,t$, be proper submersions, and let Y be the fiber product $Y:=Y_t\times_X\cdots\times_X Y_1$ with projections $\varpi_j:Y\to Y_j$ and $\pi:Y\to X$. Let α_1 be a current on Y_1, and let $\alpha_2,\dots,\alpha_t$ be smooth forms on $Y_2,\dots,Y_t$, respectively. Then*

$$\pi_*\big(\varpi_t^*\alpha_t\wedge\cdots\wedge\varpi_1^*\alpha_1\big)=(\pi_t)_*\alpha_t\wedge\cdots\wedge(\pi_1)_*\alpha_1. \tag{6.3}$$

Proof. By induction it is enough to prove the case $t=2$. It is also enough to prove (6.3) locally. We may therefore assume that $Y_j\cong X\times Z_j$, where Z_j is a manifold for $j=1,2$. It is readily verified that

$$\pi_1^*(\pi_2)_*\alpha_2=(\varpi_1)_*\varpi_2^*\alpha_2 \tag{6.4}$$

since the pushforwards on both sides are just integration along Z_2. By (2.1), (2.2), (6.4), and the fact that $\pi_1\circ\varpi_1=\pi$, we obtain that

$$\begin{aligned}(\pi_2)_*\alpha_2\wedge(\pi_1)_*\alpha_1&=(\pi_1)_*(\pi_1^*(\pi_2)_*\alpha_2\wedge\alpha_1)=(\pi_1)_*((\varpi_1)_*\varpi_2^*\alpha_2\wedge\alpha_1)\\&=(\pi_1)_*(\varpi_1)_*(\varpi_2^*\alpha_2\wedge\varpi_1^*\alpha_1)=\pi_*(\varpi_2^*\alpha_2\wedge\varpi_1^*\alpha_1).\end{aligned}$$

□

6.3. **The cohomology class, proof of statement** (1) **in Theorem 1.1.** Note in view of (1.8) that to prove statement (1) in Theorem 1.1 it is enough to prove that $s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta)$ is cohomologous to $s_{k_t}(E,g)\wedge\cdots\wedge s_{k_1}(E,g)$, where g is a smooth metric on E.

From Proposition 4.3 we have that

$$\begin{aligned}s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta)&=(-1)^k\pi_*\big([dd^c\varphi_t]_{\theta_t}^{k_t+r-1}\wedge\cdots\wedge[dd^c\varphi_1]_{\theta_1}^{k_1+r-1}\big)=\\&(-1)^k\pi_*\big(\theta_t^{k_t+r-1}\wedge\cdots\wedge\theta_1^{k_1+r-1}\big)+(-1)^k dd^c\pi_*S,\end{aligned}$$

for some current S; here we have used the notation from Section 6.1.

By applying Lemma 6.3 to $\tilde{\theta}_j^{k_j+r-1}$, noting that $\theta_j^{k_j+r-1}=\varpi_j^*\tilde{\theta}_j^{k_j+r-1}$, we get that

$$\pi_*\big(\theta_t^{k_t+r-1}\wedge\cdots\wedge\theta_1^{k_1+r-1}\big)=(\pi_t)_*\tilde{\theta}_t^{k_t+r-1}\wedge\cdots\wedge(\pi_1)_*\tilde{\theta}_1^{k_1+r-1}. \tag{6.5}$$

Let η be the metric on $\mathcal{O}_{\mathbf{P}(E)}(1)$ associated with g. Then θ is cohomologous to $dd^c\eta$ and since π_* commutes with exterior differentiation, it follows from (1.1) that $(-1)^{k_j}(\pi_j)_*\tilde{\theta}_j^{k_j+r-1}$ is a form in the class of $s_{k_j}(E,g)$. It follows that $(-1)^k$ times right hand side of (6.5) is cohomologous to $s_{k_t}(E,g)\wedge\cdots\wedge s_{k_1}(E,g)$, and we conclude that so is $s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta)$.

6.4. **Lelong numbers, proof of statement** (3) **in Theorem 1.1.** We begin by recalling the definition of the Lelong number of a closed positive current. We assume that we are on a complex manifold X of dimension n and that around a given point $a \in X$, we have local coordinates z. If T is a closed positive (p,p)-current on X, then the *Lelong number* of T at a can be defined as

$$\nu(T,a) := \int \mathbf{1}_{\{a\}}(dd^c \log|z-a|^2)^{n-p} \wedge T, \tag{6.6}$$

which is independent of the local coordinate system, see for example [De3, Definition III.5.4 and Corollary III.7.2]. Since $L(\log|z-a|^2) = \{a\}$, which has codimension n, the product in the integrand is indeed well-defined.

Note that the definition of Lelong numbers can be extended to currents that are locally of the form $T_+ - T_-$, where $T_\pm$ are closed positive currents, through $\nu(T,a) := \nu(T_+,a) - \nu(T_-,a)$ if $T = T_+ - T_-$ in a neighborhood of a. In particular, in view of Lemma 6.2 the Lelong numbers are defined for the currents $s_{k_t}(E,h,\theta) \wedge \cdots \wedge s_{k_1}(E,h,\theta)$, and thus in particular for $c_k(E,h,\theta)$.

Remark 6.4. Let us consider (6.6). For simplicity, assume that $a = 0$. Note that by the dimension principle for any (p,p)-current T that is (locally) the difference of two closed positive currents,

$$(dd^c \log|z|^2)^{n-p} \wedge T = dd^c \log|z|^2 \mathbf{1}_{X\setminus\{0\}} \wedge \cdots \wedge dd^c \log|z|^2 \mathbf{1}_{X\setminus\{0\}} \wedge T,$$

cf. Proposition 2.2.

Now assume that $T = s_{k_t}(E,h,\theta) \wedge \cdots \wedge s_{k_1}(E,h,\theta)$, i.e., $T = (-1)^k \pi_* \mu$, where

$$\mu = [dd^c \varphi_t]_{\theta_t}^{k_t+r-1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta_1}^{k_1+r-1}$$

and we are using the notation from Section 6.1. Notice that $\log|\pi^* z|^2$ is psh with analytic singularities on Y with unbounded locus $Z := \pi^{-1}\{0\}$. Thus, in view of Lemma 4.2, arguing as in the proof of Lemma 6.2 (regarding $\log|\pi^* z|^2$ as a metric on the trivial line bundle over Y), one gets that

$$dd^c \log|\pi^* z|^2 \mathbf{1}_{Y\setminus Z} \wedge \cdots \wedge dd^c \log|\pi^* z|^2 \mathbf{1}_{Y\setminus Z} \wedge \mu$$

is a globally defined current that in a neighborhood of Z is the difference of two closed positive currents with analytic singularities.

Next, note that if $u^{(\iota)}$ is a sequence of smooth psh functions decreasing to $\log|z|^2$, then $\pi^* u^{(\iota)}$ is a sequence of smooth psh functions decreasing to $\log|\pi^* z|^2$. Using (2.1), Propositions 2.3 and 3.2, and (2.4) we conclude that

$$\mathbf{1}_{\{0\}}(dd^c \log|z|^2)^{n-k} \wedge T = \pi_*\big(\mathbf{1}_Z dd^c \log|\pi^* z|^2 \mathbf{1}_{Y\setminus Z} \wedge \cdots \wedge dd^c \log|\pi^* z|^2 \mathbf{1}_{Y\setminus Z} \wedge \mu\big).$$

□

Proof of statement (3) *in Theorem 1.1.* Let us choose coordinates so that $a = 0$. Since Lelong numbers are locally defined, cf. (6.6), we may assume that we are in a neighborhood $0 \in \mathcal{U} \subset X$ as in the proof of Lemma 6.2. Let θ and θ' be two first Chern forms on $\mathcal{O}_{\mathbf{P}(E)}(1)$ corresponding to smooth metrics ψ and ψ', respectively, and let $\mu = [dd^c \varphi_t]_{\theta_t}^{k_t+r-1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta_1}^{k_1+r-1}$ and $\mu' = [dd^c \varphi_t]_{\theta'_t}^{k_t+r-1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta'_1}^{k_1+r-1}$ be the corresponding currents on Y, where we use the notation from Section 6.1 and θ'_j is defined analogously to θ_j.

We claim that in $\pi^{-1}\mathcal{U}$

$$\mu - \mu' = \sum d\beta_i \wedge \mu_i, \tag{6.7}$$

where β_i are smooth forms and μ_i are closed positive currents with analytic singularities. Using the notation from Remark 6.4, let $S \wedge T$ denote the operator $T \mapsto \mathbf{1}_Z dd^c \log|\pi^* z|^2 \mathbf{1}_{Y\setminus Z} \wedge \cdots \wedge dd^c \log|\pi^* z|^2 \mathbf{1}_{Y\setminus Z} \wedge T$. Since β_i is smooth, applying S commutes with multiplication with $d\beta_i$, and thus, since $S \wedge \mu_i$ is closed, we get

$$S \wedge d\beta_i \wedge \mu_i = d\beta_i \wedge S \wedge \mu_i = d(\beta_i \wedge S \wedge \mu_i) =: d\tau_i,$$

where $\tau_i = \beta_i \wedge S \wedge \mu_i$ has support on Z.

Hence, in view of Remark 6.4, taking the claim for granted,

$$(-1)^k\Big(\nu\big(s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta),0\big)-\nu\big(s_{k_t}(E,h,\theta')\wedge\cdots\wedge s_{k_1}(E,h,\theta'),0\big)\Big)=$$
$$\nu(\pi_*\mu,0)-\nu(\pi_*\mu',0)=\int\pi_*(S\wedge(\mu-\mu'))=\sum_i\int d\pi_*\tau_i=0,$$

where the last equality follows by Stokes' theorem since the $\pi_*\tau_i$ have support on $\pi(Z)=\{0\}$. This proves (3) in Theorem 1.1.

It remains to prove the claim. First note that, since $[dd^c\varphi_1]^0_{\theta_1}=[dd^c\varphi_1]^0_{\theta_1'}=1$, $[dd^c\varphi_1]^0_{\theta_1}-[dd^c\varphi_1]^0_{\theta_1'}$ vanishes and is in particular of the form (6.7). Next assume that we have proven that

$$T-T':=[dd^c\varphi_\kappa]^{m_\kappa-1}_{\theta_\kappa}\wedge\cdots\wedge[dd^c\varphi_1]^{m_1}_{\theta_1}-[dd^c\varphi_\kappa]^{m_\kappa-1}_{\theta_\kappa'}\wedge\cdots\wedge[dd^c\varphi_1]^{m_1}_{\theta_1'}=\sum_i d\gamma_i\wedge T_i$$

for some smooth forms γ_i and closed positive currents with analytic singularities T_i, where $m_\kappa\geq 1$.

By the assumption on $\mathcal{U}$, $T=T_+-T_-$ in $\pi^{-1}\mathcal{U}$, where $T_\pm$ are closed positive currents with analytic singularities. Now

$$[dd^c\varphi_\kappa]^{m_\kappa}_{\theta_\kappa}\wedge\cdots\wedge[dd^c\varphi_1]^{m_1}_{\theta_1}-[dd^c\varphi_\kappa]^{m_\kappa}_{\theta_\kappa'}\wedge\cdots\wedge[dd^c\varphi_1]^{m_1}_{\theta_1'}=[dd^c\varphi_\kappa]_{\theta_\kappa}\wedge T-[dd^c\varphi_\kappa]_{\theta_\kappa'}\wedge T'=$$
$$dd^c\varphi_\kappa\wedge\mathbf{1}_{Y\setminus Z_\kappa}(T-T')+(\theta_\kappa-\theta_\kappa')\wedge\mathbf{1}_{Z_\kappa}T+\theta_\kappa'\wedge\mathbf{1}_{Z_\kappa}(T-T')=$$
$$\sum_i d\gamma_i\wedge dd^c\varphi_\kappa\wedge\mathbf{1}_{Y\setminus Z_\kappa}T_i+dd^c(\psi_\kappa-\psi_\kappa')\wedge\mathbf{1}_{Z_\kappa}(T_+-T_-)+\sum_i d(\gamma_i\wedge dd^c\psi_\kappa')\wedge\mathbf{1}_{Z_\kappa}T_i,$$

which is of the form in the right hand side of (6.7) since ψ_κ and ψ_κ' are smooth. Here we have used that set of closed positive currents with analytic singularities is closed under multiplication with $\mathbf{1}_U$, where U is an constructible set and Remark 3.3. The claim now follows by induction. □

7. Comparison with [LRRS], Proof of Theorem 1.2

Assume that h is a singular Griffiths positive (negative) metric on a holomorphic vector bundle $E\to X$ over a complex manifold X of dimension n, such that that the degeneracy locus of h is contained in a variety $V\subset X$. In [LRRS] the first three authors together with Ruppenthal defined Chern and Segre currents, $c_k(E,h)$ and $s_k(E,h)$, for $k\leq\operatorname{codim}V$. Let us briefly recall the construction. Locally, h can be approximated by an increasing (decreasing) sequence h_ε of Griffiths positive (negative) smooth metrics, see, e.g., [BP, Proposition 3.1] or [R, Proposition 1.3]. Theorem 1.11 in [LRRS] asserts that the iterated limit[4]

$$\lim_{\varepsilon_t\to 0}\cdots\lim_{\varepsilon_1\to 0}s_{k_t}(E,h_{\varepsilon_t})\wedge\cdots\wedge s_{k_1}(E,h_{\varepsilon_1})\tag{7.1}$$

exists as a current and is independent of the choice of h_ε for $k_1+\cdots+k_t\leq\operatorname{codim}V$; in particular, it follows that $s_{k_t}(E,h)\wedge\cdots\wedge s_{k_1}(E,h)$, locally given as (7.1), defines a global current on X. Moreover, the Chern currents $c_k(E,h)$, defined from $s_{k_t}(E,h)\wedge\cdots\wedge s_{k_1}(E,h)$ analogously to (1.8), and the Segre currents $s_k(E,h)$ coincide with the corresponding Chern and Segre forms where h is smooth, and are in the classes $c_k(E)$ and $s_k(E)$, respectively, when X is compact.

Assume that h is Griffiths positive and let φ_ε be the smooth metric on $\pi:\mathbf{P}(E)\to X$ induced by h_ε. Then φ_ε is a sequence of smooth positive metrics on $\mathcal{O}_{\mathbf{P}(E)}(1)$ decreasing to φ. Let ω_ε be the first Chern form of φ_ε. Then $s_{k_t}(E,h)\wedge\cdots\wedge s_{k_1}(E,h)$ satisfies the following recursion for $t\geq 0$:

$$s_{k_t}(E,h)\wedge\cdots\wedge s_{k_1}(E,h)=\lim_{\varepsilon\to 0}(-1)^{k_t}\pi_*(\omega_\varepsilon^{k_t+r-1})\wedge s_{k_{t-1}}(E,h)\wedge\cdots\wedge s_{k_1}(E,h).\tag{7.2}$$

[4]In [LRRS] the limit is taken over certain subsequences of h_ε, but this is in fact not necessary; see the end of the proof of Proposition 4.6 in [LRRS].

Remark 7.1. Assume that h is Griffiths positive. Let us use the notation from Section 6.1, and denote the sequences of positive metrics on Y induced by h_ε by $\varphi_{j,\varepsilon}$. Moreover assume that we are outside the degeneracy locus of h. Then the induced φ_j are locally bounded and thus, by Proposition 2.3,

$$(-1)^k s_{k_t}(E,h)\wedge\cdots\wedge s_{k_1}(E,h) = \lim_{\varepsilon_t\to 0}\cdots\lim_{\varepsilon_1\to 0}\ \pi_*\big((dd^c\varphi_{t,\varepsilon_t})^{k_t+r-1}\wedge\cdots\wedge(dd^c\varphi_{1,\varepsilon_1})^{k_1+r-1}\big)$$
$$= \pi_*\big((dd^c\varphi_t)^{k_t+r-1}\wedge\cdots\wedge(dd^c\varphi_1)^{k_1+r-1}\big).$$

□

To prove Theorem 1.2 we need to recall some auxiliary results from [LRRS]. First, following [LRRS] we say that a smooth $(n-k,n-k)$-form β is a *bump form* at a point $x\in X$ if it is strongly positive, and such that for some (or equivalently for any) Kähler form ω defined near x, there exists a constant $C>0$ such that $C\omega^{n-k}\le\beta$ as strongly positive forms in a neighborhood of x.

Lemma 7.2. *Let $V\subset X$ be a subvariety. Then for each $k\le \operatorname{codim} V$ and each point $x\in V$, there exists a bump form β at x of bidegree $(n-k,n-k)$ with arbitrarily small support such that $dd^c\beta$ has support in $X\setminus V$.*

Proof. We construct the bump form β as in the proof of Lemma 4.3 in [LRRS] (with k equal to $k+q$ in that proof). By that proof, one may write β as a sum of terms, such that each term in some local coordinate system $(z',z'')\in\mathbf{C}^{n-k}\times\mathbf{C}^k$ is of the form $\chi_1\chi_2\beta_0$, where $\beta_0 = idz'_1\wedge d\bar z'_1\wedge\cdots\wedge idz'_{n-k}\wedge d\bar z'_{n-k}$ and χ_1 and χ_2 are cutoff functions in the variables z' and z'', respectively, such that χ_2 is constant in a neighborhood of $(\operatorname{supp}\chi_1\chi_2)\cap V$. It then suffices to prove that $d(\chi_1\chi_2\beta_0)$ has support in $X\setminus V$. This holds since β_0 has full degree in the z'-variables so that $d(\chi_1\chi_2\beta_0)=\chi_1 d\chi_2\wedge\beta_0$, and $\chi_1 d\chi_2$ has support in the set where $\chi_1\chi_2\not\equiv 0$ and χ_2 is not constant, which is contained in $X\setminus V$. □

The next result is Lemma 4.5 in [LRRS].

Lemma 7.3. *Let S and T be two closed, positive (k,k)-currents on X such that $S=T$ outside a subvariety V with $\operatorname{codim} V\ge k$, and assume that for each point of $x\in V$, there exists an $(n-k,n-k)$ bump form β at x with arbitrarily small support such that*

$$\int_X S\wedge\beta = \int_X T\wedge\beta.$$

Then $S=T$ everywhere.

Proof of Theorem 1.2. We will proceed by induction. Let us first choose $t\ge 2$ and assume that we have proved that

$$s_{k_{t-1}}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta) = s_{k_{t-1}}(E,h)\wedge\cdots\wedge s_{k_1}(E,h). \tag{7.3}$$

Let us use the notation from Section 6.1. Moreover, let Y' be the fiber product

$$Y' = \mathbf{P}(E_{t-1})\times_X\cdots\times_X\mathbf{P}(E_1),$$

with projections $\varpi'_j: Y'\to\mathbf{P}(E_j)$, $\pi': Y'\to X$, and $p: Y\to Y'$. Then $Y=\mathbf{P}(E_t)\times_X Y'$.

Let φ'_j denote the pullback metric $(\varpi'_j)^*\tilde\varphi_j$ on $L'_j := (\varpi'_j)^*L_j$ and let $\theta'_j = (\varpi'_j)^*\tilde\theta_j$. Let $\tilde\varphi_\epsilon$ denote the metric on $\widetilde L_t$ induced by h_ε, let φ_ε denote the pullback $\varpi_t^*\tilde\varphi_\varepsilon$ to Y, and let $\tilde\omega_\varepsilon$ and ω_ε denote the corresponding first Chern forms. Let

$$\mu' = [dd^c\varphi'_{t-1}]_{\theta'_{t-1}}^{k_{t-1}+r-1}\wedge\cdots\wedge[dd^c\varphi'_1]_{\theta'_1}^{k_1+r-1}$$

and let $\mu=p^*\mu'$; by regularization

$$\mu = [dd^c\varphi_{t-1}]_{\theta_{t-1}}^{k_{t-1}+r-1}\wedge\cdots\wedge[dd^c\varphi_1]_{\theta_1}^{k_1+r-1}.$$

Now, using the induction hypothesis (7.3) and Lemma 6.3 we can rewrite (7.2) as

$$s_{k_t}(E,h)\wedge\cdots\wedge s_{k_1}(E,h)=(-1)^k\lim_{\varepsilon\to 0}(\pi_t)_*\tilde{\omega}_\varepsilon^{k_t+r-1}\wedge\pi'_*\mu'=(-1)^k\lim_{\varepsilon\to 0}\pi_*(\omega_\varepsilon^{k_t+r-1}\wedge\mu).$$

Moreover,

$$s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta)=(-1)^k\pi_*\big([dd^c\varphi_t]_{\theta_t}^{k_t+r-1}\wedge\mu\big).$$

Since $k\le\operatorname{codim}V$, by Lemma 7.2, for each $x\in V$ there is a bump form β at x of bidegree $(n-k,n-k)$ with arbitrarily small support such that $dd^c\beta$ vanishes in a neighborhood of V. Note that $\pi_t(L(\tilde\varphi_t))\subset V$ in view of Lemma 5.6. It follows that

$$Z_t=\varpi_t^{-1}L(\tilde\varphi_t)\subset\varpi_t^{-1}\pi_t^{-1}V=\pi^{-1}V$$

and thus $dd^c\pi^*\beta$ vanishes in a neighborhood of $Z_t\subset Y$. Hence, by Lemma 4.4 (applied to $T=\mu$)

$$\int_X s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta)\wedge\beta=(-1)^k\int_Y[dd^c\varphi_t]_{\theta_t}^{k_t+r-1}\wedge\mu\wedge\pi^*\beta=$$
$$(-1)^k\lim_{\varepsilon\to0}\int_Y\omega_\varepsilon^{k_t+r-1}\wedge\mu\wedge\pi^*\beta=\int_X s_{k_t}(E,h)\wedge\cdots\wedge s_{k_1}(E,h)\wedge\beta.$$

In view of Remarks 6.1 and 7.1, (1.9) holds outside V, and thus by Lemma 7.3 it holds everywhere.

It remains to prove (1.9) for $t=1$. This follows, in fact, by an easier version of the argument above. If β is a bump form as above, then by Lemma 4.4 (with $T=1$)

$$\int_X s_{k_1}(E,h,\theta)\wedge\beta=(-1)^{k_1}\int_{\mathbf{P}(E_1)}[dd^c\varphi_1]_\theta^{k_1+r-1}\wedge\pi_1^*\beta=$$
$$(-1)^{k_1}\lim_{\varepsilon\to0}\int_{\mathbf{P}(E_1)}\omega_\varepsilon^{k_1+r-1}\wedge\pi_1^*\beta=\int_X s_{k_1}(E,h)\wedge\beta$$

and again (1.9) follows from Lemma 7.3. □

8. Remarks and examples

Let us start by discussing the uniqueness of the Chern and Segre currents. Assume that X is a complex manifold and that $V\subset X$ is a subvariety of pure codimension p. Moreover assume that T_1 and T_2 are closed positive (p,p)-currents on X that coincide outside V, and that the Lelong numbers of T_1 and T_2 coincide at each $x\in V$. We claim that then $T_1=T_2$. Indeed, since $T:=T_1-T_2$ is a closed normal (p,p)-current with support on V it follows that $T=\sum a_j[V_j]$, where V_j are the irreducible components of V, see, e.g., [De3, Corollary III.2.14]. Next, by assumption the Lelong number of T at each point in V is zero and therefore $a_j=0$ for each j. If T_1 and T_2 are closed positive (k,k)-currents that coincide outside V, where $k<p$, then $\mathbf{1}_VT_j$ vanishes for $j=1,2$ by the dimension principle, and hence $T_1=T_2$.

Now assume that we are in the situation of Theorem 1.1 and that $L(\log\det h^*)\subset V$. Then by Remark 6.1, $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ are independent of θ outside V. Since they are of bidegree (k,k) and (locally) differences of closed positive currents it follows in view of (3) that they are independent of θ for $k\le p$. Note that if h is smooth outside V then $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ are uniquely determined by the condition (2) for $k<p$.

On the other hand if $k>p$, let α and β be real smooth forms of bidegree $(k-p,k-p)$ such that $\alpha-\beta$ is exact. Then $(\alpha-\beta)\wedge[V]\neq0$ has zero Lelong numbers everywhere, is cohomologous to zero, and vanishes outside V. Thus there is no reason to expect $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ to be independent of θ for $k>p$ in general.

Let us consider some simple examples, where we can compute the Segre and Chern currents explicitly.

Example 8.1. Let $L \to X$ be a line bundle and $e^{-\varphi}$ a Griffiths positive metric with analytic singularities. Then $\mathcal{O}_{\mathbf{P}(L)}(1) = L$ and $e^{-\varphi} = h$, and thus

$$s_k(L, h, \theta) = [dd^c\varphi]^k_\theta = (dd^c\varphi)^k + \sum_{\ell=0}^{k-1} \theta^{k-\ell} \wedge \mathbf{1}_Z (dd^c\varphi)^\ell,$$

where Z is the unbounded locus of φ. Classically, by the Bedford-Taylor-Demailly theory, for a general φ, $(dd^c\varphi)^k$ is well-defined only for $k = 1$; if φ has analytic singularities it is well-defined for $k \leq \operatorname{codim} Z =: p$.

In fact, it is not hard to find examples of psh functions u with analytic singularities and sequences $u^{(\iota)}$ of psh functions decreasing to u where the corresponding sequences $(dd^c u^{(\iota)})^k$ converge to different positive currents for $k > p$, see, e.g., [ABW, Example 3.2]. In particular, this implies that the construction in [LRRS] cannot extend to $k > p$ in general. □

Example 8.2. Let $X = \mathbf{P}^n$, $L = \mathcal{O}_{\mathbf{P}^n}(1)$, and $h = e^{-\varphi}$, where $\varphi = \log |s|^2$ and s is a non-trivial global holomorphic section of L, cf. Example 4.1. Then the unbounded locus of φ is the hyperplane $Z = \{s = 0\} \subset \mathbf{P}^n$ and thus $(dd^c\varphi)^m$ is defined classically by the Bedford-Taylor-Demailly theory only for $m = 1$, cf. Example 8.1. By the Poincaré-Lelong formula, $dd^c\varphi = [s = 0] = [Z]$, cf. [De4, Example 2.2]. It follows that

$$(dd^c\varphi)^2 = dd^c(\varphi \mathbf{1}_{X \setminus Z} dd^c\varphi) = 0$$

and thus $(dd^c\varphi)^m = 0$ for all $m > 1$. Hence, if θ is the first Chern form of a smooth metric on L, then

$$[dd^c\varphi]^m_\theta = \theta^{m-1} \wedge [s = 0].$$

Since L is a line bundle, $\mathbf{P}(L) = X$ and $(\mathcal{O}_{\mathbf{P}(L)}(1), h) = (L, e^{-\varphi})$. Moreover, $Y = X$ and $\varphi_j = \varphi$ and $\theta_j = \theta$ for each j. Thus

$$s_{k_t}(L, h, \theta) \wedge \cdots \wedge s_{k_1}(L, h, \theta) = (-1)^k [dd^c\varphi]^k_\theta = (-1)^k\ \theta^{k-1} \wedge [s = 0]. \tag{8.1}$$

In particular, (8.1) depends on θ as soon as $k > 1$. In this case it is easy to see that the Lelong numbers are independent of θ, since θ is smooth.

Note that $s_k(E, h)$ and $c_k(E, h)$ are well-defined in the classical or [LRRS] sense only for $k \leq 1$; it holds that

$$c_1(E, h) = -s_1(E, h) = dd^c\varphi = [s = 0].$$

□

Example 8.3. Let E be a trivial rank 2 bundle over $X = \mathbf{C}^2$ with coordinates $x = (x_1, x_2)$ and let h be the singular metric $h = e^{-\log |x|^2}\,\mathrm{Id}$. In view of Section 5, in the open set $\mathcal{U}_i$ the induced metric φ on $\mathcal{O}_{\mathbf{P}(E)}(1)$ is given by

$$\varphi_i(x, [\xi]) = \log |x|^2 + \log |\xi/\xi_i|^2.$$

In particular, it follows that h is Griffiths positive with analytic singularities.

Since the unbounded locus of φ, $Z = \{x = 0\}$, has codimension 2 in $\mathbf{P}(E)$, $(dd^c\varphi)^m$ is classically well-defined for $m \leq 2$. Note that $dd^c\varphi = dd^c \log |x|^2 + \omega_{\mathrm{FS}}$, where ω_{FS} is the Fubini-Study metric on the fibers $\pi^{-1}(x) \cong \mathbf{P}^1_\xi$, and

$$(dd^c\varphi)^2 = (dd^c \log |x|^2)^2 + 2\omega_{\mathrm{FS}} \wedge dd^c \log |x|^2 = [x = 0] + 2\omega_{\mathrm{FS}} \wedge dd^c \log |x|^2 \tag{8.2}$$

since ω^2_{FS} vanishes for degree reasons. It follows that

$$\begin{aligned}(dd^c\varphi)^3 = dd^c(\varphi \mathbf{1}_{\mathbf{P}(E)\setminus Z}(dd^c\varphi)^2) = dd^c(\varphi\ 2\omega_{\mathrm{FS}} \wedge dd^c \log |x|^2) = \\ (\omega_{\mathrm{FS}} + dd^c \log |x|^2) \wedge (2\omega_{\mathrm{FS}} \wedge dd^c \log |x|^2) = 2\omega_{\mathrm{FS}} \wedge [x = 0],\end{aligned}$$

where we have again used that $\omega^2_{\mathrm{FS}} = 0$. Thus if θ is the first Chern form of a smooth metric on $\mathcal{O}_{\mathbf{P}(E)}(1)$,

$$[dd^c\varphi]^3_\theta = (dd^c\varphi)^3 + \theta \wedge \mathbf{1}_Z (dd^c\varphi)^2 + \theta^2 \wedge \mathbf{1}_Z dd^c\varphi = 2\omega_{\mathrm{FS}} \wedge [x = 0] + \theta \wedge [x = 0],$$

where the last term in the middle expression vanishes by the dimension principle since codim $Z = 2$. Therefore

$$s_2(E,h,\theta) = \pi_*[dd^c\varphi]^3_\theta = 3[0].$$

In view of (8.2), $c_1(E,h,\theta) = -s_1(E,h,\theta) = 2dd^c\log|x|^2$, and thus by (1.8) we get that $c_2(E,h,\theta) = [0]$. □

A naive attempt would be to define Segre currents as the pushforward of $(dd^c\varphi)^{k+r-1}$ instead of $[dd^c\varphi]^{k+r-1}_\theta$. Since $(dd^c\varphi)^m$ coincides with the classical Bedford-Taylor-Demailly Monge-Ampère product where φ is locally bounded, $\pi_*(dd^c\varphi)^{k+r-1}$ coincides with $s_k(E,h)$ where h is smooth, cf. Lemma 5.6. Example 8.2, however, shows that the current $\pi_*(dd^c\varphi)^{k+r-1}$ is not in $s_k(E)$ in general; in that example $(dd^c\varphi)^m = 0$ for $m > 1$, whereas $s_k(E) \neq 0$ for $0 \leq k \leq n$. Moreover, Example 8.3 shows that the Lelong number of $\pi_*(dd^c\varphi)^{k+r-1}$ is not equal to the Lelong number of $s_k(E,h,\theta)$ in general. Indeed, note that in that example $\pi_*(dd^c\varphi)^3$ equals $2[0]$ and thus has Lelong number 2 at the origin, whereas the Lelong number at the origin of $s_2(E,h,\theta)$ is 3.

The following example shows that the products (1.7) of Segre currents are not commutative in general.

Example 8.4. Let X be the unit ball in $\mathbf{C}^3$ with coordinates $x = (z,\zeta_1,\zeta_2)$, and let $E = X\times\mathbf{C}^2 \to X$ be the trivial vector bundle of rank 2. Let h be the singular hermitian metric on E whose dual metric h^* on E^* is given by the matrix $\begin{bmatrix} 0 & 0 \\ 0 & |z|^2 \end{bmatrix}$. Then in view of Section 5, the induced metric φ on $\mathcal{O}_{\mathbf{P}(E)}(1)$ is given by $\varphi_1(x,[\xi]) = \log|z|^2 + \log|\xi_2/\xi_1|^2$ and $\varphi_2 = \log|z|^2$ in $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. It follows that

$$dd^c\varphi = dd^c(\log|z|^2 + \log|\xi_2|^2) = [Z] + [W],$$

where $Z = \{z = 0\}$ and $W = \{\xi_2 = 0\}$.

Moreover let g be the smooth metric on E given by the matrix $\begin{bmatrix} 1 & |\zeta|^2 \\ |\zeta|^2 & 1 \end{bmatrix}$. A computation yields that the curvature form at $\zeta = 0$ is $\Theta^g|_{\zeta=0} = \begin{bmatrix} 0 & \bar\partial\partial|\zeta|^2 \\ \bar\partial\partial|\zeta|^2 & 0 \end{bmatrix}$ so that $\frac{i}{2\pi}\Theta^g|_{\zeta=0} = -\begin{bmatrix} 0 & dd^c|\zeta|^2 \\ dd^c|\zeta|^2 & 0 \end{bmatrix}$. Thus at $\zeta = 0$, in view of (1.2),

$$s_1(E,g) = -c_1(E,g) = 0,\ c_2(E,g) = -(dd^c|\zeta|^2)^2,\ s_2(E,g) = c_1(E,g)^2 - c_2(E,g) = (dd^c|\zeta|^2)^2.$$

Let θ be the first Chern form of the smooth metric ψ on $\mathcal{O}_{\mathbf{P}(E)}(1)$ induced by g. Then at $(x,[\xi]) \in \mathbf{P}(E)$

$$\theta = dd^c\psi = \omega^{g^*}_{\mathrm{FS}} - \frac{i}{2\pi|\xi|_g}\Theta^{g^*}_{\xi\bar\xi},$$

where Θ^{g^*} is the curvature form on E^*_x and $\omega^{g^*}_{\mathrm{FS}}$ is the induced Fubini-Study metric on the fiber $\pi^{-1}(x) = \mathbf{P}(E^*_x) \cong \mathbf{P}^1$, see, e.g., the beginning of the proof of Proposition 3.1 in [G] or the beginning of Section 2 in [Di].

Note that at $\zeta = 0$, g is just the standard Euclidean metric on $\mathbf{C}^2$, so that $\omega^{g^*}_{\mathrm{FS}}$ is just the standard Fubini-Study metric ω_{FS} on $\mathbf{P}^1$. Moreover, $\Theta^{g^*}|_{\zeta=0} = -(\Theta^g)^T|_{\zeta=0} = -\begin{bmatrix} 0 & \bar\partial\partial|\zeta|^2 \\ \bar\partial\partial|\zeta|^2 & 0 \end{bmatrix}$, where T denotes transpose. In particular, for $(x,[\xi])$ such that $\zeta = 0$ and $\xi_2 = 0$, $\Theta^{g^*}_{\xi\bar\xi} = 0$. Hence at $\zeta = 0$,

$$\theta\wedge[W] = \omega_{\mathrm{FS}}\wedge[W] = 0, \tag{8.3}$$

where the last equality follows for degree reasons. Therefore, for $m > 1$, noting that $(dd^c\varphi)^m = 0$, $[dd^c\varphi]^m_\theta = \theta^{m-1}\wedge\big([Z]+[W]\big) = \theta^{m-1}\wedge[Z]$ at $\zeta = 0$. More generally, let E_1 and E_2 be copies of

E and $Y = \mathbf{P}(E_1) \times_X \mathbf{P}(E_2)$, and let us use the notation from Section 6.1. Then a computation using (8.3), yields that for $m_1, m_2 > 1$, at $\zeta = 0$,

$$[dd^c\varphi_2]_{\theta_2}^{m_2} \wedge [dd^c\varphi_1]_{\theta_1}^{m_1} = \theta_2^{m_2} \wedge \theta_1^{m_1-1} \wedge [Z].$$

In view of (2.2), (6.2), and (6.3) it follows that

$$\pi_*\big(\theta_2^{k_2+1} \wedge \theta_1^{k_1+1} \wedge [Z]\big) = (-1)^{k_1+k_2} s_{k_2}(E,g) \wedge s_{k_1}(E,g) \wedge [Z].$$

Hence at $\zeta = 0$

$$s_1(E,h,\theta) \wedge s_2(E,h,\theta) = -\pi_*\big([dd^c\varphi_2]_{\theta_2}^2 \wedge [dd^c\varphi_1]_{\theta_1}^3\big) = \\ -\pi_*\big(\theta_2^2 \wedge \theta_1^2 \wedge [Z]\big) = -s_1(E,g) \wedge s_1(E,g) \wedge [Z] = 0,$$

and similarly

$$s_2(E,h,\theta) \wedge s_1(E,h,\theta) = -s_2(E,g) \wedge s_0(E,g) \wedge [Z] = -(dd^c|\zeta|^2)^2 \wedge [Z] \neq 0.$$

Thus $s_1(E,h,\theta) \wedge s_2(E,h,\theta) \neq s_2(E,h,\theta) \wedge s_1(E,h,\theta)$ in this case. □

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