DEGREE GROWTH OF MONOMIAL MAPS AND MCMULLEN'S POLYTOPE ALGEBRA

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Abstract. We compute all dynamical degrees of monomial maps by interpreting them as mixed volumes of polytopes. By exploiting further the isomorphism between the polytope algebra of P. McMullen and the universal cohomology of complete toric varieties, we construct invariant positive cohomology classes when the dynamical degrees have no resonance.

1. Introduction

Some of the most basic information associated to a rational dominant map $f : \mathbb{P}^d \dashrightarrow \mathbb{P}^d$ is provided by its degrees $\deg_k(f) := \deg f^{-1}(L_k)$, where L_k is a generic linear subspace of $\mathbb{P}^d$ of codimension k, see p.14 below for a formal definition. From a dynamical point of view, it is important to understand the behaviour of the sequence $\deg_k(f^n)$ as $n \to \infty$. It is not difficult to see that $\deg_k(f^{m+n}) \leq \deg_k(f^m) \deg_k(f^n)$, and thus following Russakovskii-Shiffman [RS] we can define the *k-th dynamical degree of f* as $\lambda_k(f) := \lim_n \deg_k(f^n)^{1/n}$. Basic properties of dynamical degrees can be found in [RS, DS]. Our main objective is to describe the sequence of degrees $\deg_k(f^n)$ in the special case of monomial maps f, but for arbitrary k.

Controlling the degrees of iterates of a rational map is a quite delicate problem. Up to now, most investigations have been focused on the case $d = 2$ and $k = 1$, see [DF, FJ] and the references therein. There are also various interesting families of examples for $k = 1$ in arbitrary dimensions in e.g. [AABM, AMV, BK1, BK3, BHM, N]. In particular, the case of monomial maps and $k = 1$ is treated in [BK2, Fa, HP, JW, L]. On the other hand, there are only few references in the literature concerning the case $2 \leq k \leq d - 2$, see [Og, DN]. An essential problem arises from the difficulty to explicitly compute $\deg_k(f)$ even in concrete examples. This can be overcome in the case of monomial maps, since tools from convex geometry allow one to compute these numbers in terms of (mixed) volumes of polytopes. This technique has already been used to compute all degrees of the standard Cremona transformation in arbitrary dimensions by Gonzalez-Springer and Pan [GSP].

Monomial maps on $\mathbb{P}^d$ correspond to integer valued $d \times d$ matrices, $M(d, \mathbb{Z})$. Given $A \in M(d, \mathbb{Z})$ we write ϕ_A for the corresponding monomial map $\phi_A(x_1, \ldots, x_d) = (\prod_j x_j^{a_{j1}}, \ldots, \prod_j x_j^{a_{jd}})$ with $(x_1, ..., x_d) \in (\mathbb{C}^*)^d$. This mapping is holomorphic on the torus $(\mathbb{C}^*)^d$ and extends as a rational map to the standard equivariant compactification $\mathbb{P}^d \supset (\mathbb{C}^*)^d$. Moreover ϕ_A is dominant precisely if $\det(A) \neq 0$. Observe that $\phi_A^n = \phi_{A^n}$

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for all n. If a_n and b_n are sequences of positive real numbers, we write $a_n \asymp b_n$ if $C^{-1} \leq a_n/b_n \leq C$ for some $C > 1$ and all n.

Theorem A. *Let $A \in M(d, \mathbb{Z})$ and let $\phi_A : \mathbb{P}^d \dashrightarrow \mathbb{P}^d$ be the corresponding rational map. Assume that $\det(A) \neq 0$ (so that ϕ_A is dominant). Then, for $0 \leq k \leq d$,*

$$\deg_k(\phi_A^n) \asymp \| \wedge^k A^n \|, \tag{1.1}$$

where $\wedge^k A : \wedge^k \mathbb{R}^d \to \wedge^k \mathbb{R}^d$ is the natural linear map induced by A and $\| \cdot \|$ is any norm on $\mathrm{End}(\wedge^k \mathbb{R}^d)$.

Corollary B. *Let ϕ_A and A be as in Theorem A. Order the eigenvalues of A in decreasing order, $|\rho_1| \geq |\rho_2| \geq \ldots \geq |\rho_d|$. Then the k-th dynamical degree of the monomial map ϕ_A is equal to $\prod_1^k |\rho_j|$.*

Recall that the topological entropy of a rational map $f : X \dashrightarrow X$ on a projective variety is defined as the asymptotic rate of growth of (n, ε)-separated sets outside the indeterminacy set of iterates of ϕ, see [DS] for details. On the one hand, the topological entropy of a monomial map is greater than its restriction to the compact real torus $\{|x_i| = 1\} \subset (\mathbb{C}^*)^d$ which is equal to $\log(\prod_1^d \max\{1, |\rho_i|\})$, see [HP, Sect. 5]. On the other hand, it is a general result due to Gromov [G], and Dinh-Sibony [DS] that $\max_k \log \lambda_k$ is an upper bound for the topological entropy. By Corollary B, $\log(\prod_1^d \max\{1, |\rho_i|\}) = \max_k \log \lambda_k$. Thus we have

Corollary C. *Let X be a projective smooth toric variety, let A be as in Theorem A, and let $\phi_A : X \dashrightarrow X$ be the induced rational map. Then the topological entropy of ϕ_A is equal to $\max \log \lambda_k$.*

We note that Theorem A and its two corollaries have been obtained independently by Jan-Li Lin, [L2], by different but related methods. His approach relies on the notion of Minkowski weights.

By Khovanskii-Teissier's inequalities, the sequence $k \mapsto \log \deg_k(f)$ is concave so that we always have $\lambda_k^2(f) \geq \lambda_{k-1}(f)\, \lambda_{k+1}(f)$ for any $1 \leq k \leq d-1$. Our next result gives a more precise control of the degrees when the asymptotic degrees are strictly concave. It can be seen as an analogue of [BFJ, Main Theorem] in the case of monomial maps but in arbitrary dimensions.

Theorem D. *Let $A \in M(d, \mathbb{Z})$ and let $\phi_A : \mathbb{P}^d \dashrightarrow \mathbb{P}^d$ be the associated rational monomial map. Write $\lambda_k = \lambda_k(\phi_A)$. Assume that $\det(A) \neq 0$ and that for some $1 \leq k \leq d-1$ the dynamical degrees satisfy*

$$\lambda_k^2 > \lambda_{k-1}\, \lambda_{k+1} \ . \tag{1.2}$$

Then there exists a constant $C > 0$ and an integer $r \geq 0$ such that, for this k,

$$\deg_k(\phi_A^n) = C \lambda_k^n + \mathcal{O}\left(n^r \left(\frac{\lambda_{k-1}\, \lambda_{k+1}}{\lambda_k} \right)^n \right). \tag{1.3}$$

Theorems A and D (and thus Corollary B) hold true for $\mathbb{P}^d$ replaced by a projective smooth variety, cf. Remark 6.2.

For $k = 1$ Theorem A is proven in [HP], and Theorem D is due to Lin [L, Thms 6.6-7]. In fact, there are finer estimates for the growth of ϕ_A. For example, Bedford-Kim [BK2] gave a description of when $\deg_1(\phi_A^n)$ satisfies a linear recurrence; in particular, it happens if $|\rho_1| > |\rho_2|$. We do not know if the assumption in Theorem D is sufficient for $\deg_k(\phi_A^n)$ to satisfy a linear recursion. This problem is related to the construction of a toric model $X(\Delta)$ dominating $\mathbb{P}^d$ such that the induced action $\phi_A^* : H^k(X(\Delta), \mathbb{R}) \to H^k(X(\Delta), \mathbb{R})$ of the monomial map ϕ_A commutes with iteration; or, in the terminology of Fornaess-Sibony, a model in which the map induced by ϕ_A is *stable*. This very delicate problem is treated in detail in various papers by Bedford-Kim [BK2], the first author [Fa] , Hasselblatt-Propp [HP], Jonsson and the second author [JW], and Lin [L] in the case $k = 1$. We do not address this problem here.

Let us briefly explain the idea of the proofs of Theorems A and D. The degree $\deg_k(\phi_A)$ can be naturally interpreted as an intersection number $\deg_k(\phi_A) = \phi_A^*\mathcal{O}(1)^k \cdot \mathcal{O}(1)^{d-k}$. Recall that any polytope P with integral vertices determines a toric variety $X(\Delta_P)$ and a line bundle L_P over $X(\Delta_P)$. Conversely, any line bundle on a toric variety that is generated by its global sections determines a polytope. Using this correspondence, one can compute intersection products of line bundles in a toric variety in terms of mixed volumes of polytopes, see [Od, p.79]. More precisely, given any two polytopes P, Q giving rise to two line bundles L_P, L_Q on the same toric variety, then the intersection product $L_P^k \cdot L_Q^{d-k}$ is given as the *mixed volume* $d!\,\mathrm{Vol}(P[k], Q[d-k])$, i.e., (a constant times) the coefficient of t^k in the polynomial $\mathrm{Vol}(tP + Q)$. Since $\mathcal{O}(1)$ over $\mathbb{P}^d$ corresponds to the standard simplex $\Sigma_d \subset \mathbb{Q}^d$, and the action of ϕ_A^* on L_P corresponds to the linear action of A on P it turns out that the proofs of Theorems A and D amounts to controlling the growth of mixed volumes under the action of the linear map A:

$$\deg_k(\phi_A^n) = d!\,\mathrm{Vol}\left(A^n(\Sigma_d)[k], \Sigma_d[d-k]\right). \tag{1.4}$$

The computation of mixed volumes is in general quite difficult. However, since we are interested in the asymptotic behaviour of $\deg_k$ we may replace Σ_d by a ball, which allows us to apply the Cauchy-Crofton formula, see Section 5.3. When A is diagonalizable, one can avoid this geometric-integral formula and work in a basis of diagonalization of A to estimate directly the growth of $\deg_k(\phi_A^n)$, see Section 5.1.

For dynamical applications it is often crucial to construct invariant cohomology classes with nice positivity properties. In Section 7, we explain how to construct such invariant classes for monomial maps satisfying the assumptions of Theorem D. Our Corollary 7.2 is an analog of [BFJ, Corollary 3.6] in the toric setting and in arbitrary dimensions. However these classes do not live in the cohomology groups of a particular toric variety, but rather on the inductive limit of all cohomology groups $\underrightarrow{\mathrm{H}}^k := \varinjlim H^{2k}(X(\Delta), \mathbb{R})$ over *all* toric models. This idea has already been fruitfully used in dynamics in [C, BFJ], and we propose a general framework to study the action of monomial maps on these spaces.

To this end, we rely on a beautiful interpretation of $\underrightarrow{\mathrm{H}}^*$ in terms of convex geometry due to Fulton-Sturmfels [FS], see also [B]. Namely, the classes in $\underrightarrow{\mathrm{H}}^*$ are in one-to-one correspondence with the classes in P. McMullen's *polytope algebra*. The polytope algebra Π is the $\mathbb{R}$-algebra generated by classes $[P]$ of polytopes with vertices in $\mathbb{Q}^d$, with relations $[P + v] = [P]$ for $v \in \mathbb{Q}^d$ and $[P \cup Q] + [P \cap Q] = [P] + [Q]$ whenever $P \cup Q$ is convex. It is endowed with multiplication $[P] \cdot [Q] := [P + Q]$, where $P + Q$ denotes the Minkowski sum. To any polytope P, we can attach the Chern character of its associated line bundle

L_P, and this defines a linear map $\mathrm{ch} : \Pi \to \underrightarrow{\mathrm{H}}^*$, which is, in fact, an isomorphism of algebras, [B, FS]. It holds that $\phi_A^* \mathrm{ch}[P] = \mathrm{ch}[A(P)]$.

The isomorphism $\mathrm{ch} : \Pi \to \underrightarrow{\mathrm{H}}^*$ extends by duality to an isomorphism between the space of linear forms on Π and $\mathcal{H}^*$. We will call elements in the former space *currents*. The invariant cohomology classes alluded to above correspond to a very special type of currents obtained by taking the volume of the projection of the polytope on suitable linear subspaces.

Although strictly not needed for the proofs, the formalism of currents on the polytope algebra is mainly motivated by our endeavour of constructing invariant classes for monomial maps. However, we note that the space of currents contains classical objects from convex geometry, such as valuations in the sense of [MS]. We think that it would be interesting to further explore this space; e.g. investigate positivity properties of currents and define (under reasonable geometric conditions) the intersection product of currents.

The paper is organized as follows. Sections 2 and 3 contain basics on toric varieties and the polytope algebra, respectively. In Section 4 we discuss dynamical degrees on toric varieties and, in particular, we derive (1.4). The proof(s) of Theorem A (and Corollary B) occupies Section 5, whereas Theorem D is proved in Section 6, and invariant classes of monomial maps are constructed in Section 7.

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2. Toric varieties

A toric variety X over $\mathbb{C}$ is a normal irreducible algebraic variety endowed with an action of the multiplicative torus $\mathbb{G}_m^d := (\mathbb{C}^*)^d$ which admits an open and dense orbit. This section contains the necessary material from toric geometry that will be needed for the proof of our results. Our basic references are [Fu, Od].

2.1. **Fans and toric varieties.** Let $N \simeq \mathbb{Z}^d$ be a lattice, i.e., a free abelian group, of rank d, denote by $M = \mathrm{Hom}(N, \mathbb{Z})$ its dual lattice, set $N_\mathbb{Q} := N \otimes_Z \mathbb{Q}$ and $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$, and analogously define $M_\mathbb{Q}$ and $M_\mathbb{R}$.

A *rational polyhedral strictly convex cone* $\sigma \subset N_\mathbb{R}$ is a closed convex cone generated by finitely many vectors lying in N, and such that $\sigma \cap -\sigma = \{0\}$. Its dual cone $\check{\sigma} := \{m \in M_\mathbb{R},\, u(m) \geq 0 \text{ for all } u \in \sigma\}$ is a finitely generated semi-group. Thus σ defines an affine variety $U_\sigma := \mathrm{Spec}\, \mathbb{C}[\check{\sigma} \cap M]$. The torus $\mathbb{G}_m^d = \mathrm{Spec}\, \mathbb{C}[M]$ is contained as a dense orbit in U_σ and the action by $\mathbb{G}_m^d$ on itself extends to U_σ, which makes U_σ a toric variety. Conversely, any affine toric variety can be obtained in this way.

If σ is *simplicial*, i.e., it is generated by vectors linearly independent over $\mathbb{R}$, then U_σ has at worst quotient singularities. The toric variety U_σ is smooth if and only if σ is simplicial and generated by d vectors $e_1, \ldots, e_d$ forming a basis of N as an abelian group; such a σ is said to be *regular*.

A *fan* Δ is a finite collection of rational polyhedral strictly convex cones in $N_\mathbb{R}$ such that each face of a cone in Δ belongs to Δ and the intersection of two cones in Δ is a face of both of them. A fan Δ determines a toric variety $X(\Delta)$, obtained by patching together the affine toric varieties $\{U_\sigma\}_{\sigma\in\Delta}$ along their intersections in a natural way. If all cones in Δ are simplicial then Δ is said to be *simplicial* and if all cones are regular, then Δ is said to be *regular*; $X(\Delta)$ is smooth if and only if Δ is regular. If $\bigcup_{\sigma\in\Delta}\sigma = N_{\mathbb{R}}$, then Δ is said to be *complete*. The toric variety $X(\Delta)$ is compact if and only if Δ is complete. Unless otherwise stated, we will assume that all fans in this paper are complete.

There is an one-to-one correspondence between cones of Δ of dimension k and orbits of the action of $\mathbb{G}_m^d$ on $X(\Delta)$ of codimension k. We denote the closure of the orbit associated with a cone $\sigma \in \Delta$ in $X(\Delta)$ by $X(\sigma)$. In particular, 1-dimensional cones correspond to (irreducible) $\mathbb{G}_m^d$-invariant divisors.

A fan Δ' *refines* another fan Δ if each cone in Δ' is included in a cone in Δ.

2.2. **Equivariant (holomorphic) maps.** Given a group morphism $A : M \to M$, we will write A also for the induced linear maps $M_{\mathbb{Q}} \to M_{\mathbb{Q}}$ and $M_{\mathbb{R}} \to M_{\mathbb{R}}$. Morever, we let $\check{A}$ denote the dual map $N \to N$, as well as the dual linear maps $N_{\mathbb{Q}} \to N_{\mathbb{Q}}$ and $N_{\mathbb{R}} \to N_{\mathbb{R}}$. It turns out to be convenient to use this notation rather than writing A for the map on N and $\check{A}$ for the map on M.

A *map of fans* $\check{A} : (N, \Delta_2) \to (N, \Delta_1)$ is a linear map $\check{A} : N_{\mathbb{R}} \to N_{\mathbb{R}}$ that preserves N and satisfies that the fan $\check{A}(\Delta_2) := \{\check{A}(\sigma) : \sigma \in \Delta_2\}$ refines Δ_1. If $\sigma_1 \in \Delta_1$ and $\sigma_2 \in \Delta_2$ satisfy that $\check{A}(\sigma_2) \subseteq \sigma_1$, then the dual map $A : M \to M$ maps $\check{\sigma}_1$ to $\check{\sigma}_2$ and induces a map $A : \mathbb{C}[\check{\sigma}_1 \cap M] \to \mathbb{C}[\check{\sigma}_2 \cap M]$, which, in turn, induces a map $\phi_A : U_{\sigma_2} \to U_{\sigma_1}$. These maps can be patched together to a holomorphic map $\phi_A : X(\Delta_2) \to X(\Delta_1)$ which is *equivariant* in the following sense: Denote by $\rho_A : \mathbb{G}_m^d \to \mathbb{G}_m^d$ the natural group morphism induced by the ring morphism $A : \mathbb{C}[M] \to \mathbb{C}[M]$. Then for any $x \in X(\Delta_2)$, and any $g \in \mathbb{G}_m^d$, one has $\phi_A(g \cdot x) = \rho_A(g) \cdot \phi_A(x)$. Conversely any equivariant holomorphic map $X(\Delta_2) \to X(\Delta_1)$ is determined by a map of fans $\check{A} : (N, \Delta_2) \to (N, \Delta_1)$.

The map ϕ_A is dominant if and only $\det(A) \neq 0$ and the topological degree of ϕ_A equals $|\det(A)|$.

2.3. **Universal cohomology of toric varieties.** Let Δ be a complete simplicial fan. Then $X(\Delta)$ has at worst quotient singularities, its cohomology groups $H^j(X(\Delta)) := H^j(X(\Delta), \mathbb{R})$ with values in $\mathbb{R}$ vanish whenever j is odd, and $H^*(X(\Delta))$ is generated as an algebra by the $\mathbb{G}_m^d$-invariant divisors $[X(\sigma)]$, where σ runs over the 1-dimensional cones in Δ.

We let $\mathfrak{D}$ denote the set of all complete simplicial fans in N and endow it with a partial ordering by imposing $\Delta \prec \Delta'$ if (and only if) Δ' refines Δ. For any two fans $\Delta, \Delta' \in \mathfrak{D}$, one can find a third fan Δ'' refining both; hence $\mathfrak{D}$ is a directed set. Assume $\Delta \prec \Delta'$. Then the identity map on N induces a map of fans $\check{\mathrm{id}}_{\Delta',\Delta} : (N, \Delta') \to (N, \Delta)$, and thus yields a natural birational morphism $\pi := \phi_{\mathrm{id}_{\Delta',\Delta}} : X(\Delta') \to X(\Delta)$. This map induces linear actions on cohomology, $\pi^* : H^*(X(\Delta)) \to H^*(X(\Delta'))$ and $\pi_* : H^*(X(\Delta')) \to H^*(X(\Delta))$ that satisfy $\pi_*\pi^* = \mathrm{id}$; in particular, the map π_* is surjective and π^* is injective.

The pushforward π_* and pullback π^* arrows make $\mathfrak{D}$ into an inverse and directed set, respectively, and so the limits

$$\mathcal{H}^* := \varprojlim_{\mathfrak{D}} H^*(X(\Delta)) \text{ and } \underrightarrow{\mathrm{H}}^* := \varinjlim_{\mathfrak{D}} H^*(X(\Delta))$$

are well-defined infinite dimensional graded real vector spaces. We will refer to $\mathcal{H}^*$ and $\underrightarrow{\mathrm{H}}^*$ as the *universal (inverse* respectively, *direct) cohomology* of toric varieties.

In concrete terms, an element $\omega \in \mathcal{H}^*$ is a collection of *incarnations* $\omega_\Delta \in H^*(X(\Delta))$ for each $\Delta \in \mathfrak{D}$, such that $\pi_*(\omega_{\Delta'}) = \omega_\Delta$ if $\Delta \prec \Delta'$ and $\pi = \phi_{\mathrm{id}_{\Delta',\Delta}}$. An element

$\omega \in \underset{\rightarrow}{\mathrm{H}}^*$ is determined by some class $\omega_\Delta \in H^*(X(\Delta))$, and two classes $\omega_{\Delta_i} \in H^*(X(\Delta_i))$, $i = 1, 2$ determine the same class in $\underset{\rightarrow}{\mathrm{H}}^*$ if and only if there exists a common refinement $\Delta' \succ \Delta_i$ such that $\pi_1^*(\omega_{\Delta_1}) = \pi_2^*(\omega_{\Delta_2})$ if $\pi_i = \phi_{\mathrm{id}_{\Delta',\Delta_i}}$. Note that the map that sends $\omega_\Delta \in H^*(X(\Delta))$ to the class it determines in $\underset{\rightarrow}{\mathrm{H}}^*$ is injective.

We endow $\mathcal{H}^*$ with its projective limit topology so that $\omega_j \to \omega$ if and only if $\omega_{j,\Delta} \to \omega_\Delta$ for each fan $\Delta \in \mathfrak{D}$. Then $\underset{\rightarrow}{\mathrm{H}}^*$ is dense in $\mathcal{H}^*$.

Each cohomology space $H^*(X(\Delta))$ has a ring structure coming from the intersection product which respects the grading such that $\omega_\Delta \cdot \eta_\Delta \in H^{2(i+j)}(X(\Delta))$ if $\omega_\Delta \in H^{2i}(X(\Delta))$ and $\eta_\Delta \in H^{2j}(X(\Delta))$. Given classes ω and η in $\underset{\rightarrow}{\mathrm{H}}^*$, pick $\Delta \in \mathfrak{D}$ such that they are determined by ω_Δ and η_Δ, respectively, and let $\omega \cdot \eta$ be the class in $\underset{\rightarrow}{\mathrm{H}}^*$ determined by $\omega_\Delta \cdot \eta_\Delta$. It is not difficult to check that this definition of $\omega \cdot \eta$ is independent of the choice of Δ. Hence, in this way, $\underset{\rightarrow}{\mathrm{H}}^*$ is endowed with a natural structure of a graded $\mathbb{R}$-algebra.

More generally, given $\omega \in \mathcal{H}^*$ and $\eta \in \underset{\rightarrow}{\mathrm{H}}^*$, pick Δ such that η is determined by η_Δ and let $\omega \cdot \eta$ be the class in $\mathcal{H}^*$ determined by $\omega_\Delta \cdot \eta_\Delta$. Again, this product is well-defined and independent of the choice of Δ and so $\mathcal{H}^*$ is a $\underset{\rightarrow}{\mathrm{H}}^*$-module. Note, however, that it is not possible to define a ring structure on $\mathcal{H}^*$ that continuously extends the one on $\underset{\rightarrow}{\mathrm{H}}^*$.

Since the intersection product $H^{2j}(X(\Delta)) \times H^{2(d-j)}(X(\Delta)) \to \mathbb{R}$ is a perfect pairing for each $\Delta \in \mathfrak{D}$ by Poincaré duality, the pairing $\mathcal{H}^{2j} \times \underset{\rightarrow}{\mathrm{H}}^{2(d-j)} \to \mathbb{R}$ is also perfect and thus $\mathcal{H}^*$ and $\underset{\rightarrow}{\mathrm{H}}^*$ are naturally dual one to the other.

2.4. **Toric line bundles.** The Picard group $\operatorname{Pic} X(\Delta)$ of a toric variety $X(\Delta)$ is generated by classes of $\mathbb{G}_m^d$-invariant Cartier divisors. These divisors can in turn be described in terms of functions on $N_\mathbb{R}$ as follows. Let $\mathrm{PL}(\Delta)$ be the set of all continuous real-valued functions h on $|\Delta| \subset N_\mathbb{R}$ that are *piecewise linear with respect to* Δ, i.e., such that for any cone $\sigma \in \Delta$ there exists $m = m(\sigma) \in M$ with $h|_\sigma = m$. Given any 1-dimensional cone σ of Δ, the associated *primitive vector* is the first lattice point u_i met along $\sigma = \mathbb{R}_{\geq 0} u_i$. Let $\Delta(1) = \{u_i\} \subset N$ be the set of primitive vectors of 1-dimensional cones in Δ. With $h \in \mathrm{PL}(\Delta)$, we associate the Cartier divisor $D(h) := \sum h(u_i)\, X(\mathbb{R}_{\geq 0} u_i)$. The map sending $h \in \mathrm{PL}(\Delta)$ to $\mathcal{O}(-D(h)) \in \operatorname{Pic}(X(\Delta))$ is surjective and the kernel is the space of linear functions $M \subset \mathrm{PL}(\Delta)$. By taking the first Chern class, we get a linear map:

$$\Theta_1 : \mathrm{PL}(\Delta) \to H^2\left(X(\Delta)\right),\ h \mapsto \Theta_1(h) = c_1(\mathcal{O}(-D(h)))\ . \tag{2.1}$$

When $X(\Delta)$ is smooth, the kernel of Θ_1 is M and the image is precisely $H^2(X(\Delta), \mathbb{Z})$. Note that Θ_1 extends by linearity to $\mathrm{PL}_\mathbb{Q}(\Delta) := \mathrm{PL}(\Delta) \otimes_\mathbb{Z} \mathbb{Q}$, corresponding to $\mathbb{Q}$-line bundles, with image $H^2(X(\Delta), \mathbb{Q})$.

Let $\check{A} : (N, \Delta_2) \to (N, \Delta_1)$ be a map of fans, inducing a holomorphic map $\phi_A : X(\Delta_2) \to X(\Delta_1)$, and pick $h \in \mathrm{PL}(\Delta_1)$. Then the pullback $\phi_A^* D(h)$ is a well-defined Cartier divisor on $X(\Delta_2)$, equal to $D(h \circ \check{A})$. It follows that

$$\phi_A^* \Theta_1(h) = \Theta_1(h \circ \check{A}). \tag{2.2}$$

There is a link between positivity properties of the classes in $H^2(X(\Delta))$ and convex geometry. A function $h \in \mathrm{PL}(\Delta)$ is said to be *strictly convex (with respect to* Δ*)* if it is convex and defined by different elements $h|_\sigma \in M$ for different d-dimensional cones $\sigma \in \Delta$. Recall that on a complete algebraic variety, a Cartier divisor D is nef if $D \cdot C \geq 0$ for any curve C. The line bundle $\mathcal{O}(-D(h))$ over $X(\Delta)$ is nef if (and only if) $h \in \mathrm{PL}(\Delta)$ is convex and it is ample if (and only if) h is strictly convex.

A function h in $\mathrm{PL}(\Delta)$ determines a (non-empty) polyhedron

$$P(h) := \{m \in M_{\mathbb{R}},\, m \le h\} \subset M_{\mathbb{R}}\ .$$

If h is strictly convex with respect to some fan, then $P(h)$ is a compact *lattice polytope* in $M_{\mathbb{R}}$, i.e., it is the convex hull of finitely many points in the lattice M, and it has non-empty interior. Conversely, if $P \subset M_{\mathbb{R}}$ is a lattice polytope, then the function

$$h_P(u) := \sup\{m(u),\, m \in P\} \tag{2.3}$$

is a piecewise linear convex function on $N_{\mathbb{R}}$. If Δ_P denotes the *normal fan* of P (see [Fu, Section 1.5] for a definition) then $h_P \in \mathrm{PL}(\Delta)$ precisely if Δ refines Δ_P and it is strictly convex with respect to Δ_P. If $h_P \in \mathrm{PL}(\Delta)$ then Δ is said to be *compatible* with P.

If $\check{A} : (N, \Delta_2) \to (N, \Delta_1)$ is a map of fans, note that $P(h \circ \check{A}) = AP(h)$. If Δ is compatible with P and Q, then the intersection product is

$$\mathcal{O}(-D(h_P))^k \cdot \mathcal{O}(-D(h_Q))^{d-k} = d!\,\mathrm{Vol}(P[k], Q[d-k]), \tag{2.4}$$

where Vol is the mixed volume as defined in Section 3.2, see [Od, p. 79].

Example 2.1. Take a basis $e_1, \dots, e_d$ of N with dual basis $e_1^*, \dots, e_d^*$, and set $e_0 := -\sum_1^d e_i$. Let Δ be the unique fan whose d-dimensional cones are the $d+1$ cones $\sigma_i = \sum_{j \ne i} \mathbb{R}_{\ge 0} e_j$. Then $X(\Delta)$ is isomorphic to the projective space $\mathbb{P}^d$. Let h be the unique function in $\mathrm{PL}(\Delta)$ that satisfies $h(e_0) = 1$ and $h(e_i) = 0$ if $i \ge 1$; note that h is strictly convex with respect to Δ. Then $\mathcal{O}(-D(h)) = \mathcal{O}_{\mathbb{P}^d}(1)$ and moreover the polytope $P_{D(h)}$ is the standard simplex

$$\Sigma_d := \{u = \sum_1^d s_i e_i^*,\ s_i \le 0,\ \sum_1^d s_i \ge -1\} \subset M_{\mathbb{R}}\ .$$

2.5. **Piecewise polynomial functions.** Higher cohomology classes of toric varieties in $\mathfrak{D}$ can be encoded in terms of (piecewise) polynomial functions on $N_{\mathbb{R}}$.

Given a fan Δ, let $\mathrm{PP}(k, \Delta)$ be the set of *piecewise polynomial functions (with respect to Δ) of degree* k, i.e, continuous functions $h : N_{\mathbb{R}} \to \mathbb{R}$ such that for each cone $\sigma \in \Delta$, the restriction $h|_\sigma = \sum m_{i_1} \otimes \cdots \otimes m_{i_k}$, $m_{i_j} \in M$, is a homogeneous polynomial of degree k. Note that $\mathrm{PP}(1, \Delta) = \mathrm{PL}(\Delta)$. Moreover, note that $h \otimes h' \in \mathrm{PP}(k + k', \Delta)$ if $h \in \mathrm{PP}(k, \Delta)$ and $h' \in \mathrm{PP}(k', \Delta)$, so that $\mathrm{PP}(\Delta) := \oplus_k \mathrm{PP}(k, \Delta)$ is a graded ring.

From now on, assume $\Delta \in \mathfrak{D}$, and let σ be a j-dimensional cone in Δ. Since σ is simplicial it is generated by exactly j primitive vectors in $\Delta(1) = \{u_1, \dots, u_N\}$, say $u_1, \dots, u_j$, and so $x \in \sigma = \sum_{i=1}^j \mathbb{R}_{\ge 0} u_i$ admits a unique representation $x = \sum_{i=1}^j x_i u_i$ with $x_i \ge 0$. It follows that $h|_\sigma$ has a unique expansion $h|_\sigma(x) = \sum a_I x_{i_1} \cdots x_{i_k}$, where the sum ranges over all $I = \{i_1, \dots, i_k\} \subset \{1, \dots, j\}$.

We can now define a linear map $\Theta_k : \mathrm{PP}(k, \Delta) \to H^{2k}(X(\Delta))$ by $\Theta_k(h) := \sum a_I X_I$, where the sum ranges over all $I = \{i_1, \dots, i_k\} \subset \{1, \dots, N\}$, X_I is (the class of) the intersection product $X(\mathbb{R}_{\ge 0} u_{i_1}) \cdot \ldots \cdot X(\mathbb{R}_{\ge 0} u_{i_k})$, and a_I is the coefficent of $x_{i_1} \cdots x_{i_k}$ in $h|_\sigma$ defined above if $\sigma = \sum \mathbb{R}_{\ge 0} u_{i_\ell}$ is a cone in Δ and $a_I = 0$ otherwise.

By patching the maps Θ_k together, we obtain a graded map $\Theta : \mathrm{PP}(\Delta) \to H^*(X(\Delta)), h \mapsto \sum_k \Theta_k(h_k)$, where h_k is the k-th graded piece of h. Note that Θ is also a ring morphism since $\Theta(h\,h') = \Theta(h)\,\Theta(h')$ for any $h, h' \in \mathrm{PP}(\Delta)$. If $X(\Delta)$ is smooth the image of Θ is $H^*(X(\Delta),\mathbb{Z})$. As for Θ_1 we can extend Θ to $\mathrm{PP}_{\mathbb{Q}}(\Delta) := \mathrm{PP}(\Delta)\otimes_{\mathbb{Z}}\mathbb{Q}$, with image $H^*(X(\Delta),\mathbb{Q})$.

2.6. **Equivariant rational morphisms.** Let $A : M_{\mathbb{R}} \to M_{\mathbb{R}}$ be a linear map that preserves M. Take $\Delta, \Delta' \in \mathfrak{D}$ such that Δ' refines Δ and $\check{A}^{-1}(\sigma)$ is a union of cones in Δ' for each $\sigma \in \Delta$. Then $\check{A} : (N,\Delta') \to (N,\Delta)$ is a map of fans, inducing a holomorphic equivariant map $f_A : X(\Delta') \to X(\Delta)$. Let $\pi : X(\Delta') \to X(\Delta)$ be the equivariant birational map induced by $\check{\mathrm{id}}_{\Delta',\Delta} : (N,\Delta') \to (N,\Delta)$, and let $\phi_A := f_A \circ \pi^{-1}$. Then $\phi_A : X(\Delta) \dashrightarrow X(\Delta)$ is a rational map that is equivariant under the action of $\mathbb{G}_m^d$. Conversely, any equivariant rational self-map on $X(\Delta)$ arises in this way. The map ϕ_A is holomorphic precisely if $\Delta \prec \check{A}(\Delta)$ and it is dominant precisely if $\det(A) \neq 0$.

Let $e_1,\dots,e_d$ be a basis of M and let $e_1^*,\dots,e_d^*$ be the corresponding basis of N. Then $A = \sum a_{ij} e_i \otimes e_j^*$, for some $a_{ij} \in \mathbb{Z}$. If $x_1,\dots,x_d$ are the induced coordinates on $\mathbb{G}_m^d$, then ϕ_A restricted to $\mathbb{G}_m^d$ is the (holomorphic) monomial map $\phi_A(x_1,\dots,x_d) = (\prod x_j^{a_{j1}},\dots,\prod x_j^{a_{jd}})$.

Recall that a dominant holomorphic map $\phi : X' \to X$ induces linear actions on cohomology $\phi^* : H^*(X) \to H^*(X')$ and $\phi_* : H^*(X') \to H^*(X)$. Assume that ϕ_A is dominant. Then we define the *pushforward* $(\phi_A)_\bullet : H^*(X(\Delta)) \to H^*(X(\Delta))$ as the composition $(\phi_A)_\bullet := (f_A)_* \circ \pi^*$, and the *pullback* $\phi_A^\bullet : H^*(X(\Delta)) \to H^*(X(\Delta)$ as $\phi_A^\bullet := \pi_* \circ f_A^*$. It is readily verified that $(\phi_A)_\bullet$ and $\phi_A^\bullet$ do not depend on the choice of Δ'. We insist on writing $(\phi_A)_\bullet$, $\phi_A^\bullet$ instead of $(\phi_A)_*$, ϕ_A^* since one does not have good functorial properties, e.g. $(\phi_B \circ \phi_A)_\bullet \neq (\phi_B)_\bullet \circ (\phi_A)_\bullet$ in general.

The linear map A also induces natural linear actions $\phi_A^* : \underrightarrow{\mathrm{H}}^* \to \underrightarrow{\mathrm{H}}^*$ and $(\phi_A)_* : \mathcal{H}^* \to \mathcal{H}^*$, defined as follows. Suppose η is a class in $\underrightarrow{\mathrm{H}}^*$ determined by $\eta_\Delta \in H^*(X(\Delta))$. Pick $\mathfrak{D} \ni \Delta' \succ \Delta$ such that the map $f_A : X(\Delta') \to X(\Delta)$ induced by $\check{A}$ is holomorphic, and define $\phi_A^*\eta$ to be the class in $\underrightarrow{\mathrm{H}}^*$ determined by $f_A^*\eta_\Delta \in H^*(X(\Delta'))$. Next, suppose $\omega \in \mathcal{H}^*$. The incarnation of $(\phi_A)_*\omega$ in $H^*(X(\Delta))$ for a given $\Delta \in \mathfrak{D}$ is defined as $(\phi_A)_*\omega_\Delta := (f_A)_*\omega_{\Delta'}$, where Δ' is choosen as above. It is not hard to check that ϕ_A^* and $(\phi_A)_*$ are independent of the choice of refinement Δ', and moreover that $(\phi_A)_*$ is continuous on $\mathcal{H}^*$, $\phi_{B\circ A}^* = \phi_A^* \circ \phi_B^*$, $(\phi_{B\circ A})_* = (\phi_B)_* \circ (\phi_A)_*$, $(\phi_A)_* \circ (\phi_A)^* = |\det(A)|$, and $(\phi_A)_*\omega \cdot \eta = \omega \cdot (\phi_A)^*\eta$ for any two classes $\omega \in \mathcal{H}^{2j}$, $\eta \in \underrightarrow{\mathrm{H}}^{2(d-j)}$.

Given $h \in \mathrm{PL}(\Delta)$, the first Chern class $\Theta_1(h) \in H^2(X(\Delta))$ determines a class in $\underrightarrow{\mathrm{H}}^*$, also denoted by $\Theta_1(h)$, that satisfies

$$\phi_A^*(\Theta_1(h)) = \Theta_1(h \circ \check{A}) \text{ in } \underrightarrow{\mathrm{H}}^* , \tag{2.5}$$

which follows in light of (2.2).

3. The polytope algebra

3.1. **Definition.** Given any finite collection of convex sets $K_1,\dots,K_s \subset M_{\mathbb{R}}$, we let $K_1+\cdots+K_s$ denote the *Minkowski sum* $K_1+\cdots+K_s := \{x_1+\cdots+x_s \mid x_j \in K_j\}$. For any $r \in \mathbb{R}_{\geq 0}$, we also write $rK_j := \{rx \mid x \in K_j\}$. A *polytope in* $M_{\mathbb{Q}}$ is the convex hull of finitely many points in $M_{\mathbb{Q}}$.

We now introduce the *polytope algebra* $\Pi = \Pi(M_{\mathbb{R}})$ which is a variant of the original construction of P. McMullen [M]. It is the $\mathbb{R}$-algebra with a generator $[P]$ for each polytope

P in $M_{\mathbb{Q}}$, with $[\emptyset] =: 0$. The generators satisfy the relations $[P \cup Q] + [P \cap Q] = [P] + [Q]$ whenever $P \cup Q$ is convex, and $[P+t] = [P]$ for any $t \in M_{\mathbb{Q}}$. The multiplication in Π is given by $[P] \cdot [Q] := [P+Q]$, with multiplicative unit $1 := [\{0\}]$. The polytope algebra admits a grading $\Pi = \bigoplus_{k=0}^{d} \Pi_k$ such that $\Pi_k \cdot \Pi_l \subset \Pi_{k+l}$. The k-th graded piece Π_k is the $\mathbb{R}$-vector space spanned by all elements of the form $(\log[P])^k$, where $\log[P] := \sum_{r=1}^{d} \frac{(-1)^{r+1}}{r}([P]-1)^r$ and P runs over all polytopes in $M_{\mathbb{Q}}$. The top-degree part Π_d is one-dimensional, and multiplication gives non-degenerate pairings $\Pi_j \times \Pi_{d-j} \to \Pi_d$. Given $\alpha \in \Pi$, we will write α_k for its homogeneous part of degree k.

The lattice M determines a (canonical) volume element on $M_{\mathbb{R}}$, which we denote by Vol. It is normalized by the convention $\mathrm{Vol}(P) = 1$ for any parallelogram $P = \{\sum s_i e_i^*,\ 0 \le s_i \le 1\}$ such that $e_1^*, \ldots, e_n^*$ is a basis of the lattice M. In particular, the volume of the standard simplex $\mathrm{Vol}(\Sigma_d)$ is $1/d!$. There is a canonical linear map $\mathrm{Vol} : \Pi \to \mathbb{R}$ defined by $\mathrm{Vol}([P]) = \mathrm{Vol}(P)$. This map is zero on all pieces Π_k for $k \le d-1$, and it induces an isomorphism $\mathrm{Vol} : \Pi_d \xrightarrow{\simeq} \mathbb{R}$.

Let $A : M_{\mathbb{Q}} \to M_{\mathbb{Q}}$ be a linear map. Then A induces a linear map $\Pi \to \Pi$, defined by $[P] \mapsto [A(P)]$; we shall denote it by $A_* : \Pi \to \Pi$. Note that A_* is actually a ring homomorphism since $A_*([P] \cdot [Q]) = [A(P+Q)] = A_*[P] \cdot A_*[Q]$ for polytopes P and Q in $M_{\mathbb{Q}}$, and A_* preserves the grading on Π since $A_*(\log[P]) = \log[A(P)]$. Also, it is clear that $(A \circ B)_* = A_* \circ B_*$ for any two linear maps A and B.

An important example is given by the homothety $A = r \times \mathrm{id}$, $r \in \mathbb{Q}_{\ge 0}$; we denote the corresponding map on Π by $\mathcal{D}(r)$. Note that if P is a polytope and $r \in \mathbb{Z}$, then $\mathcal{D}(r)[P] = [P]^r$. It is proved in [M, Lemma 20] that

$$\alpha \in \Pi_k \text{ if and only if } \mathcal{D}(r)\alpha = r^k \alpha \ , \text{ for any } r \in \mathbb{Q}_{\ge 0} \ . \tag{3.1}$$

If $\det(A) \ne 0$, there is a well-defined *pullback* map $A^* : \Pi \to \Pi$ by $A^* := \mathcal{D}\left(|\det(A)|^{1/k}\right) \circ (A^{-1})_*$ on Π_k; in particular,

$$A^*[P] = |\det(A)| [A^{-1}(P)]$$

for any polytope P. Moreover $(A \circ B)^* = B^* \circ A^*$ for any two linear maps A and B with non-zero determinant. Beware that A^* is not a ring homomorphism on Π. On the other hand, $A^*(A_*(\alpha)) = |\det(A)|\, \alpha$ for any $\alpha \in \Pi$.

3.2. **Mixed volumes.** Let $K_1, \ldots, K_s \subset M_{\mathbb{R}}$ be convex compact sets and pick $r_1, \ldots, r_s \in \mathbb{R}_{\ge 0}$. A theorem by Minkowski and Steiner asserts that $\mathrm{Vol}(r_1 K_1 + \cdots + r_s K_s)$ is a homogeneous polynomial of degree d in the variables $r_1, \ldots, r_s$. In particular, there is a unique expansion:

$$\mathrm{Vol}\left(r_1 K_1 + \cdots + r_s K_s\right) = \sum_{k_1 + \cdots + k_s = d} \binom{d}{k_1, \ldots, k_s} \mathrm{Vol}\left(K_1[k_1], \ldots, K_s[k_s]\right) r_1^{k_1} \cdots r_s^{k_s}, \tag{3.2}$$

the coefficients $\mathrm{Vol}(K_1[k_1], \ldots, K_s[k_s]) \in \mathbb{R}$ are called *mixed volumes*. Here the notation $K_j[k_j]$ refers to the repetition of K_j k_j times. It is a fact that $\mathrm{Vol}(K_1[k_1], \ldots, K_s[k_s])$ is non-negative, multilinear symmetric in the variables K_j, and increasing in each variable, meaning that

$$\mathrm{Vol}\left(K_1[k_1], K_2[k_2], \ldots, K_s[k_s]\right) \le \mathrm{Vol}\left(K_1'[k_1], K_2[k_2], \ldots, K_s[k_s]\right) \text{ whenever } K_1 \subseteq K_1'. \tag{3.3}$$

Note that $\mathrm{Vol}(K_1[d], K_2[0], \ldots, K_s[0]) = \mathrm{Vol}(K_1)$. There is in general no simple geometric description of mixed volumes, unless the K_j has some symmetries, cf. Section 5.1 and (5.9) below.

Since $\mathrm{Vol}(K)$ is invariant under translation of K,

$$\mathrm{Vol}\left((K_1+t)[k_1], K_2[k_2], \ldots, K_s[k_s]\right) = \mathrm{Vol}\left(K_1[k_1], K_2[k_2], \ldots, K_s[k_s]\right) \ , \tag{3.4}$$

for any $t \in M_{\mathbb{R}}$. Moreover, $\mathrm{Vol}(K_1[k_1], \ldots, K_s[k_s]) \in \mathbb{R}$ is additive in the sense that

$$\mathrm{Vol}\left((K_1 \cup K_1')[k_1], K_2[k_2], \ldots, K_s[k_s]\right) + \mathrm{Vol}\left((K_1 \cap K_1')[k_1], K_2[k_2], \ldots, K_s[k_s]\right) = \mathrm{Vol}\left(K_1[k_1], K_2[k_2], \ldots, K_s[k_s]\right) + \mathrm{Vol}\left(K_1'[k_1], K_2[k_2], \ldots, K_s[k_s]\right);$$

as soon as $K_1 \cup K_1'$ is convex. It follows that the mixed volumes extend to the polytope algebra Π as multilinear functionals:

$$\Pi^s \ni (\alpha_1, \ldots, \alpha_s) \mapsto \mathrm{Vol}\left(\alpha_1[k_1], \ldots, \alpha_s[k_s]\right) \in \mathbb{R} \ ,$$

so that, in particular, $\mathrm{Vol}\left([P_1][k_1], \ldots, [P_s][k_s]\right) = \mathrm{Vol}\left(P_1[k_1], \ldots, P_s[k_s]\right)$. Equation (3.2) translates into

$$\mathrm{Vol}\left(\mathcal{D}(r_1)\alpha_1 \cdot \ldots \cdot \mathcal{D}(r_s)\alpha_s\right) = \sum_{k_1+\ldots+k_s=d} \binom{d}{k_1, \ldots, k_s} \mathrm{Vol}\left(\alpha_1[k_1], \ldots, \alpha_s[k_s]\right) r_1^{k_1} \cdots r_s^{k_s} \ , \tag{3.5}$$

which holds for $r_1, \ldots, r_s \in \mathbb{Q}_{\geq 0}$. Note that (3.5) implies the following homogeneity:

$$\mathrm{Vol}\left(\mathcal{D}(r_1)\alpha_1[k_1], \ldots, \mathcal{D}(r_s)\alpha_s[k_s]\right) = \mathrm{Vol}\left(\alpha_1[k_1], \ldots, \alpha_s[k_s]\right) r_1^{k_1} \cdots r_s^{k_s} \ . \tag{3.6}$$

Lemma 3.1. *Let $\alpha_1, \ldots, \alpha_s$ be homogeneous elements in the polytope algebra of degrees $\ell_1, \ldots, \ell_s$, respectively. Then* $\mathrm{Vol}\left(\alpha_1[k_1], \ldots, \alpha_s[k_s]\right) = 0$ *unless $\ell_j = k_j$ for all j, in which case it is equal to* $\binom{d}{\ell_1, \ldots, \ell_s}^{-1} \mathrm{Vol}\left(\alpha_1 \cdot \ldots \cdot \alpha_s\right)$.

Proof. By (3.1), and the linearity of $\mathrm{Vol} : \Pi \to \mathbb{R}$,

$$\mathrm{Vol}\left(\mathcal{D}(r_1)\alpha_1 \cdot \ldots \cdot \mathcal{D}(r_s)\alpha_s\right) = \mathrm{Vol}\left(r_1^{\ell_1}\alpha_1 \cdot \ldots \cdot r_s^{\ell_s}\alpha_s\right) = \mathrm{Vol}\left(\alpha_1 \cdot \ldots \cdot \alpha_s\right) r_1^{\ell_1} \cdots r_s^{\ell_s};$$

in particular, the only non-vanishing mixed volume is $\mathrm{Vol}\left(\alpha_1[\ell_1], \ldots, \alpha_s[\ell_s]\right) = \binom{d}{\ell_1, \ldots, \ell_s}^{-1} \mathrm{Vol}\left(\alpha_1 \cdot \ldots \cdot \alpha_s\right)$. □

Lemma 3.1 implies that if $\alpha_1, \ldots, \alpha_s \in \Pi$, and $\alpha_{j,\ell}$ denotes the ℓ-th graded part of α_j, then

$$\mathrm{Vol}\left(\alpha_1[k_1], \ldots, \alpha_s[k_s]\right) = \binom{d}{k_1, \ldots, k_s}^{-1} \mathrm{Vol}\left(\alpha_{1,k_1} \cdot \ldots \cdot \alpha_{s,k_s}\right) \tag{3.7}$$

Lemma 3.2. *Let $A : M_{\mathbb{Q}} \to M_{\mathbb{Q}}$ be a linear map such that* $\det(A) \neq 0$. *Then*

$$\mathrm{Vol}\left(A^*\alpha_1[k], \alpha_2[d-k]\right) = \mathrm{Vol}\left(\alpha_1[k], A_*\alpha_2[d-k]\right),$$

for any two elements $\alpha_1, \alpha_2 \in \Pi$.

Proof. By multlinearity, we may assume that $\alpha_i = [P_i]$ for some polytopes P_i. Note that for $r_i \in \mathbb{Q}_{\geq 0}$,

$$\mathrm{Vol}\left(r_1 A^{-1}(P_1) + r_2 P_2\right) = \sum_k \binom{d}{k} \mathrm{Vol}\left(A^{-1}(P_1)[k], P_2[d-k]\right) r_1^k r_2^{d-k},$$

and

$$|\det(A)|\ \mathrm{Vol}\left(r_1 A^{-1}(P_1) + r_2 P_2\right) = \mathrm{Vol}\left(A(r_1 A^{-1}(P_1) + r_2 P_2)\right) =$$
$$\mathrm{Vol}\left(r_1 P_1 + r_2 A(P_2)\right) = \sum_k \binom{d}{k} \mathrm{Vol}\left(P_1[k], A(P_2)[d-k]\right)\ r_1^k r_2^{d-k}.$$

By identification of the coefficients of $r_1^k r_2^{d-k}$ we get

$$|\det(A)|\ \mathrm{Vol}\left(A^{-1}P_1[k], P_2[d-k]\right) = \mathrm{Vol}\left(P_1[k], A(P_2)[d-k]\right). \tag{3.8}$$

The right hand side of (3.8) is precisely $\mathrm{Vol}\left([P_1][k], A_*[P_2][d-k]\right)$, and in light of Lemma 3.1, the left hand side is equal to

$$|\det(A)|\ \mathrm{Vol}\left([A^{-1}(P_1)]_k[k], P_2[d-k]\right) = \mathrm{Vol}\left[\mathcal{D}\left(|\det(A)|^{1/k}\right)[A^{-1}(P_1)]_k[k], P_2[d-k]\right] =$$
$$\mathrm{Vol}\left(A^*[P_1]_k[k], P_2[d-k]\right) = \mathrm{Vol}\left(A^*[P_1][k], P_2[d-k]\right),$$

which concludes the proof. Here we have used (3.6), the definition of A^*, and Lemma 3.1 for the first, second, and last equalities, respectively. □

3.3. **Currents on the polytope algebra.** In order to simplify computations and relate the polytope algebra to the universal cohomology of toric varieties it is convenient to introduce the following terminology. A *current* is a linear form on the polytope algebra. We denote the space of currents on Π by $\mathcal{C}$ and endow it with the topology of pointwise convergence. Moreover, we write $\langle T, \beta\rangle \in \mathbb{R}$ for the action of $T \in \mathcal{C}$ on $\beta \in \Pi$.

A current $T \in \mathcal{C}$ is said to be of *degree k* if $T|_{\Pi_j} = 0$ for $j \neq d-k$. Let $\mathcal{C}_k$ denote the subspace of $\mathcal{C}$ of currents of degree k. Note that $T \in \mathcal{C}$ admits a unique decomposition $T = \sum T_k$, where $T_k \in \mathcal{C}_k$. (In fact, T_k is the trivial extension to Π of the restriction of the linear form T to Π_k.)

Any invertible linear map $A : M_{\mathbb{Q}} \to M_{\mathbb{Q}}$ induces actions on $\mathcal{C}$, dual to the pullback and pushforward on Π, defined by $\langle A_*T, \beta\rangle := \langle T, A^*\beta\rangle$ and $\langle A^*T, \beta\rangle := \langle T, A_*\beta\rangle$ for $T \in \mathcal{C}$ and $\beta \in \Pi$. It is not difficult to see that $T \in \mathcal{C}$ is homogeneous of degree k if and only if $\mathcal{D}(r)^*T = r^k T$ for any $r \in \mathbb{Q}_{\geq 0}$.

Let us describe some important examples of currents.

Example 3.3. Pick $\alpha \in \Pi$, and let T_α be the current defined by $T_\alpha(\beta) := \mathrm{Vol}(\alpha \cdot \beta)$ for $\beta \in \Pi$. The map $\alpha \mapsto T_\alpha$ gives a linear injective map $\Pi \to \mathcal{C}$ that sends Π_k to currents of degree k.

In general, for $\alpha = \sum \alpha_k \in \Pi$, with $\alpha_k \in \Pi_k$, $T_{\alpha_k}(\beta) = \binom{d}{k}\mathrm{Vol}(\alpha[k], \beta[k-d])$, which follows immediately from Lemma 3.1. Moreover, by Lemma 3.2, the actions of an invertible linear map $A : M_{\mathbb{Q}} \to M_{\mathbb{Q}}$ on Π and $\mathcal{C}$ are compatible so that $T_{A^*\alpha} = A^*T_\alpha$ and $T_{A_*\alpha} = A_*T_\alpha$ for any class $\alpha \in \Pi$.

Example 3.4. Given a vector space V, we define the *convex body algebra* $\mathcal{K}(V)$ as the polytope algebra, but with a generator $[K]$ for each compact convex set $K \subset V$ and with the relations $[K+t] = [K]$ for any $t \in V$, and $[K \cup L] + [K \cap L] = [K] + [L]$ whenever $K \cup L$ is convex. A *(continuous translation-invariant) valuation* is a linear map on $\mathcal{K}(V)$ that is continuous for the Hausdorff metric on compact sets, see e.g. [Sc, Sect. 3.4]. Let $\mathrm{Val}(V)$ denote the space of valuations on the space of convex bodies in V. Restricting the action of valuations on $M_{\mathbb{R}}$ to Π gives an injective morphism $0 \to \mathrm{Val}(M_{\mathbb{R}}) \to \mathcal{C}$. The construction of the current T_α in Example 3.3 can be extended to $\alpha, \beta \in \mathcal{K}(M_\mathbb{R})$, and the mapping $\alpha \mapsto T_\alpha$ embeds $\mathcal{K}(M_\mathbb{R})$ into $\mathrm{Val}(M_\mathbb{R})$. Thus $\Pi \subset \mathcal{K}(M_\mathbb{R}) \subset \mathrm{Val}(M_\mathbb{R}) \subset \mathcal{C}$.

Example 3.5. Endow $M_\mathbb{R}$ with an arbitrary euclidean metric g. Let H be a linear subspace of $M_\mathbb{R}$ of codimension k, and Vol_H be the volume element on H induced by g. Consider any linear projection $p : M_\mathbb{R} \to H$ onto H. Since $p(P \cap Q) = p(P) \cap p(Q)$ whenever $P \cup Q$ is convex, p can be extended to a function $\Pi \to \Pi$, defined by $p[P] := [p(P)]$, and the linear map $\alpha \mapsto \mathrm{Vol}_H(p(\alpha))$ is a valuation of degree k that we shall denote by $[H, p]$.

3.4. **Relations to the inverse cohomology of toric varieties.** Each polytope P in $M_\mathbb{Q}$ determines a class $\mathrm{ch}(P)$ in $\underset{\rightarrow}{\mathrm{H}}^*$, defined as follows: Let h_P be defined as in (2.3) and choose $\Delta \in \mathfrak{D}$ so that $h_P \in \mathrm{PL}_\mathbb{Q}(\Delta)$. Now $\mathrm{ch}(P)$ is determined by the Chern character of the associated $\mathbb{Q}$-line bundle

$$(\mathrm{ch}(P))_\Delta := \sum_{k=0}^{d} \frac{1}{k!} \Theta_1(h_P)^k = \Theta\left(\sum_{k=0}^{d} \frac{1}{k!} h_P^k\right) = \Theta(\exp(h_P)) \in H^*(X(\Delta)) \ , \tag{3.9}$$

where Θ_1 and Θ are as in Sections 2.4 and 2.5, respectively.

The Chern character induces a linear map from the vector space $\oplus_P \mathbb{R}[P]$ to $\underset{\rightarrow}{\mathrm{H}}^*$, defined by $\mathrm{ch}(\sum t_j[P_j]) = \sum t_j \,\mathrm{ch}(P_j)$. We claim, in fact, ch is a well-defined ring homomorphism from Π to $\underset{\rightarrow}{\mathrm{H}}^*$. To see this, first note that $h_{P+t} = h_P + t$ for $t \in M_\mathbb{Q}$. It follows that $D(h_{P+t})$ and $D(h_P)$ are linearly equivalent, see Section 2.4. In particular, $\Theta_1(h_{P+t}) = \Theta_1(h_P)$, which implies $\mathrm{ch}(P + t) = \mathrm{ch}(P)$. Next, if P and Q are polytopes in $M_\mathbb{Q}$ such that $P \cup Q$ is convex, then $h_{P\cup Q} = \max\{h_P, h_Q\}$ and $h_{P\cap Q} = \min\{h_P, h_Q\}$. Thus $h^k_{P\cup Q} + h^k_{P\cap Q} = h^k_P + h^k_Q$ for any $k \geq 0$, and by linearity of Θ, $\mathrm{ch}([P \cap Q] + [P \cup Q]) = \mathrm{ch}([P] + [Q])$. Thus $\mathrm{ch} : [P] \mapsto \mathrm{ch}(P)$ is well-defined. Next, note that if P and Q are polytopes in $M_\mathbb{Q}$, then $h_{P+Q} = h_P + h_Q$. Hence $\mathrm{ch}([P] \cdot [Q]) = \mathrm{ch}([P + Q]) = \mathrm{ch}(P + Q) = \mathrm{ch}(P)\,\mathrm{ch}(Q) = \mathrm{ch}([P])\,\mathrm{ch}([Q])$; and so the claim is proved.

Let $\deg : \underset{\rightarrow}{\mathrm{H}}^* \to \mathbb{R}$ be the linear *degree* map that is 0 on $\underset{\rightarrow}{\mathrm{H}}^k$ for $k < d$ and sends the class determined by a point in $X(\Delta)$ to 1. The following theorem is due to Fulton-Sturmfels [FS, Sect. 5] and Brion [B, Sect. 5].

Theorem 3.6. *The Chern character map* $\mathrm{ch} : [P] \mapsto \mathrm{ch}(P)$ *is an isomorphism of graded algebras* $\mathrm{ch} : \Pi \to \underset{\rightarrow}{\mathrm{H}}^*$. *It holds that* $\deg(\mathrm{ch}(\alpha)) = \mathrm{Vol}(\alpha)$ *for* $\alpha \in \Pi$.

By duality, we get a continuous isomorphism $\mathrm{coh} : \mathcal{H}^* \to \mathcal{C}$, *defined by* $\langle \mathrm{coh}(\omega), \beta\rangle := \omega \cdot \mathrm{ch}(\beta)$ *for* $\omega \in \mathcal{H}^*$ *and* $\beta \in \Pi$.

Proposition 3.7. *Let* $A : M_\mathbb{Q} \to M_\mathbb{Q}$ *be a linear map with* $\det(A) \neq 0$. *Then*

$$\mathrm{ch}(A_*\alpha) = \phi_A^* \,\mathrm{ch}(\alpha) \tag{3.10}$$

for any $\alpha \in \Pi$. *Similarly, for any* $\eta \in \mathcal{H}^*$,

$$\mathrm{coh}((\phi_A)_*\eta) = A^*(\mathrm{coh}(\eta)). \tag{3.11}$$

Proof. By linearity we may assume that $\alpha = [P]$ for some polytope P in $M_\mathbb{Q}$. By definition, $A_*[P] = [A(P)]$ and $\mathrm{ch}([A(P)])$ is the class in $\underset{\rightarrow}{\mathrm{H}}^*$ determined by $\Theta(\exp(h_{A(P)}))$. Now

$$h_{A(P)} = \sup\{m,\ m \in A(P)\} = \sup\{m \circ \check{A},\ m \in P\} = h_P \circ \check{A}.$$

In light of (2.5) and (3.9), it follows that $\mathrm{ch}([A(P)])$ is the pullback under ϕ_A of the class determined by $\Theta(\exp(h_P))$, i.e., $\mathrm{ch}([A(P)]) = \phi_A^* \mathrm{ch}([P])$.

Now (3.11) follows from (3.10) by duality. Indeed, for $\eta \in \mathcal{H}^*$ and $\alpha \in \Pi$, we have

$$\langle \mathrm{coh}\left((\phi_A)_*\eta\right), \alpha\rangle = (\phi_A)_*\eta\cdot \mathrm{ch}\,\alpha = \eta\cdot\phi_A^*\,\mathrm{ch}\,\alpha = \eta\cdot\mathrm{ch}(A_*\alpha) = \langle \mathrm{coh}(\eta), A_*\alpha\rangle = \langle A^*\,\mathrm{coh}(\eta), \alpha\rangle.$$

Here we have used the definition of coh for the first and fourth equality and (3.10) for the third equality. Moreover, the second and the last equality follow by Sections 2.6 and 3.3, respectively. ☐

4. Dynamical degrees on toric varieties

Let Δ be a complete simplicial fan and let h be a strictly convex piecewise linear function with respect to Δ. Furthermore, let $A : M \to M$ be a group morphism and let $\phi := \phi_A : X(\Delta) \dashrightarrow X(\Delta)$ be the corresponding rational equivariant map. The *k-th degree of ϕ with respect to* the ample divisor $D := D(h)$ is defined as

$$\deg_{D,k}(\phi) := \phi^\bullet D^k \cdot D^{d-k} \in \mathbb{R}_{\geq 0}.$$

If $X(\Delta) = \mathbb{P}^d$ and $\mathcal{O}(D) = \mathcal{O}_{\mathbb{P}^d}(1)$, then $\deg_{D,k}(\phi)$ coincides with the *k-th degree of ϕ* $\deg_k(\phi)$ as defined in the introduction (Section 1).

The following result is a key ingredient in the proofs of Theorems A and D.

Proposition 4.1. *Let Δ be a complete simplicial fan and let D be an ample $\mathbb{G}_m^d$-invariant divisor on $X(\Delta)$ with corresponding polytope P_D. Moreover, let $A : M \to M$ be a group morphism with $\det(A) \neq 0$, and let $\phi_A : X(\Delta) \dashrightarrow X(\Delta)$ be the corresponding equivariant rational map. Then*

$$\deg_{D,k}(\phi_A) = d!\,\mathrm{Vol}(A(P_D)[k], P_D[d-k]) \ . \tag{4.1}$$

Recall from Example 2.1 that if $X(\Delta) = \mathbb{P}^d$ and $D = \mathcal{O}_{\mathbb{P}^d}(1)$, then P_D is the standard simplex Σ_d. In this case (4.1) reads

$$\deg_k(\phi_A) = d!\,\mathrm{Vol}(A(\Sigma_d)[k], \Sigma_d[d-k]) \ .$$

Proof. Pick a complete simplicial fan $\Delta' \succ \Delta$ such that the dual $\check{A} : N \to N$ of A is a map of fans $\check{A} : (N, \Delta') \to (N, \Delta)$, let $f_A : X(\Delta') \to X(\Delta)$ be the corresponding equivariant map, and let $\pi : X(\Delta') \to X(\Delta)$ be the map induced by $\check{\mathrm{id}}_{\Delta',\Delta}$.

Recall from Section 2.6 that then $\phi_A^\bullet = \pi_* \circ f_A^*$. Hence

$$\deg_{D,k}(\phi_A) = \pi_* \circ f_A^*(D)^k \cdot D^{d-k} = f_A^*(D)^k \cdot \pi^*(D^{d-k}) = (f_A^* D)^k \cdot (\pi^* D)^{d-k}. \tag{4.2}$$

By Section 2.4, and in particular (2.4), the right hand side of (4.2) equals

$$d!\,\mathrm{Vol}(A(P_D)[k], P_D[d-k]),$$

which concludes the proof. ☐

Let us collect some basic properties of k-th degrees. These results are well-known and valid for arbitrary rational maps, see [DS]. However the case of toric maps is particularly simple.

Proposition 4.2. *Let Δ_1 and Δ_2 be complete simplicial fans, and let D_1 and D_2 be ample $\mathbb{G}_m^d$-invariant divisors on $X(\Delta_1)$ and $X(\Delta_2)$, respectively. Then there exists a constant C such that for any group morphism $A : M \to M$, one has*

$$C^{-1} \deg_{D_2,k}(\phi_{A_2}) \leq \deg_{D_1,k}(\phi_{A_1}) \leq C \deg_{D_2,k}(\phi_{A_2}) \tag{4.3}$$

where $\phi_{A_i} : X(\Delta_i) \dashrightarrow X(\Delta_i)$, $i = 1, 2$ denote the respective induced maps.

Proof. We claim that there is a constant C such that

$$C^{-1} \operatorname{Vol}(A(P_{D_2})[k], P_{D_2}[d-k]) \leq \operatorname{Vol}(A(P_{D_1})[k], P_{D_1}[d-k]) \leq C \operatorname{Vol}(A(P_{D_2})[k], P_{D_2}[d-k]).$$

Then (4.3) follows immediately from Proposition 4.1.

Since D_1 and D_2 are ample, and since Vol is translation invariant, (3.4), in order to prove the claim we may assume that P_{D_1} and P_{D_2} contain the origin in $M_{\mathbb{R}}$ in their interior. Then for some C_0 large enough, $C_0^{-1} P_{D_2} \subset P_{D_1} \subset C_0 P_{D_2}$. It follows that the claim holds for $C = C_0^d$ since Vol is multilinear and monotone, (3.3). □

Proposition 4.3. *Let Δ be a complete simplicial fan and let D be an ample $\mathbb{G}_m^d$-invariant divisor on $X(\Delta)$. Then there exists a constant C such that for any group morphisms $A_1, A_2 : M \to M$, one has*

$$\deg_{D,k}(\phi_{A_1} \circ \phi_{A_2}) \leq C \deg_{D,k}(\phi_{A_1}) \deg_{D,k}(\phi_{A_2}).$$

Proof. Given a rational map $f : \mathbb{P}^d \dashrightarrow \mathbb{P}^d$, we denote by $C(f)$ the set of points $p \in \mathbb{P}^d$ that are either indeterminate or critical for f, and by $PC(f) := \pi_2 \pi_1^{-1}(C(f))$ where π_1, π_2 denote the two projections of the graph of f onto $\mathbb{P}^d$. This defines two proper algebraic subsets of $\mathbb{P}^d$.

If Z is a variety of pure codimension k in $\mathbb{P}^d$, we denote by $f^{-1}(Z)$ the closure in $\mathbb{P}^d$ of $f^{-1}(Z \cap PC(f))$. Note that by construction, $f^{-1}(Z)$ is of codimension k (or empty). We have the general inequality $\deg(f^{-1}(Z)) \leq \deg_k(f) \deg(Z)$, and for a generic choice of Z, $\deg(f^{-1}(Z)) = \deg_k(f) \deg(Z)$. In particular, if L is a generic linear subspace of $\mathbb{P}^d$ of codimension k, then $\deg_k(f) = \deg(f^{-1}(L))$.

We always have $\phi_{A_1}^{-1}(\phi_{A_2}^{-1}(L)) = (\phi_{A_1} \circ \phi_{A_2})^{-1}(L)$ outside $W := C(\phi_{A_2}) \cup \phi_{A_2}^{-1} C(\phi_{A_1})$. For a generic choice of L, the closure of $(\phi_{A_1} \circ \phi_{A_2})^{-1}(L) \cap W$ is equal to $(\phi_{A_1} \circ \phi_{A_2})^{-1}(L)$. Whence

$$\deg((\phi_{A_1} \circ \phi_{A_2})^{-1}(L)) = \deg(\phi_{A_2}^{-1}(\phi_{A_1}^{-1}(L))) \leq \deg_k(\phi_{A_2}) \deg(\phi_{A_1}^{-1}(L)) \tag{4.4}$$

Since L is generic the left hand side of (4.4) equal $\deg_k(\phi_{A_1} \circ \phi_{A_2})$ and the right hand side equals $\deg_k(\phi_{A_2}) \deg_k(\phi_{A_1})$. Thus $\deg_k(\phi_{A_1} \circ \phi_{A_2}) \leq \deg_k(\phi_{A_1}) \deg_k(\phi_{A_2})$, and applying Proposition 4.2 to $D_1 = \mathcal{O}_{\mathbb{P}^d}(1)$ and $D_2 = D$, we get

$$\begin{aligned}\deg_{D,k}(\phi_{A_1} \circ \phi_{A_2}) \leq C \deg_k(\phi_{A_1} \circ \phi_{A_2}) \leq C \deg_k(\phi_{A_1}) \deg_k(\phi_{A_2}) \leq \\ C^3 \deg_{D,k}(\phi_{A_1}) \deg_{D,k}(\phi_{A_2}),\end{aligned}$$

which concludes the proof. □

Pick a group morphism $A : M \to M$, a fan $\Delta \in \mathfrak{D}$, and an ample $\mathbb{G}_m^d$-invariant divisor D on $X(\Delta)$. Then Proposition 4.3 implies

$$C \deg_{D,k}(\phi_A^{n+m}) \leq (C \deg_{D,k}(\phi_A^n)) (C \deg_{D,k}(\phi_A^m)) .$$

Since the sequence $\{C \deg_{D,k}(\phi_A^n)\}_n$ is sub-multiplicative, and $C^{1/n} \to 1$, we can define the *k-th dynamical degree of ϕ_A with respect to D*,

$$\lambda_{D,k}(\phi_A) := \lim_n \deg_{D,k}(\phi_A^n)^{1/n}.$$

Assume $\Delta_1, \Delta_2 \in \mathfrak{D}$ and that D_1, D_2 are ample $\mathbb{G}_m^d$-invariant divisors on $X(\Delta_1)$ and $X(\Delta_2)$, respectivly. Let $A : M \to M$ be a group morphism, and let $\phi_{A_i} : X(\Delta_i) \dashrightarrow X(\Delta_i)$, $i = 1, 2$ denote the induced equivariant morphisms. Then Proposition 4.2 implies that $\lambda_{D_1,k}(\phi_{A_1}) = \lambda_{D_2,k}(\phi_{A_2})$. We shall write $\lambda_k(\phi_A)$ for the *k-th dynamical degree of ϕ_A* (computed in any toric model, and with respect to any ample divisor).

For the record, we mention the following properties of the dynamical degrees. Proposition 4.1 applied to P_D yields that $\deg_{D,0}(\phi_A^n) = d! \operatorname{Vol}(P_D)$ for all n and $\deg_{D,d}(\phi_A^n) = d! \operatorname{Vol}(A^n(P_D)) = d! |\det(A)|^n \operatorname{Vol}(P_D)$; therefore $\lambda_0(\phi_A) = 1$ and $\lambda_d(\phi_A) = |\det(A)|$. Moreover $\lambda_1(\phi_A) = \rho(A)$, where $\rho(A)$ is the spectral radius of A, i.e., the largest modulus of an eigenvalue of A; a proof is given in [HP, Sect. 6].

Proposition 4.4. *Let $A : M \to M$ be a group morphism and $\Delta \in \mathfrak{D}$, and denote by $\phi_A : X(\Delta) \dashrightarrow X(\Delta)$ the induced equivariant morphism. Then, for any $0 \le k, l \le d$,*

$$\lambda_{k+l}(\phi_A) \le \lambda_k(\phi_A)\, \lambda_l(\phi_A). \tag{4.5}$$

Proof. By the Aleksandrov–Fenchel inequality, see (6.4.5) on p. 334 of [Sc],

$$\operatorname{Vol}(A(P_D)[k+l], P_D[d-k-l]) \operatorname{Vol}(P_D) \le \operatorname{Vol}(A(P_D)[k], P_D[d-k]) \operatorname{Vol}(A(P_D)[l], P_D[d-l])$$

which, in light of Proposition 4.1 implies (4.5). $\square$

Note that Proposition 4.4 also immediately follows from Corollary B, taking it for granted.

5. Degree growth - Proof of Theorem A

The proof of Theorem A can be reduced to controlling the growth of mixed volumes of convex bodies under the action of a linear map. Indeed, let $A : M \to M$ be a group morphism and let D be a divisor on a toric variety $X(\Delta)$, where Δ is a complete simplicial fan. Then, by Proposition 4.1, $\deg_{D,k}(\phi_A) = d! \operatorname{Vol}(A(P_D)[k], P_D[d-k])$. In particular, $\deg_k(\phi_A) = d! \operatorname{Vol}(A(\Sigma_d)[k], \Sigma_d[d-k])$, where Σ_d is the standard simplex, see Example 2.1. Now Theorem A follows immediately from the following result.

Theorem 5.1. *Let $A : M \to M$ be a group morphism such that $\det(A) \neq 0$. Then for any $0 \le k \le d$, and any convex sets $K, L \subset M_{\mathbb{R}}$ with non-empty interiors,*

$$\operatorname{Vol}(A^n(K)[k], L[d-k]) \asymp \| \wedge^k A^n \|, \tag{5.1}$$

where $\wedge^k A^n$ denotes the natural induced linear map on $\wedge^k M_{\mathbb{R}}$, and $\| \cdot \|$ is any norm on $\operatorname{End}(\wedge^k M_{\mathbb{R}})$.

It remains to prove Theorem 5.1. We first present a simple proof in the case when A is diagonalizable over $\mathbb{R}$ in Section 5.1. To deal with the general case, we rely on the Cauchy-Crofton formula. Some basic material on the geometry of the affine Grassmannian is given in Section 5.2, and the proof of Theorem 5.1 is then given in Section 5.3.

Note that, since all norms on $\operatorname{End}(\wedge^k M_{\mathbb{R}})$ are equivalent, it suffices to prove (5.1) for one particular choice of $\| \cdot \|$.

5.1. **Proof of Theorem 5.1 in the diagonalizable case.** Assume that A is diagonalizable over $\mathbb{R}$, and denote by $\rho_1, \dots, \rho_d$ its eigenvalues, ordered so that $|\rho_1| \geq \dots \geq |\rho_d|$.

Let us first compute $\| \wedge^k A^n \|$. We fix a basis $e_1, \dots, e_d$ of $M_{\mathbb{R}}$ that diagonalizes A so that $Ae_j = \rho_j e_j$ for all j. For any k-tuple $I = \{i_1, \dots, i_k\}$ of distinct elements in $\{1, \dots, d\}$, we write $e_I := e_{i_1} \wedge \dots \wedge e_{i_k}$ and $\rho^I := \prod_1^k \rho_{i_j}$. Then $(\wedge^k A)(e_I) = \rho^I \, e_I$ and the collection of e_I's forms a basis of $\wedge^k M_{\mathbb{R}}$ that diagonalizes $\wedge^k A$. If $\| \cdot \|_{\sup}$ is the supremum norm with respect to this basis, then

$$\| \wedge^k A^n \|_{\sup} = \prod_{j=1}^k |\rho_j|^n.$$

We now turn to the computation of the mixed volume $\mathrm{Vol}(A^n(K)[k], L[d-k])$. First, fix a Euclidean metric g on $M_{\mathbb{R}}$ such that the basis $e_1, \dots e_d$ is orthonormal, and let Vol_g denote the induced volume element. Then there exists a constant $C > 0$ such that $\mathrm{Vol}_g(K) = C \; \mathrm{Vol}(K)$ for any convex body $K \subset M_{\mathbb{R}}$. It follows that

$$\mathrm{Vol}\left(A^n(K)[k], L[d-k]\right) = C^{-1} \, \mathrm{Vol}_g\left(A^n(K)[k], L[d-k]\right).$$

Since K and L have non-empty interiors, by arguments as in the proof of Proposition 4.2, one can show that

$$\mathrm{Vol}_g\left(A^n(K)[k], L[d-k]\right) \asymp \mathrm{Vol}_g\left(A^n(K')[k], K'[d-k]\right), \tag{5.2}$$

where K' is any convex set with non-empty interior.

We will compute the right hand side of (5.2) when K' is a polydisk. For $r = (r_1, \dots, r_d) \in \mathbb{R}^d_{\geq 0}$, let $\mathbb{D}_r$ be the polydisk $\mathbb{D}_r := \{\sum x_j e_j, \, |x_j| \leq r_j/2\} \subset M_{\mathbb{R}}$. Note that $t\mathbb{D}_r + \tau \mathbb{D}_s = \mathbb{D}_{tr + \tau s}$ for $r, s \in \mathbb{R}^d_{\geq 0}$ and $t, \tau \in \mathbb{R}_{\geq 0}$. It follows that $\mathrm{Vol}_g(t\mathbb{D}_r + \tau \mathbb{D}_s) = \prod_{j=1}^d (tr_j + \tau s_j)$. Thus by (3.2), $\mathrm{Vol}_g(\mathbb{D}_r[k], \mathbb{D}_s[d-k]) = \binom{d}{k}^{-1} \sum r^I s^{I^C}$, where the sum runs over all multi-indices $\{i_1, \dots, i_k\} \subset \{1, \dots, d\}$, $r^I := \prod_1^k r_{i_j}$, and $I^C := \{1, \dots, d\} \setminus I$. Let $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}_{\geq 0}$. Then $A^n \mathbb{D}_{\mathbf{1}} = \mathbb{D}_{(|\rho_1|^n, \dots, |\rho_d|^n)}$, and

$$\mathrm{Vol}_g\left(A^n(\mathbb{D}_{\mathbf{1}})[k], \mathbb{D}_{\mathbf{1}}[d-k]\right) = \binom{d}{k}^{-1} \sum_{|I|=k} |\rho^I|^n \asymp \max_I |\rho^I|^n = \prod_{j=1}^k |\rho_j|^n.$$

This concludes the proof of Theorem 5.1 in the diagonalizable case.

Remark 5.2. The same idea can be used to treat the case A is diagonalizable over $\mathbb{C}$. In this case, one chooses K' to be an adequate product of segments and two-dimensional disks.

5.2. **The affine Grassmannian.** For $k = 1, \dots, d-1$, we denote by $\mathrm{Gr}(d,k)$ the Grassmannian of linear subspaces of $M_{\mathbb{R}}$ of dimension k, and by $\mathrm{Graff}(d,k)$ the Grassmanian of affine k-dimensional subspaces. Then $\mathrm{Gr}(d,k)$ and $\mathrm{Graff}(d,k)$ are smooth manifolds, and there is a natural projection map $\varpi : \mathrm{Graff}(d,k) \to \mathrm{Gr}(d,k)$ sending an affine subspace to the unique linear subspace that is parallel to it. The preimage $\varpi^{-1}(H)$ of $H \in \mathrm{Gr}(d,k)$ is canonically identified with $M_{\mathbb{R}}/H$, and hence we can view $\mathrm{Graff}(d,k)$ as the total space of a rank $d-k$ vector bundle over $\mathrm{Gr}(d,k)$. For $v \in M_{\mathbb{R}}$, and $H \in \mathrm{Gr}(d,k)$, we write $v + H \in \mathrm{Graff}(d,k)$ for the affine space obtained by translating H by v, so that $\varpi(v + H) = H$. Note that the zero section of ϖ is the natural inclusion map $\mathrm{Gr}(d,k) \hookrightarrow \mathrm{Graff}(d,k)$ given by viewing a linear space as an affine one.

The tangent space $T_H \operatorname{Gr}(d,k)$ is canonically isomorphic to $\operatorname{Hom}(H, M_{\mathbb{R}}/H)$, see [Sh, Ex. VI.4.1.3]. It follows that at any point $v+H$ in the affine Grassmannian, we have the exact sequence

$$0 \to M_{\mathbb{R}}/H \to T_{v+H} \operatorname{Graff}(d,k) \to \operatorname{Hom}(H, M_{\mathbb{R}}/H) \to 0 \ . \tag{5.3}$$

Suppose we are given an invertible *affine map* $A_{\mathrm{aff}} : M_{\mathbb{R}} \to M_{\mathbb{R}}$, with linear part A, i.e., $A_{\mathrm{aff}} = A + w$ for some $w \in M_{\mathbb{R}}$. Then A_{aff} induces smooth maps $A_{\mathrm{aff}} : \operatorname{Graff}(d,k) \to \operatorname{Graff}(d,k)$ and $A : \operatorname{Gr}(d,k) \to \operatorname{Gr}(d,k)$. For any tangent vector $\tau \in T_H \operatorname{Gr}(d,k)$, interpreted as a linear map $\tau : H \to M_{\mathbb{R}}/H$, we have

$$dA_H(\tau) = A \circ \tau \circ A^{-1} \in T_{A(H)} \operatorname{Gr}(d,k).$$

The differential of A_{aff} at $v + H \in \operatorname{Graff}(d,k)$ is computed analogously using (5.3).

Let us fix a Euclidean metric g on $M_{\mathbb{R}}$, and denote by Vol_g the induced volume element. Note that $\varpi^{-1}(H) \simeq M_{\mathbb{R}}/H$ is canonically identified with $H^\perp$. We will see that there are natural induced Riemannian metrics on $\operatorname{Gr}(d,k)$ and $\operatorname{Graff}(d,k)$. First, note that there is a natural action of the orthogonal group $\mathrm{O}(M_{\mathbb{R}}) \simeq \mathrm{O}(d)$ on $\operatorname{Gr}(d,k)$ sending (ϕ, H) to $\phi(H)$. Since the stabilizer of a k-dimensional subspace $H \subset M_{\mathbb{R}}$ is $\mathrm{O}(H) \times \mathrm{O}(H^\perp) \simeq \mathrm{O}(k) \times \mathrm{O}(d-k)$, the Grassmanian $\operatorname{Gr}(d,k)$ is diffeomorphic to the homogeneous space $\mathrm{O}(d)/(\mathrm{O}(k) \times \mathrm{O}(d-k))$. There is a Riemannian metric on $\mathrm{O}(d)$ that is both left and right invariant by the action of $O(d)$; it is given by the pairing $(X,Y) \mapsto -\operatorname{Tr}(XY)$ in its Lie algebra. This metric induces a Riemannian metric g_{Gr} on $\operatorname{Gr}(d,k)$ invariant by the action of $\mathrm{O}(d)$, see [GHL, Thm. 2.42].

We saw above that $\varpi : \operatorname{Graff}(d,k) \to \operatorname{Gr}(d,k)$ identifies $\operatorname{Graff}(d,k)$ as the total space of a vector bundle over $\operatorname{Gr}(d,k)$. Any fixed affine k-plane has a canonical representation $v + H$ with $v \in H^\perp$. The section $\operatorname{Gr}(d,k) \to \operatorname{Graff}(d,k)$ sending H' to $v + H'$ determines a subspace ξ_{v+H} in the tangent space of $\operatorname{Graff}(d,k)$ at $v+H$ such that $d\varpi : \xi_{v+H} \to T_H \operatorname{Gr}(d,k)$ is an isomorphism, and $T_{v+H} \operatorname{Graff}(d,k) = \ker d\varpi \oplus \xi_{v+H}$. We may therefore endow $\operatorname{Graff}(d,k)$ with the unique metric g_{Graff} making this decomposition orthogonal, such that the restriction $d\varpi : \xi_{v+H} \to T_H \operatorname{Gr}(d,k)$ and the isomorphism $M_{\mathbb{R}}/H \simeq H^\perp$ are isometries. Note that this metric is invariant by $\mathrm{O}(d)$ but *not* by translations. However, any translation preserves the fibers of ϖ and their restriction to each fiber is an isometry.

Recall that any Riemannian metric h on a manifold M defines a volume element Vol_h on M (and a volume form dVol_h on M if it is oriented), see [GHL, Sect. 2.7]. If $x = (x_1, \dots, x_s)$ are local coordinates on M, then locally $h = \sum h_{ij} dx_i \otimes dx_j$ and $\mathrm{dVol}_h = \sqrt{|\det(h_{ij})|}\, |dx_1 \wedge \ldots \wedge dx_s|$. We will denote by $\mathrm{Vol}_{\mathrm{Gr}}$ and $\mathrm{Vol}_{\mathrm{Graff}}$ the volume elements defined by the metrics g_{Gr} and g_{Graff}, respectively. In fact, $\mathrm{Vol}_{\mathrm{Gr}}$ is the unique (up to a scaling factor) volume element that is invariant under the action of $\mathrm{O}(d)$.

Recall that an affine map is an *affine orthogonal transformation* if (and only if) its linear part is orthogonal.

Proposition 5.3. *The volume element* $\mathrm{Vol}_{\mathrm{Graff}}$ *on* $\operatorname{Graff}(d,k)$ *is invariant by the group of all affine orthogonal transformations. Moreover it satisfies the following Fubini-type property:*

$$\int_{\operatorname{Graff}(d,k)} h \ \mathrm{dVol}_{\mathrm{Graff}} = \int_{H \in \operatorname{Gr}(d,k)} \left(\int_{H^\perp} h \ \mathrm{dVol}_{g|_{H^\perp}} \right) \mathrm{dVol}_{\mathrm{Gr}}, \tag{5.4}$$

for any Borel function h on $\mathrm{Graff}(d,k)$.

Proof. Let us first prove (5.4). Pick $H \in \mathrm{Gr}(d,k)$ and a neighborhood $H \in \mathcal{U} \subseteq \mathrm{Gr}(d,k)$ with local coordinates $y = (y_1, \dots, y_{k(d-k)})$. Then locally in $\mathcal{U}$, g_{Gr} is of the form $g_{\mathrm{Gr}} = \sum b_{ij}(y)\, dy_i \otimes dy_j$. Moreover we can choose a local trivialization $\mathcal{U} \times \mathbb{R}^{d-k}$ of $\mathrm{Graff}(d,k) \to \mathrm{Gr}(d,k)$ with coordinates (x,y), where $\varpi(x,y) = y$ and $x = (x_1, \dots, x_{d-k})$ are coordinates in $\mathbb{R}^{(d-k)} \simeq H^\perp$. Since $\varpi : \mathrm{Graff}(d,k) \to \mathrm{Gr}(d,k)$ is a Riemannian submersion, and the restriction of g_{Graff} to the fiber $\varpi^{-1}(H) \simeq H^\perp$ is the constant metric $g|_{H^\perp} = \sum a_{ij}(H)\, dx_i \otimes dx_j$, locally in the trivialization, $g_{\mathrm{Graff}}(x,y) = \sum a_{ij}(y)\, dx_i \otimes dx_j + \sum b_{ij}(y)\, dy_i \otimes dy_j$. Consequently $d\,\mathrm{Vol}_{\mathrm{Graff}} = \sqrt{\det(a_{ij}(y))}\sqrt{\det(b_{ij}(y))}\, dx \wedge dy$. After a partition of unity we may assume that h has support in a small neighborhood of $v + H \in \mathrm{Graff}(d,k)$ and thus

$$\int h \;\mathrm{dVol}_{\mathrm{Graff}(d,k)} = \int \left(\int h \sqrt{\det(b_{ij}(y))}\, dx \right) \wedge \sqrt{\det(a_{ij}(y))}\, dy,$$

which proves (5.4).

To prove the first part of the proposition we need to show that $\mathrm{Vol}_{\mathrm{Graff}}$ is invariant under linear orthogonal transformations and translations. First, since g_{Graff} is invariant under $O(d)$, so is $\mathrm{Vol}_{\mathrm{Graff}}$. Next, let $A_w := \mathrm{id} + w : \mathrm{Graff}(d,k) \to \mathrm{Graff}(d,k)$ be the translation by $w \in M_{\mathbb{R}}$. Since g, and thus $g|_{H^\perp}$, is invariant under translations on $H^\perp$,

$$\int_{v \in H^\perp} h \circ A_w(v)\; \mathrm{dVol}_{g|_{H^\perp}} = \int_{v \in H^\perp} h(v + w')\; \mathrm{dVol}_{g|_{H^\perp}} = \int_{v \in H^\perp} h(v)\; \mathrm{dVol}_{g|_{H^\perp}},$$

where w' is the orthogonal projection of w onto $H^\perp$. Thus, using (5.4), $\int (h \circ A_w)\; \mathrm{dVol}_{\mathrm{Graff}} = \int h\; \mathrm{dVol}_{\mathrm{Graff}}$; this shows that $\mathrm{Vol}_{\mathrm{Graff}}$ is invariant under translations. □

Let $A : \mathrm{Gr}(d,k) \to \mathrm{Gr}(d,k)$ and $A_{\mathrm{aff}} : \mathrm{Graff}(d,k) \to \mathrm{Graff}(d,k)$ be the maps induced by an invertible affine map $A_{\mathrm{aff}} : M_{\mathbb{R}} \to M_{\mathbb{R}}$ with linear part A. Recall that the *Jacobians* of A and A_{aff}, respectively, are the unique smooth functions $JA : \mathrm{Gr}(d,k) \to \mathbb{R}_{\geq 0}$ and $JA_{\mathrm{aff}} : \mathrm{Graff}(d,k) \to \mathbb{R}_{\geq 0}$ that satisfy the change of variables formula, i.e.,

$$\int_{\mathrm{Gr}(d,k)} h \;\mathrm{dVol}_{\mathrm{Gr}} = \int_{\mathrm{Gr}(d,k)} (h \circ A)\, JA \;\mathrm{dVol}_{\mathrm{Gr}}, \tag{5.5}$$

$$\int_{\mathrm{Graff}(d,k)} h \;\mathrm{dVol}_{\mathrm{Graff}} = \int_{\mathrm{Graff}(d,k)} (h \circ A_{\mathrm{aff}})\, JA_{\mathrm{aff}} \;\mathrm{dVol}_{\mathrm{Graff}} \tag{5.6}$$

for any integrable functions h on $\mathrm{Graff}(d,k)$ and $\mathrm{Gr}(d,k)$, respectively.

Given a linear map $A : M_{\mathbb{R}} \to M_{\mathbb{R}}$, let $\Phi_A : \mathrm{Gr}(d,k) \to \mathbb{R}_{\geq 0}$ be the map that maps H to (absolute value of) the Jacobian of the induced linear map $A : M_{\mathbb{R}}/H \to M_{\mathbb{R}}/A(H)$, computed with respect to the volume elements $\mathrm{Vol}_{g|_{H^\perp}}$ and $\mathrm{Vol}_{g|_{A(H)^\perp}}$ defined by g on $M_{\mathbb{R}}/H \simeq H^\perp$ and $M_{\mathbb{R}}/A(H) \simeq A(H)^\perp$, respectively. In other words, Φ_A is the unique function that satisfies

$$\int_{M_{\mathbb{R}}/A(H)} h \;\mathrm{dVol}_{g|_{A(H)^\perp}} = \int_{M_{\mathbb{R}}/H} (h \circ A)\, \Phi_A(H) \;\mathrm{dVol}_{g|_{H^\perp}} \tag{5.7}$$

The following is a key lemma in the proof of Theorem 5.1.

Lemma 5.4. *Let $A : M_{\mathbb{R}} \to M_{\mathbb{R}}$ be any invertible linear map. Then for any linear space $H \subset M_{\mathbb{R}}$, and any $v \in M_{\mathbb{R}}$, we have*

$$JA_{\mathrm{aff}}(v+H) = JA(H) \times \Phi_A(H).$$

Proof. Recall that, by the definition of g_{Graff}, the tangent space of $\mathrm{Graff}(d,k)$ at $v+H$ splits as an orthogonal sum $M_{\mathbb{R}}/H \oplus T_H \operatorname{Gr}(d,k)$. The differential of A_{aff} does not preserve this orthogonal decomposition in general but sends $M_{\mathbb{R}}/H$ to $M_{\mathbb{R}}/A(H)$; the tangent space at $A_{\mathrm{aff}}(v+H) = A_{\mathrm{aff}}(v) + A(H)$ orthogonally splits as $M_{\mathbb{R}}/A(H) \oplus T_{A(H)} \operatorname{Gr}(d,k)$. Choose (local) orthonormal bases v_j, w_j, v'_j, and w'_j of $M_{\mathbb{R}}/H \simeq H^{\perp}$, $T_H \operatorname{Gr}(d,k)$ (at H), $M_{\mathbb{R}}/A(H) \simeq A(H)^{\perp}$ and $T_{A(H)} \operatorname{Gr}(d,k)$ (at $A(H)$), respectively. Then, in light of (5.6), the Jacobian of A at $v+H$ is the absolute value of the determinant of the matrix dA with respect to the bases $\{v_j, v'_j\}$ at $v+H$ and $\{w_j, w'_j\}$ at $A(v)+A(H)$. This matrix, however, is block diagonal and so its determinant is the product of two determinants: one of the matrix $dA : M_{\mathbb{R}}/H \to M_{\mathbb{R}}/A(H)$ with respect to the bases $\{v_j\}$ and $\{w_j\}$ and one of the matrix of $dA : T_H \operatorname{Gr}(d,k) \to T_{A(H)} \operatorname{Gr}(d,k)$ in the bases $\{v'_j\}$ and $\{w'_j\}$. In light of (5.7) and (5.5), this concludes the proof. $\square$

5.3. **Proof of Theorem 5.1.** Let $\mathbb{B} \subset M_{\mathbb{R}}$ be the unit ball with respect to the metric g on $M_{\mathbb{R}}$ from last section. Then, by arguments as in Section 5.1,

$$\operatorname{Vol}\left(A^n(K)[k], L[d-k]\right) \asymp \operatorname{Vol}_g\left(A^n(\mathbb{B})[k], \mathbb{B}[d-k]\right). \tag{5.8}$$

To compute the right hand side of (5.8) we will apply the Cauchy-Crofton formula, see [Sc, formula 4.5.10], which asserts that there exists a universal constant $C > 0$ such that for any convex set $K \subset M_{\mathbb{R}}$:

$$\operatorname{Vol}_g\left(K[k], \mathbb{B}[d-k]\right) = C \operatorname{Vol}_{\mathrm{Graff}}\left\{v+H \in \mathrm{Graff}(d,d-k),\, (v+H)\cap K \neq \emptyset\right\}\ , \tag{5.9}$$

where $\operatorname{Vol}_{\mathrm{Graff}}$ is defined as in Section 5.2.

Now

$$\begin{aligned}
\frac{1}{C}\operatorname{Vol}_{\mathrm{Graff}}\left(A^n(\mathbb{B})[k], \mathbb{B}[d-k]\right) &= \int_{v+H\in \mathrm{Graff}(d,d-k),\,(v+H)\cap A^n(\mathbb{B})\neq\emptyset} \mathrm{dVol}_{\mathrm{Graff}} = \\
&\int_{v+H\in \mathrm{Graff}(d,d-k),\,(v+H)\cap \mathbb{B}\neq\emptyset} JA^n_{\mathrm{aff}}(v+H)\ \mathrm{dVol}_{\mathrm{Graff}} = \\
&\int_{v+H\in \mathrm{Graff}(d,d-k),\,(v+H)\cap \mathbb{B}\neq\emptyset} (JA^n \times \Phi_{A^n})(H)\ \mathrm{dVol}_{\mathrm{Graff}} = \\
\int_{H\in \operatorname{Gr}(d,d-k)} &\left(\int_{v\in H^{\perp},(v+H)\cap\mathbb{B}\neq\emptyset} \mathrm{dVol}_{g|_{H^{\perp}}}\right) (JA^n \times \Phi_{A^n})(H)\ \mathrm{dVol}_{\mathrm{Gr}} = \\
&V_k \int_{H\in\operatorname{Gr}(d,d-k)} \Phi_{A^n} \times JA^n\ \mathrm{dVol}_{\mathrm{Gr}} = V_k \int_{\operatorname{Gr}(d,d-k)} (\Phi_{A^n}\circ A^{-n})\ \mathrm{dVol}_{\mathrm{Gr}},
\end{aligned}$$

where V_k is the volume of the orthogonal projection of $\mathbb{B}$ onto $H^{\perp}$, i.e., the volume of the standard k-dimensional ball in Euclidean space, and $\operatorname{Vol}_{\mathrm{Gr}}$ and Φ_A are defined as in Section 5.2. Here we have used (5.9), (5.6), Lemma 5.4, (5.4), and (5.5) for the first, second, third, fourth, and last equality, respectively.

To sum up,

$$\mathrm{Vol}\,(A^n(K)[k], L[d-k]) \asymp \int_{\mathrm{Gr}(d,d-k)} (\Phi_{A^n} \circ A^{-n})\, \mathrm{dVol}_{\mathrm{Gr}}, \tag{5.10}$$

We will prove Theorem 5.1 by estimating the right hand side of (5.10). For that we will need the following two lemmas.

Lemma 5.5. *Let $H \subset M_{\mathbb{R}}$ be a linear subspace of codimension k and let $A : M_{\mathbb{R}} \to M_{\mathbb{R}}$ be a linear map with* $\det(A) \neq 0$. *Then for any $\gamma \in \wedge^{d-k} M_{\mathbb{R}}$ defining H in the sense that $\gamma \wedge v = 0$ if and only if $v \in H$, we have*

$$\Phi_A \circ A^{-1}(H) = |\det(A)| \, \frac{\| \wedge^{d-k} A^{-1}(\gamma)\|}{\|\gamma\|} \ ,$$

where $\wedge^{d-k} A^{-1}$ is the induced linear map on $\wedge^{d-k}$ and $\|\cdot\|$ is the natural norm on $\mathrm{End}(\wedge^{d-k} M_{\mathbb{R}})$ *induced by g.*

Lemma 5.6. *Let $(V, \|\cdot\|)$ be a finite dimensional normed vector space, and let $h : V \to V$ be a linear map with* $\det(h) \neq 0$. *Moreover, for $v \in V$, let*

$$\tau_h(v) := \inf_n \frac{\|h^n(v)\|}{\|v\|\,\|h^n\|}.$$

Then $\tau_h : V \to \mathbb{R}_{\geq 0}$ is an upper semicontinuous function and $\{\tau_h = 0\}$ is a proper linear subspace of V.

Set $h := \det(A)\,(\wedge^{d-k} A^{-1})$. Observe that, since the pairing $\wedge^k M_{\mathbb{R}} \times \wedge^{d-k} M_{\mathbb{R}} \to \wedge^d M_{\mathbb{R}}$ is perfect, in fact, $\|h^n\| = \| \wedge^k A^n\|$.

For any $H \in \mathrm{Gr}(d, d-k)$, we pick $\gamma(H) \in \wedge^{d-k} M_{\mathbb{R}}$ defining it. Then $\gamma(H)$ is unique up to a scalar factor, and the induced map $\mathrm{Pl} : \mathrm{Gr}(d, d-k) \to \mathbb{P}(\wedge^{d-k} M_{\mathbb{R}})$ is the Plücker embedding of $\mathrm{Gr}(d, d-k)$. Lemma 5.5 can be rephrased as follows:

$$\Phi_{A^n} \circ A^{-n}(H) = \frac{|h^n(\mathrm{Pl}(H))|}{|\,\mathrm{Pl}(H)|}. \tag{5.11}$$

Note that the right hand side of (5.11) is well defined by homogeneity. The image of $\mathrm{Gr}(d, d-k)$ under Pl is not contained in any proper linear subspace of $\wedge^{d-k} M_{\mathbb{R}}$. Therefore, by Lemma 5.6, there is a non-empty open set $\mathcal{U} \subset \mathrm{Gr}(d, d-k)$, such that τ_h restricted to $\mathcal{U}$ is strictly positive. In particular, $\mu := \int_{\mathcal{U}} \tau_h(\mathrm{Pl}(H)) > 0$. Consequently,

$$\int_{\mathrm{Gr}(d,d-k)} (\Phi_{A^n} \circ A^{-n})\ \mathrm{dVol}_{\mathrm{Gr}} \geq \|h^n\| \int_{\mathrm{Gr}(d,d-k)} \tau_h(\mathrm{Pl}(H))\, \mathrm{dVol}_{\mathrm{Gr}} \geq \mu \, \| \wedge^k A^n\| \ . \tag{5.12}$$

Now (5.1) follows from (5.10), (5.12), and the trivial upper bound $\Phi_{A^n} \circ A^{-n} \leq \|h^n\| = \| \wedge^k A^n\|$. Thus we have proved Theorem 5.1.

It remains to prove the lemmas.

Proof of Lemma 5.5. Pick $H \in \mathrm{Gr}(d, d-k)$. Choose orthonormal bases $e_1, \ldots, e_d$ and $f_1, \ldots, f_d$ of $M_{\mathbb{R}}$ such that $(e_1, \ldots, e_k) \in A^{-1}(H)^{\perp} \simeq M_{\mathbb{R}}/A^{-1}(H)$ and $(f_1, \ldots, f_k) \in H^{\perp} \simeq M_{\mathbb{R}}/H$. Then $A = \sum a_{ij} f_i \otimes e_j^*$ for some $a_{ij} \in \mathbb{R}$, and $\Phi_A \circ A^{-1}(H)$ is by definition equal to $|\det(a_{ij})_{1 \leq i,j \leq k}|$. On the other hand the vector $\gamma = f_{k+1} \wedge \cdots \wedge f_d$ defines H, and

$$\wedge^{d-k} A^{-1}(\gamma) = \frac{e_{k+1} \wedge \cdots \wedge e_d}{\det(a_{ij})_{k+1 \leq i,j \leq d}} = \pm \frac{\Phi_A \circ A^{-1}(H)}{|\det(A)|}\, e_{k+1} \wedge \cdots \wedge e_d \ .$$

We conclude noting that $|e_{k+1} \wedge \cdots \wedge e_d| = |\gamma| = 1$. $\square$

Proof of Lemma 5.6. For each n the function $v \mapsto \frac{\|h^n(v)\|}{\|v\|\,\|h^n\|}$ is continuous, and so τ_h is the infimum of a sequence of continuous functions, which implies that it is upper semicontinuous.

Let us now describe the zero locus of τ_h. First, assume that the minimal polynomial of h is of the form $(x-\rho)^{\dim(V)}$. Choose a basis of V such that the matrix (also denoted by h) of h is in Jordan normal form (over $\mathbb{C}$), and let $x_1, \dots, x_d$ with $d = \dim V$ be the corresponding coordinates. Then

$$h = \begin{bmatrix} \rho & 1 & 0 & \dots & 0 \\ 0 & \rho & 1 & \dots & 0 \\ 0 & 0 & \rho & \dots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \rho \end{bmatrix} \text{ and } h^n \asymp \begin{bmatrix} \rho^n & n\rho^n & n^2\rho^n & \dots & n^{d-1}\rho^n \\ 0 & \rho^n & n\rho^n & \dots & n^{d-2}\rho^n \\ 0 & 0 & \rho^n & \dots & n^{d-3}\rho^n \\ & & \ddots & & \\ 0 & 0 & 0 & \dots & n\rho^n \\ 0 & 0 & 0 & \dots & \rho^n \end{bmatrix}. \tag{5.13}$$

In this case, $\{\tau_h = 0\}$ is precisely the hyperplane $\{x_{\dim V} = 0\}$.

In the general case, we decompose $V = \bigoplus W_i$ into minimal h-invariant subspaces W_i. Let ρ_i be the modulus of the unique eigenvalue of $h|_{W_i}$ and write $d_i := \dim(W_i)$. Set $\rho := \max\{\rho_i\}$, $I_0 := \{i, \rho_i = \rho\}$, $\delta := \max\{d_i, i \in I_0\}$, and $I_* := \{i \in I_0, d_i = \delta\}$. Then $\tau_h = \max\{\tau_{h|_{W_i}} \circ p_{W_i}, i \in I_*\}$, where $p_{W_i} : V \to W_i$ is the natural projection, and $\{\tau_h = 0\}$ is the direct sum of the W_i with $i \notin I_*$ and the hyperplanes $\{\tau_{h|_{W_i}} = 0\} \subset W_i$ for $i \in I_*$; in particular, $\{\tau_h = 0\}$ is linear, and since I_* is non-empty, it is a proper subspace of V. $\square$

5.4. **Proof of Corollary B.** Choose a basis of $M_{\mathbb{R}}$ such that the matrix of A is in Jordan normal form, and let $\|\cdot\|_{\sup}$ be the supremum norm with respect to the induced basis on $\wedge^k M_{\mathbb{R}}$. Then $\|\wedge^k A^n\|_{\sup} \asymp n^r |\rho_1|^n \cdots |\rho_k|^n$ for some integer $0 \le r \le d-1$, cf. the proof of Lemma 5.6. Hence $\lambda_k = \lim_n (\deg_k(\phi_A^n))^{1/n} = |\rho_1| \cdots |\rho_k|$.

6. Proof of Theorem D

As well as Theorem A, Theorem D can be proved by controlling the growth of mixed volumes under the linear map $A : M_{\mathbb{R}} \to M_{\mathbb{R}}$.

We fix a euclidean metric g on $M_{\mathbb{R}}$. Given a subspace $H \subset M_{\mathbb{R}}$, let Vol_H denote the volume element on H induced by g. Moreover, let p_H denote the orthogonal projection onto H.

Theorem 6.1. *Let $A : M \to M$ be a group morphism such that $\det(A) \neq 0$, with eigenvalues $|\rho_1| \ge \dots \ge |\rho_d|$. Suppose that $\kappa := |\rho_{k+1}|/|\rho_k| < 1$, and write $V_u := \oplus_{i \le k} \ker(A - \rho_i \,\mathrm{id})^d$, and $V_s := \oplus_{i>k} \ker(A - \rho_i \,\mathrm{id})^d$. Then there exists an integer $r \ge 0$, such that for any two (non-empty) convex sets $K, L \subset M_{\mathbb{R}}$,*

$$\frac{1}{\lambda_k^n} \mathrm{Vol}\left(A^n(K)[k], L[d-k]\right) = \mathrm{Vol}_{V_u}\left(p_{V_u/V_s}(K)\right) \mathrm{Vol}_{V_u^\perp}\left(p_{V_u^\perp}(L)\right) + \mathcal{O}(n^r \kappa^n), \tag{6.1}$$

where p_{V_u/V_s} denotes the projection onto V_u parallel to V_s.

Note that, by Corollary B, the condition $\kappa < 1$ is equivalent to (1.2). Recall from Proposition 4.1 that $\deg_k(\phi_A^n) = d! \,\mathrm{Vol}(A^n(\Sigma_d)[k], \Sigma_d[d-k])$, where Σ_d is the standard simplex. Thus, noting that $\kappa\lambda_k = (\lambda_{k+1}\lambda_{k-1})/\lambda_k$, Theorem 6.1 gives (1.3) with $C = d!\,\mathrm{Vol}_{V_u}(p_{V_u/V_s}(\Sigma_d))\ \mathrm{Vol}_{V_u^\perp}(p_{V_u^\perp}(\Sigma_d)) > 0$. Taking Theorem 6.1 for granted this concludes the proof of Theorem D.

Remark 6.2. Note that Theorem 6.1 applied to $K = L = P_D$, under the assumption in Theorem D, gives the following version of (1.3):

$$\deg_{D,k}(\phi_A^n) = C\lambda_k^n + \mathcal{O}\left(n^r \left(\frac{\lambda_{k-1}\,\lambda_{k+1}}{\lambda_k}\right)^n\right),$$

where $C = d!\,\mathrm{Vol}_{V_u}(p_{V_u/V_s}(P_D))\ \mathrm{Vol}_{V_u^\perp}(p_{V_u^\perp}(P_D)) > 0$.

6.1. **Proof of Theorem 6.1.** For the proof we will need the following two lemmas on mixed volumes.

Lemma 6.3. *Let $H \subset M_\mathbb{R}$ be subspace of dimension k. Then for any convex sets $L_1, \dots, L_{d-k}$, and K in $M_\mathbb{R}$ such that $K \subset H$,*

$$\mathrm{Vol}_{M_\mathbb{R}}\left(K[k], L_1, \dots, L_{d-k}\right) = \mathrm{Vol}_H(K)\ \mathrm{Vol}_{H^\perp}\left(p_{H^\perp}(L_1), \dots, p_{H^\perp}(L_{d-k})\right). \tag{6.2}$$

Proof of Lemma 6.3. Recall that we have the following polarization formula:

$$(d-k)!\,\mathrm{Vol}(K[k], L_1, \dots, L_{d-k}) = \sum_{I\subset\{1,\dots,d-k\}} (-1)^{(d-k-|I|)}\,\mathrm{Vol}(K[k], (\sum_I L_i)[d-k])\ .$$

By multilinearity, we may thus assume that $L_1 = \ldots = L_{d-k} = L$. Fix $t \in \mathbb{R}$. Then, by Fubini's theorem,

$$\mathrm{Vol}(tK + L) = \int_{v\in H^\perp} \mathrm{Vol}_H\left((tK + L) \cap (v + H)\right) \mathrm{dVol}_{H^\perp}, \tag{6.3}$$

where we have identified the volume element induced by Vol on $v + H$ with Vol_H. Let $L_v := L \cap (v + H)$. Since K is included in H, we have $(tK + L) \cap (v + H) = tK + L_v$, and so the right hand side of (6.3) equals $\int_{v\in H^\perp} \mathrm{Vol}_H(tK + L_v)\ \mathrm{dVol}_{H^\perp}$. Note that $L_v \neq \emptyset$ if and only if $v \in p_{H^\perp}(L)$, in which case $\mathrm{Vol}_H(tK + L_v) = t^k\,\mathrm{Vol}(K) + \mathcal{O}(t^{k-1})$. We conclude that

$$\mathrm{Vol}(tK + L) = \int_{p_{H^\perp}(L)} \left(t^k\,\mathrm{Vol}(K) + \mathcal{O}(t^{k-1})\right) \mathrm{dVol}_{H^\perp} = t^k\,\mathrm{Vol}(K)\,\mathrm{Vol}_{H^\perp}(L) + \mathcal{O}(t^{k-1}),$$

which implies (6.2). ☐

Recall that the *Hausdorff distance* $d_H(K, L)$ between two (non-empty) sets $K, L \subset M_\mathbb{R}$ is the infimum of all $\epsilon \in \mathbb{R}_{\geq 0}$ such that $K \subset L + \mathbb{B}_\epsilon$ and $L \subset K + \mathbb{B}_\epsilon$, where $\mathbb{B}_r \subset M_\mathbb{R}$ is a ball with radius r and center 0. In the sequel, we write $\mathbb{B} = \mathbb{B}_1$.

Lemma 6.4. *Let $L_1, \dots, L_{d-k}$ be convex sets in $M_\mathbb{R}$. Then there exists a constant $C > 0$ such that for any (non-empty) convex sets $K, K' \subset M_\mathbb{R}$, one has*

$$\begin{aligned} &|\,\mathrm{Vol}\left(K[k], L_1, \dots, L_{d-k}\right) - \mathrm{Vol}\left(K'[k], L_1, \dots, L_{d-k}\right)| \leq \\ &\qquad C \max_{j=1}^{k}\{d_H(K, K')^j\ \mathrm{Vol}\left(K[k-j], \mathbb{B}[d-k+j]\right)\}. \end{aligned} \tag{6.4}$$

Proof of Lemma 6.4. To simplify notation, write $\mathrm{Vol}\,(\cdots, L_1, \ldots, L_{d-k}) =: \mathrm{Vol}\,(\cdots, L_i)$, and $\delta := d_H(K, K')$ so that $K \subset K' + \mathbb{B}_\delta$, and $K' \subset K + \mathbb{B}_\delta$.

Using the multilinearity of the mixed volume and (3.3) we get:

$$\mathrm{Vol}\,(K[k], L_i) - \mathrm{Vol}\,\big(K'[k], L_i\big) \leq \mathrm{Vol}\,\big((K' + \mathbb{B}_\delta)[k], L_i\big) - \mathrm{Vol}\,\big(K'[k], L_i\big) =$$
$$\sum_{\ell=1}^{k} \binom{k}{\ell} \delta^\ell \; \mathrm{Vol}\,\big(K'[k-\ell], \mathbb{B}[\ell], L_i\big) \leq \sum_{\ell=1}^{k} \binom{k}{\ell} \delta^\ell \; \mathrm{Vol}\,((K + \mathbb{B}_\delta)[k-\ell], \mathbb{B}[\ell], L_i) =$$
$$\sum_{j=1}^{k} C_j \, \delta^j \; \mathrm{Vol}\,(K[k-j], \mathbb{B}[j], L_i) \leq C \max_{j=1}^{k} \{(\delta)^j \; \mathrm{Vol}\,(K[k-j], \mathbb{B}[d-k+j])\}$$

for some constants $C_j, C > 0$; for the last inequality we have used that each L_i is contained in $\mathbb{B}_{\rho_i}$ for $\rho_i \in \mathbb{R}_{\geq 0}$ large enough. Conversely,

$$\mathrm{Vol}\,\big(K'[k], L_i\big) - \mathrm{Vol}\,(K[k], L_i) \leq \mathrm{Vol}\,((K + \mathbb{B}_\delta)[k], L_i) - \mathrm{Vol}\,(K[k], L_i) =$$
$$\sum_{\ell=1}^{k} \binom{k}{\ell} \delta^\ell \; \mathrm{Vol}\,(K[k-\ell], \mathbb{B}[\ell], L_i) \leq C \max_{j=1}^{k} \{(\delta)^j \; \mathrm{Vol}\,(K[k-j], \mathbb{B}[d-k+j])\} \;,$$

for a suitable constant $C > 0$. This proves (6.4). $\square$

We are now ready to prove Theorem 6.1. Write $p := p_{V_u/V_s}$ to simplify notation.

First, since $p(K)$, as well as $A^n \circ p(K) = p \circ A^n(K)$, is included in the k-dimensional subspace $V_u \subset M_\mathbb{R}$, by Lemma 6.3,

$$\lambda_k^n \, \mathrm{Vol}_{V_u}\,(p(K)) \; \mathrm{Vol}_{V_u^\perp}\left(p_{V_u^\perp}(L)\right) = \mathrm{Vol}_{V_u}\,(p \circ A^n(K)) \; \mathrm{Vol}_{V_u^\perp}\left(p_{V_u^\perp}(L)\right) = \mathrm{Vol}\,(p \circ A^n(K)[k], L[d-k])\,. \tag{6.5}$$

Next, note that there exists a constant $C > 0$ and an integer $r \geq 0$ such that $|A^n(v)| \leq C\, n^r \, |\rho_{k+1}|^n \, |v|$ for all $v \in V_s$; for example, this can be seen using (5.13). In particular, $|A^n(v) - p \circ A^n(v)| \leq C\, n^r |\rho_{k+1}|^n \, |v|$ for all $v \in M_\mathbb{R}$, from which we infer that

$$d_H(A^n(K), p \circ A^n(K)) \leq C' \, n^r \, |\rho_{k+1}|^n, \tag{6.6}$$

where $C' = C \max_{v \in K} |v|$. Now, applying Lemma 6.4 to $K = A^n(K)$, $K' = p \circ A^n(K)$, and $L_i = L$ for all i, and using (6.5) and (6.6), we get

$$\mathrm{Vol}\,(A^n(K)[k], L[d-k]) - \lambda_k^n \, \mathrm{Vol}_{V_u}\,(p(K)) \; \mathrm{Vol}_{V_u^\perp}\left(p_{V_u^\perp}(L)\right) \leq C'' \max_{j=1}^{k} \{n^{jr} \, |\rho_{k+1}|^{jn} \; \mathrm{Vol}\,(A^n(K)[k-j], \mathbb{B}[d-k+j])\} \tag{6.7}$$

for some constant $C'' > 0$. Furthemore, Theorem 5.1 implies that

$$\mathrm{Vol}\,(A^n(K)[k-j], \mathbb{B}[d-k+j]) \leq C_j n^{r_j} \prod_{i=1}^{k-j} |\rho_i|^n$$

for suitable constants $C_j > 0$ and integers $r_j \geq 0$, cf. Section 5.4. Since $|\rho_1| \cdots |\rho_{k-j}| |\rho_{k+1}|^j \leq \lambda_k \kappa$ for $1 \leq j \leq k$ by Corollary B, the right hand side of (6.7) is bounded from above by $C'''n^{r'}(\lambda_k\kappa)^n$ for some constant $C''' > 0$ and some integer $r' \geq 0$, which proves (6.1).

7. Invariant classes

In fact, Theorem 6.1 gives more information than Theorem D. Keeping the notation from the beginning of Section 6, consider the currents $T^- := [V_u, p_{V_u/V_s}]$ and $T^+ := [V_u^\perp, p_{V_u^\perp}]$ of degree $(d-k)$ and k, respectively, as defined in Example 3.5. Then for polytopes $P, Q \subset M_{\mathbb{Q}}$, (6.1) reads

$$\frac{1}{\lambda_k^n}\operatorname{Vol}(A_*^n[P][k],[Q][d-k]) = \langle T^-,[P]\rangle\,\langle T^+,[Q]\rangle + \mathcal{O}(n^r\,\kappa^n),$$

and, by multilinearity, using (3.7), we get:

Theorem 7.1. *Let A, κ, V_s, V_u be as in Theorem 6.1. Then there exists an integer r such that, for any $\alpha \in \Pi_k$, and $\beta \in \Pi_{d-k}$,*

$$\frac{1}{\lambda_k^n}\operatorname{Vol}(A_*^n\alpha\cdot\beta) = \binom{d}{k}\langle T^-,\alpha\rangle\,\langle T^+,\beta\rangle + \mathcal{O}(n^r\,\kappa^n)\ .$$

In particular, in the space of currents on Π, the convergence

$$\frac{1}{\lambda_k^n}A_*^n\alpha \to c(\alpha)\,T^+$$

holds for any class $\alpha \in \Pi_k$, with $c(\alpha) = \binom{d}{k}\langle T^-,\alpha\rangle$; here we identify $\gamma \in \Pi$ with $T_\gamma \in \mathcal{C}$, cf. Example 3.3. By Lemmas 3.1 and 3.2, $\operatorname{Vol}(A_*^n\alpha\cdot\beta) = \operatorname{Vol}(\alpha\cdot A^{n*}\beta)$ for $\alpha \in \Pi_k$ and $\beta \in \Pi_{d-k}$, so that, by duality

$$\frac{1}{\lambda_k^n}A^{n*}\beta \to c'(\beta)T^-$$

for any class $\beta \in \Pi_{d-k}$, with $c'(\beta) = \binom{d}{k}\langle T^+,\beta\rangle$. By Theorem 3.6, the currents T^+ and T^- induce classes in the universal cohomology of toric varieties, $\theta^+ \in \mathcal{H}^k$ and $\theta^- \in \mathcal{H}^{d-k}$, respectively.

Corollary 7.2. *Let A, κ, V_s, V_u be as in Theorem 6.1. Then there exists an integer r such that, for any complete simplicial fan Δ, and any classes $\omega \in H^{2k}(X(\Delta))$, $\eta \in H^{2(d-k)}(X(\Delta))$,*

$$\frac{1}{\lambda_k^n}(\phi_A^n)^*\omega\cdot\eta = \binom{d}{k}(\theta_\Delta^-\cdot\omega)\,(\theta_\Delta^+\cdot\eta) + \mathcal{O}(n^r\,\kappa^n).$$

Moreover, if L is an ample class in some projective toric variety, and $\omega = L^k$, respectively $\eta = L^{d-k}$, then $\langle\theta^+,\omega\rangle > 0$, respectively $\langle\theta^-,\eta\rangle > 0$.

In particular, $\frac{1}{\lambda_k^n}(\phi_A^n)^*\omega$, regarded as a class in $\underrightarrow{\mathrm{H}}^k$, converges towards $\binom{d}{k}(\theta^-\cdot\omega)\,\theta^+$ and by duality $\frac{1}{\lambda_k^n}(\phi_A^n)_*\eta$, regarded as a class in $\underrightarrow{\mathrm{H}}^{d-k}$ converges towards $\binom{d}{k}(\theta^+\cdot\eta)\,\theta^-$. This result is the analog of [BFJ, Corollary 3.6] in the context of monomial maps but in arbitrary dimensions.

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