# ON A REPRESENTATION OF THE FUNDAMENTAL CLASS OF AN IDEAL DUE TO LEJEUNE-JALABERT

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Abstract. Lejeune-Jalabert showed that the fundamental class of a Cohen-Macaulay ideal $\mathfrak{a} \subset \mathcal{O}_0$ admits a representation as a residue, constructed from a free resolution of $\mathfrak{a}$, of a certain differential form coming from the resolution. We give an explicit description of this differential form in the case where the free resolution is the Scarf resolution of a generic monomial ideal. As a consequence we get a new proof of Lejeune-Jalabert's result in this case.

## 1. Introduction

In [L-J] Lejeune-Jalabert showed that the fundamental class of a Cohen-Macaulay ideal $\mathfrak{a}$ in the ring of germs of holomorphic functions $\mathcal{O}_0$ at $0 \in \mathbf{C}^n$ admits a representation as a residue, constructed from a free resolution of $\mathfrak{a}$, of a certain differential form coming from the resolution, see also [L-J2, AL-J]. This representation generalizes the well-known fact that if $\mathfrak{a}$ is generated by a regular sequence $f = (f_1, \dots, f_n)$, then

$$\operatorname{res}_f(df_n \wedge \cdots \wedge df_1) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{df_n \wedge \cdots \wedge df_1}{f_n \cdots f_1} = \dim_{\mathbf{C}}(\mathcal{O}_0/\mathfrak{a}), \tag{1.1}$$

where $\operatorname{res}_f$ is the *Grothendieck residue* of $f_1, \dots, f_n$ and $\Gamma$ is the real $n$-cycle defined by $\{|f_j| = \epsilon\}$ for some $\epsilon$ such that $f_j$ are defined in a neighborhood of $\{|f_j| \leq \epsilon\}$ and oriented by $d(\arg f_1) \wedge \cdots \wedge d(\arg f_n) \geq 0$, see [GH, Chapter 5.2].

We will present a formulation of Lejeune-Jalabert's result in terms of currents. Recall that the *fundamental cycle* of $\mathfrak{a}$ is the cycle

$$[\mathfrak{a}] = \sum m_j [Z_j],$$

where $Z_j$ are the irreducible components of the variety $Z$ of $\mathfrak{a}$, and $m_j$ are the *geometric multiplicities* of $Z_j$ in $Z$, defined as the length of the Artinian ring $\mathcal{O}_{Z_j,Z}$, see, e.g., [F, Chapter 1.5]. In particular, if $Z = \{0\}$, then $[\mathfrak{a}] = \dim_{\mathbf{C}}(\mathcal{O}_0/\mathfrak{a})[\{0\}]$.

Assume that

$$0 \to E_p \xrightarrow{\varphi_p} \dots \xrightarrow{\varphi_2} E_1 \xrightarrow{\varphi_1} E_0 \tag{1.2}$$

is a free resolution of $\mathcal{O}_0/\mathfrak{a}$ of minimal length $p = \operatorname{codim} \mathfrak{a}$; here the $E_k$ are free $\mathcal{O}_0$-modules and $E_0 \cong \mathcal{O}_0$. In [AW] together with Andersson we constructed from (1.2) a (residue) current $R$, which has support on $Z$, takes values in $E_p$, is of bidegree $(0,p)$, and can be thought of as a current version of Lejeune-Jalabert's residue, cf. [LW2, Section 6.3]. Given bases of $E_k$, let $d\varphi_k$ be the $\operatorname{Hom}(E_k, E_{k-1})$-valued $(1,0)$-form with entries $(d\varphi_k)_{ij} = d(\varphi_k)_{ij}$ if $(\varphi_k)_{ij}$ are the entries of $\varphi_k$ and let $d\varphi$ denote the $E_p^*$-valued $(p,0)$-form

$$d\varphi := d\varphi_1 \wedge \cdots \wedge d\varphi_p.$$

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Now, identifying the fundamental cycle $[\mathfrak{a}]$ with the current of integration along $[\mathfrak{a}]$, see, e.g., [D, Chapter III.2.B], Theorem 1.1 in [LW2] states that $[\mathfrak{a}]$ admits the factorization

$$[\mathfrak{a}] = \frac{1}{p!(-2\pi i)^p} d\varphi \wedge R.^1 \tag{1.3}$$

This should be thought of as a current version of Lejeune-Jalabert's result in [L-J]; in particular, the differential form $d\varphi$ is the same form that appears in her paper. In fact, using residue theory the factorization (1.3) can be obtained from Lejeune-Jalabert's result and vice versa; for a discussion of this, as well as the formulation of Lejeune-Jalabert's result, we refer to Section 6.3 in [LW2]. In [LW2] we also give a direct proof of (1.3) that does not rely on [L-J] and that extends to pure-dimensional ideal sheaves.

Assume that $\mathfrak{a}$ is generated by a regular sequence $f = (f_1, \ldots, f_p)$ and let $E_\bullet, \varphi_\bullet$ be the associated Koszul complex, i.e., let $E$ be a free $\mathcal{O}_0$-module of rank $p$ with basis $e_1, \ldots, e_p$, let $E_k = \Lambda^k E$ with bases $e_{\mathcal{I}} = e_{i_k} \wedge \cdots \wedge e_{i_1}$, and let $\varphi_k$ be the contraction with $\sum f_j e_j^*$. Then

$$R = \bar\partial \frac{1}{f_1} \wedge \cdots \wedge \bar\partial \frac{1}{f_p} e_\emptyset^* \otimes e_{\{1,\ldots,p\}},$$

where $R_{CH}^f = \bar\partial(1/f_1) \wedge \cdots \wedge \bar\partial(1/f_p)$ is the classical *Coleff-Herrera residue current* of $f$, introduced in [CH], and $e_\emptyset$ denotes the basis element of $\Lambda^0 E \cong \mathcal{O}_0$. Moreover $d\varphi = p! df_p \wedge \cdots \wedge df_1 e_{\{1,\ldots,p\}}^* \otimes e_\emptyset$ and thus (1.3) reads

$$[\mathfrak{a}] = \frac{1}{(2\pi i)^p} \bar\partial \frac{1}{f_1} \wedge \cdots \wedge \bar\partial \frac{1}{f_p} \wedge df_p \wedge \cdots \wedge df_1. \tag{1.4}$$

This factorization of $[\mathfrak{a}]$ can be seen as a current version of (1.1) and also as a generalization of the classical Poincaré-Lelong formula

$$[f = 0] = \frac{1}{2\pi i} \bar\partial\partial \log |f|^2 = \frac{1}{2\pi i} \bar\partial \frac{1}{f} \wedge df, \tag{1.5}$$

where $[f = 0]$ is the current of integration along the zero set of $f$, counted with multiplicities. It appeared already in [CH], and in [DP] Demailly and Passare proved an extension to locally complete intersection ideal sheaves.

In [LW2] and [L-J], the factorization (1.3) is proved by comparing $R$ and $d\varphi$ to a residue and differential form, respectively, constructed from a certain Koszul complex; in [LW2] this is done using a recent comparison formula for residue currents due to Lärkäng, [L].

To explicitly describe the factors in (1.3), however, seems to be a delicate problem in general. In this note we compute the form $d\varphi$ when $E_\bullet, \varphi_\bullet$ is a certain resolution of a monomial ideal. More precisely, let $A$ be the ring $\mathcal{O}_0$ of holomorphic germs at the origin in $\mathbf{C}^n$ with coordinates $z_1, \ldots, z_n$, or let $A$ be the polynomial ring $\mathbf{C}[z_1, \ldots, z_n]$. We then give an explicit description of the form

$$d\varphi = \sum_\sigma \frac{\partial \varphi_1}{\partial z_{\sigma(1)}} dz_{\sigma(1)} \wedge \cdots \wedge \frac{\partial \varphi_n}{\partial z_{\sigma(n)}} dz_{\sigma(n)} \tag{1.6}$$

when $E_\bullet, \varphi_\bullet$ is the *Scarf resolution*, introduced in [BPS], of an Artinian, i.e., zero-dimensional, *generic* monomial ideal $M$ in $A$, see Section 3 for definitions. Here the sum is over all permutations $\sigma$ of $\{1, \ldots, n\}$. It turns out that each summand in (1.6) is a vector of monomials (times $dz_n \wedge \cdots \wedge dz_1$) whose coefficients have a neat description

[1] For a discussion of the sign in (1.3), see Section 2.6 in [LW2].

in terms of the so-called staircase of $M$ and sum up to the geometric multiplicity of $M$, see Theorem 1.1 below. This can be seen as a far-reaching generalization of the fact that the coefficient of $d(z^a)$ equals $a$, which is the geometric multiplicity of the principal ideal $(z^a)$, cf. Example 5.1 below. Thus, in a sense, the fundamental class of $M$ is captured already by the form $d\varphi$. In the case of the Scarf resolution we recently, together with Lärkäng, [LW], gave a complete description of the current $R$. Combining Theorem 1.1 below with Theorem 1.1 in [LW] we obtain a new proof of (1.3) in this case, cf. Corollary 1.2 below.

Let us describe our result in more detail. Let $M$ be an Artinian monomial ideal in $A$. By the *staircase* $S = S_M$ of $M$ we mean the set

$$S = \overline{\{(x_1, \dots, x_n) \in \mathbf{R}^n_{>0} | z_1^{\lfloor x_1 \rfloor} \cdots z_n^{\lfloor x_n \rfloor} \notin M\}} \subset \mathbf{R}^n_{>0}. \tag{1.7}$$

Here $\lfloor x \rfloor$ denotes the largest integer $\leq x$. The name is motivated by the shape of $S$, cf. Figures 5.1, 5.2, and 6.1. We will refer to the finitely many maximal elements in $S$, with respect to the standard partial order on $\mathbf{R}^n$, as *outer corners*.

The Scarf resolution $E_\bullet, \varphi_\bullet$ of $M$ is encoded in the *Scarf complex*, $\Delta_M$, which is a labeled simplicial complex of dimension $n-1$ with one vertex for each minimal monomial generator of $M$ and one top-dimensional simplex for each outer corner of $S$, see Section 3. The rank of $E_k$ equals the number of $(k-1)$-dimensional simplices in $\Delta_M$. In particular, $E_\bullet, \varphi_\bullet$ ends at level $n$ and the rank of $E_n$ equals the number of outer corners of $S$. Thus $d\varphi$ is a vector with one entry for each outer corner of $S$.

For our description of $d\varphi$ we need to introduce certain partitions of $S$. Given a permutation $\sigma$ of $\{1, \dots, n\}$ let $\geq_\sigma$ be the lexicographical order induced by $\sigma$, i.e, $\alpha = (\alpha_1, \dots, \alpha_n) \geq_\sigma \beta = (\beta_1, \dots, \beta_n)$ if for some $1 \leq k \leq n$, $\alpha_{\sigma(\ell)} = \beta_{\sigma(\ell)}$ for $1 \leq \ell \leq k-1$ and $\alpha_{\sigma(k)} > \beta_{\sigma(k)}$, or $\alpha = \beta$. If $\alpha \geq_\sigma \beta$ and $\alpha \neq \beta$ we write $\alpha >_\sigma \beta$. Let $\alpha^1 \geq_\sigma \cdots \geq_\sigma \alpha^k \geq_\sigma \cdots$ be the total ordering of the outer corners induced by $\geq_\sigma$, and define inductively

$$\begin{aligned} S_{\sigma,\alpha^1} &= \{x \in S \mid x \leq \alpha_1\} \\ &\vdots \\ S_{\sigma,\alpha^k} &= \{x \in S \setminus (S_{\sigma,\alpha^1} \cup \cdots \cup S_{\sigma,\alpha^{k-1}}) \mid x \leq \alpha_k\} \\ &\vdots \end{aligned}$$

For a fixed $\sigma$, $\{S_{\sigma,\alpha}\}_\alpha$ provides a partition of $S$, cf. Section 2.

**Theorem 1.1.** *Let $M$ be an Artinian generic monomial ideal in $A$, and let*

$$0 \to E_n \xrightarrow{\varphi_n} \dots \xrightarrow{\varphi_2} E_1 \xrightarrow{\varphi_1} E_0 \tag{1.8}$$

*be the Scarf resolution of $A/M$. Then*

$$d_\sigma\varphi := \frac{\partial \varphi_1}{\partial z_{\sigma(1)}} dz_{\sigma(1)} \wedge \cdots \wedge \frac{\partial \varphi_n}{\partial z_{\sigma(n)}} dz_{\sigma(n)} \tag{1.9}$$

*has one entry $(d_\sigma\varphi)_\alpha$ for each outer corner $\alpha$ of the staircase $S$ of $M$ and*

$$(d_\sigma\varphi)_\alpha = \operatorname{sgn}(\alpha) \operatorname{Vol}(S_{\sigma,\alpha}) z^{\alpha - \mathbf{1}} dz, \tag{1.10}$$

*where $\operatorname{sgn}(\alpha) = \pm 1$ comes from the orientation of the Scarf complex, $dz = dz_n \wedge \cdots \wedge dz_1$, and $z^{\alpha-\mathbf{1}} = z_1^{\alpha_1 - 1} \cdots z_n^{\alpha_n - 1}$ if $\alpha = (\alpha_1, \dots, \alpha_n)$.*

The sign $\operatorname{sgn}(\alpha)$ will be specified in Section 3.1 below.

Theorem 5.1 in [LW] asserts that the residue current associated with the Scarf resolution has one entry

$$R_\alpha = \operatorname{sgn}(\alpha)\ \bar\partial\frac{1}{z_1^{\alpha_1}} \wedge \cdots \wedge \bar\partial\frac{1}{z_n^{\alpha_n}} \tag{1.11}$$

for each outer corner $\alpha$ of $S$. Since $(1/2\pi i)\bar\partial(1/z^a) \wedge z^{a-1}dz = [z = 0]$, cf. (1.5), we conclude from (1.10) and (1.11) that

$$\frac{1}{(-2\pi i)^n} d_\sigma\varphi \wedge R = \sum_\alpha \operatorname{Vol}(S_{\sigma,\alpha})[0] = \operatorname{Vol}(S)[0].$$

Note that $\operatorname{Vol}(S)$ equals the number of monomials that are not in $M$. Since these monomials form a basis for $A/M$, $\operatorname{Vol}(S)$ equals the geometric multiplicity $\dim_{\mathbf{C}}(A/M)$ of $M$. Thus we get the following version of (1.3).

**Corollary 1.2.** *Let $M$ and* (1.8) *be as in Theorem 1.1 and let $R$ be the associated residue current. Then*

$$\frac{1}{(-2\pi i)^n}\frac{\partial\varphi_1}{\partial z_{\sigma(1)}}dz_{\sigma(1)} \wedge \cdots \wedge \frac{\partial\varphi_n}{\partial z_{\sigma(n)}}dz_{\sigma(n)} \wedge R = [M]. \tag{1.12}$$

Summing over all permutations $\sigma$ we get back (1.3). In fact, as was recently pointed out to us by Jan Stevens, if (1.8) is any free resolution of an Artinian ideal and $R$ is the associated residue current, then the left hand side of (1.12) is independent of $\sigma$, see Proposition 6.3.

The core of the proof of Theorem 1.1 is an alternative description of the $S_{\sigma,\alpha}$ as certain cuboids, see Lemma 4.1. Given this description it is fairly straightforward to see that the volumes of the $S_{\sigma,\alpha}$ are precisely the coefficients of the monomials in $d_\sigma\varphi$; this is done in Section 4.2.

We suspect that Theorem 1.1 extends to a more general setting than the one above. If $M$ is an Artinian non-generic monomial ideal, we can still construct the partitions $\{S_{\sigma,\alpha}\}_\alpha$. The elements $S_{\sigma,\alpha}$ will, however, no longer be cuboids in general. Also, the computation of $d\varphi$ is more delicate in general. In Example 6.1 we compute $d\varphi$ for a non-generic monomial ideal for which the *hull resolution*, introduced in [BS], is minimal, and show that Theorem 1.1 holds in this case. On the other hand, in Example 6.2 we consider a monomial ideal for which the hull resolution is not a minimal resolution and where Theorem 1.1 fails to hold.

The paper is organized as follows. In Section 2 and 3 we provide some background on staircases of monomial ideals and the Scarf complex, respectively. The proof of Theorem 1.1 occupies Section 4 and in Section 5 we illustrate the theorem and its proof by some examples. Finally, in Section 6 we consider resolutions of non-generic Artinian monomial ideals and look at some examples. We also show that the left hand side of (1.12) is independent of $\sigma$ in general.

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## 2. Staircases

We let $\geq$ denote the standard partial order on $\mathbf{R}^n$, i.e., $a = (a_1, \dots, a_n) \geq b = (b_1, \dots, b_n)$ if $a_\ell \geq b_\ell$ for $\ell = 1, \dots, n$. If $a \geq b$ and $a \neq b$ we write $a > b$. If $a_\ell > b_\ell$ for all $\ell$ we write $a \succ b$. Throughout we let $A = A_n$ denote the ring $\mathbf{C}[z_1, \dots, z_n]$ or the ring $\mathcal{O}_0$ of holomorphic germs at $0 \in \mathbf{C}^n_{z_1,\dots,z_n}$. For $a = (a_1, \dots, a_n) \in \mathbf{N}^n$, where $\mathbf{N} = 0, 1, \dots$, we use the shorthand notation $z^a$ for the monomial $z_1^{a_1} \cdots z_n^{a_n}$ in $A$. For a general reference on (resolutions of) monomial ideals, see, e.g., [MS].

Unless otherwise stated $M$ will be a monomial ideal in $A$, i.e., an ideal generated by monomials, and $S$ will be the staircase of $M$ as defined in (1.7). Note that $M$ is Artinian if and only if there are generators of the form $z_i^{a_i}$, $a_i > 0$, for $i = 1, \dots, n$, which is equivalent to that $S \subset \{x \in \mathbf{R}^n_{>0} \mid x_i \leq a_i, i = 1, \dots, n\}$ for some $a_i$, which in turn is equivalent to that $S$ is bounded. Recall that the closure in (1.7) is taken in $\mathbf{R}^n_{>0}$; we will however often consider $S$ as a subset of $\mathbf{R}^n$. As in the introduction we will refer to the maximal elements of $S$ as *outer corners*. The minimal elements of $\overline{\mathbf{R}^n_{>0} \setminus S} \subset \mathbf{R}^n$ we will call *inner corners*. Unless otherwise mentioned the closure $\overline{A}$ of a set $A$ is taken in $\mathbf{R}^n$.

One can check that for any monomial ideal $M$ there is a unique minimal set of exponents $B \subset \mathbf{N}^n$ such that the monomials $\{z^a\}_{a \in B}$ generate $M$. We refer to these monomials as *minimal monomial generators* of $M$. Moreover

$$S = \mathbf{R}^n_{>0} \setminus \bigcup_{a \in B} (a + \mathbf{R}^n_{>0}). \tag{2.1}$$

In particular, the inner corners of $S$ are precisely the elements in $B$.

Dually, $M$ can be described as an intersection of so-called *irreducible* monomial ideals, i.e., ideals generated by powers of variables; such an ideal can be described as $\mathfrak{m}^\alpha := (z_i^{\alpha_i} | \alpha_i \geq 1)$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$. (More generally an ideal is irreducible if it cannot be written as a non-trivial intersection of two ideals.) For every monomial ideal $M$ there is a unique minimal set $C \subset \mathbf{N}^n$ such that

$$M = \bigcap_{\alpha \in C} \mathfrak{m}^\alpha, \tag{2.2}$$

see, e.g., [MS, Theorem 5.27]. The ideal $M$ is Artinian if and only if each $\alpha \in C$ satisfies $\alpha \succ 0$ (i.e., $\alpha_\ell > 0$ for each $\ell$). If $\alpha \succ 0$, then note that a monomial $z^b \notin \mathfrak{m}^\alpha$ if and only if $b \prec \alpha$. It follows that, if $M$ is Artinian, then

$$S = \bigcup_{\alpha \in C} \{x \in \mathbf{R}^n_{>0} \mid x \leq \alpha\}. \tag{2.3}$$

In particular, the outer corners of $S$ are precisely the elements in $C$. If $M$ is not Artinian, then $S$ is not bounded in $\mathbf{R}^n$ and the representation (2.3) fails to hold. Note that (2.3) guarantees that for a fixed $\sigma$, $\{S_{\sigma,\alpha}\}_\alpha$, as defined in the introduction, is a partition of $S$.

Inspired by (2.1) we will call any set of this form a staircase: Let $H$ be an affine subspace of $\mathbf{R}^n$ of the form

$$H = \{x_{\ell_1} = a_1, \dots, x_{\ell_k} = a_k\}$$

where $a_j \in \mathbf{Z}$. For $a \in \mathbf{Z}^n \cap H$ let $U_a = \{x \in H \mid x \succ a\}$. Note that $U_0^{\mathbf{Z}^n}$ is just the first (open) orthant $\mathbf{R}^n_{>0}$ in $\mathbf{R}^n$. We say that a set $S \subset H$ is a *staircase* if it is of the form

$$S = U_{a^0} \setminus \bigcup_{j=1}^{s} U_{a^j}$$

for some $a^0, a^1, \dots, a^s \in \mathbf{Z}^n \cap H$. We say that $a_0$ is the *origin* of $S$ and if $a^1, \dots, a^s$ are chosen so $a^1, \dots, a^s \geq a^0$ and $a_j \not\leq a_k$ for $j \neq k$, $1 \leq j, k \leq s$ we call them the *inner corners* of $S$. We call the maximal elements of $S$ the *outer corners* of $S$. Since $\mathbf{Z}^n \cap H$ is a lattice, the outer corners are in $\mathbf{Z}^n \cap H$. Note that $S$ is a closed subset of $U_{a^0}$.

If $\pi$ is the projection $\mathbf{R}^n \to \mathbf{R}^{n-k}$ that maps $(x_1, \dots, x_n)$ to $(x_{j_1}, \dots, x_{j_{n-k}})$ if $\{j_1, \dots, j_{n-k}\} = \{1, \dots, n\} \setminus \{\ell_1, \dots, \ell_k\}$ and $\rho : \mathbf{R}^n \to \mathbf{R}^{n-k}$ is the affine map $\rho(x) = \pi(x - a^0)$, let $M(S) = M_\rho(S)$ be the monomial ideal in $A_k$ that is generated by $z^{\rho(a^j)}$, where $a^j$ are the inner corners of $S$. Then the staircase of $M(S)$ equals $\rho(S)$.

For $\alpha \in \mathbf{Z}^n \cap H$, let $V_\alpha = \{x \in H \mid x \leq \alpha\}$. If $M(S)$ is Artinian, then $S$ admits a representation analogous to (2.3),

$$S = U_{a^0} \cap \bigcup_\alpha V_\alpha, \tag{2.4}$$

where the union is taken over all outer corners of $S$.

Note that any set $S$ of the form (2.4) with $a^0, \alpha \in \mathbf{Z}^n \cap H$ is a staircase; indeed since $\mathbf{Z}^n \cap H$ is a lattice the minimal elements of $\overline{U_{a^0} \setminus S}$ are in $\mathbf{Z}^n \cap H$.

## 3. The Scarf complex

For $a, b \in \mathbf{R}^n$, we will denote by $a \vee b$ the *join* of $a$ and $b$, i.e., the unique $c$ such that $c \geq a, b$, and $c \leq d$ for all $d \geq a, b$.

Let $M$ be an Artinian monomial ideal in $A$, with minimal monomial generators $m_1 = z^{a^1}, \dots, m_r = z^{a^r}$. The *Scarf complex* $\Delta = \Delta_M$ of $M$ was introduced by Bayer-Peeva-Sturmfels, [BPS], based on previous work by H. Scarf. It is the collection of subsets $\mathcal{I} = \{i_1, \dots, i_k\} \subset \{1, \dots, r\}$ whose corresponding least common multiple $m_\mathcal{I} := \operatorname{lcm}(m_{i_1}, \dots, m_{i_k}) = z^{a^{i_1} \vee \dots \vee a^{i_k}}$ is unique, that is,

$$\Delta = \{\mathcal{I} \subset \{1, \dots, r\} | m_\mathcal{I} = m_{\mathcal{I}'} \Rightarrow \mathcal{I} = \mathcal{I}'\}.$$

Clearly the vertices of $\Delta$ are the minimal monomial generators of $M$, i.e., the inner corners of $S_M$. One can prove that the Scarf complex is a simplicial complex of dimension at most $n - 1$. We let $\Delta(k)$ denote the set of simplices in $\Delta$ with $k$ vertices, i.e., of dimension $k - 1$. Moreover we label the faces $\mathcal{I} \subset \Delta$ by the monomials $m_\mathcal{I}$. We will sometimes be sloppy and identify the faces in $\Delta$ with their labels or exponents of the labels and write $m_\mathcal{I}$ or $\alpha$ for the face with label $m_\mathcal{I} = z^\alpha$ and $\{z^{a^{i_1}}, \dots, z^{a^{i_k}}\}$ or $\{a^{i_1}, \dots, a^{i_k}\}$ for $\mathcal{I} = \{i_1, \dots, i_k\}$.

The ideal $M$ is said to be *generic* in the sense of [BPS, MSY] if whenever two distinct minimal generators $m_i$ and $m_j$ have the same positive degree in some variable, then there is a third generator $m_k$ that *strictly divides* $m_{\{i,j\}}$, which means that $m_k$ divides $m_{\{i,j\}}/z_\ell$ for all variables $z_\ell$ dividing $m_{\{i,j\}}$. In particular, $M$ is generic if no two generators have the same positive degree in any variable.

If $M$ is generic, then $\Delta$ has precisely dimension $n-1$; it is a regular triangulation of the $(n-1)$-dimensional simplex, see [BPS, Corollary 5.5]. The (labels of the) top-dimensional faces of $\Delta$ are precisely the exponents $\alpha$ in the minimal irreducible decomposition (2.2) of $M$, i.e., the outer corners of $S_M$, see [BPS, Theorem 3.7].

For $k = 0, \dots, n$, let $E_k$ be the free $A$-module with basis $\{e_\mathcal{I}\}_{\mathcal{I} \in \Delta(k)}$ and let the differential $\varphi_k : E_k \to E_{k-1}$ be defined by

$$\varphi_k : e_\mathcal{I} \mapsto \sum_{j=1}^k (-1)^{j-1} \frac{m_\mathcal{I}}{m_{\mathcal{I}_j}} e_{\mathcal{I}_j}, \tag{3.1}$$

where $\mathcal{I}_j$ denotes $\{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_k\}$ if $\mathcal{I} = \{i_1, \dots, i_k\}$. Then the complex $E_\bullet, \varphi_\bullet$ is exact and thus gives a free resolution of the cokernel of $\varphi_0$, which with the identification $E_0 = A$ equals $A/M$, see [BPS, Theorem 3.2]. In fact, this so-called *Scarf resolution* is a minimal resolution of $A/M$, i.e., for each $k$, $\varphi_k$ maps a basis of $E_k$ to a minimal set of generators of $\operatorname{Im}\varphi_k$, see, e.g., [E, Corollary 1.5]. Originally, in [BPS], the situation $A = \mathbf{C}[z_1, \dots, z_n]$ was considered. However, since $\mathcal{O}_0$ is flat over $\mathbf{C}[z_1, \dots, z_n]$, see, e.g., [T, Theorem 13.3.5], the complex $E_\bullet, \varphi_\bullet$ is exact for $A = \mathcal{O}_0$ if and only if it is exact for $A = \mathbf{C}[z_1, \dots, z_n]$.

3.1. **The sign** $\operatorname{sgn}(\alpha)$**.** Let $\mathcal{I} = \{i_1, \dots, i_n\}$ be a top-dimensional simplex in $\Delta$ with label $\alpha$. Then there is a unique permutation $\eta = \eta(\alpha)$ of $\{1, \dots, n\}$ such that for each $1 \le \ell \le n$ $i_{\eta(\ell)}$ is the unique vertex of $\mathcal{I}$ such that $\alpha_\ell = a_\ell^{i_{\eta(\ell)}}$; we will refer to this vertex as the *$x_\ell$-vertex* of $\mathcal{I}$. To see this, first of all, since $\alpha = \operatorname{lcm}(a^i)$, $a^i_\ell \le \alpha_\ell$ and therefore there must be at least one vertex $i$ of $\mathcal{I}$ such that $a^i_\ell = \alpha_\ell$. Assume that $i$ and $j$ are vertices of $\mathcal{I}$ such that $a^i_\ell = a^j_\ell = \alpha_\ell$. Then, since $M$ is generic, there is a generator $z^b$ of $M$ that strictly divides $z^{a^i \vee a^j}$. But then $a^i \vee a^j = a^i \vee a^j \vee b$ and so $\{i, j\}$ is not in $\Delta$, which contradicts that $i$ and $j$ are both vertices of $\mathcal{I}$.

We let $\operatorname{sgn}(\alpha)$ denote the sign of the permutation $\eta$. This is the sign that appears in (1.10) in Theorem 1.1 as well as in (1.11). We should remark that we use a different sign convention in this paper than in [LW, LW2], which corresponds to a different orientation of the $(n-1)$-simplex or, equivalently, to a different choice of bases for the modules $E_k$, cf. [AW, LW2].

3.2. **The subcomplex** $\Delta_{\sigma, a^1, \dots, a^k}$**.** Given a permutation $\sigma$ of $\{1, \dots, n\}$, and vertices $a^1, \dots, a^k$ of $\Delta$, let $\Delta_{\sigma, a^1, \dots, a^k}$ be the (possibly empty) subcomplex of $\Delta$, with top-dimensional simplices $\alpha$ that satisfy $\alpha_{\sigma(\ell)} = a^\ell_{\sigma(\ell)}$ for $\ell = 1, \dots, k$. In other words, the top-dimensional simplices in $\Delta_{\sigma, a^1, \dots, a^k}$ are the ones that have $a^\ell$ as $x_{\sigma(\ell)}$-vertex for $\ell = 1, \dots, k$.

Note that for each choice of permutation $\sigma$, $k \in \{1, \dots, n\}$, and $\alpha \in \Delta(n)$ there is a unique sequence $a^1, \dots, a^k$ such that $\alpha \in \Delta_{\sigma, a^1, \dots, a^k}$. Moreover, $\Delta_{\sigma, a^1, \dots, a^n}$ is the unique simplex in $\Delta(n)$ that satisfies $a^\ell_{\sigma(\ell)} = \alpha_{\sigma(\ell)}$ for $\ell = 1, \dots, n$ if $\alpha$ is the label of $\Delta_{\sigma, a^1, \dots, a^n}$.

We will write $\Delta^*_{\sigma, a^1, \dots, a^k}$ for the subcomplex of $\Delta_{\sigma, a^1, \dots, a^k}$ consisting of all faces in $\Delta_{\sigma, a^1, \dots, a^k}$ that do not contain $a^1, \dots, a^{k-1}$, or $a^k$. Note that since $\Delta_{\sigma, a^1, \dots, a^k}$ is simplicial, $\{b^1, \dots, b^\ell\}$ is a face of $\Delta^*_{\sigma, a^1, \dots, a^k}$ if and only $\{a^1, \dots, a^k, b^1, \dots, b^\ell\}$ is a face of $\Delta_{\sigma, a^1, \dots, a^k}$.

## 4. Proof of Theorem 1.1

To prove the theorem we will first give an alternative description of the $S_{\sigma,\alpha}$ as certain cuboids. Throughout this section we will assume that $E_\bullet, \varphi_\bullet$ is the Scarf resolution of a generic Artinian monomial ideal $M$ and we will use the notation from above.

**Lemma 4.1.** *Assume that $\alpha$ is the label of the face $\mathcal{I} = \{i_1, \dots, i_n\} \in \Delta(n)$. Let $\eta$ be the permutation of $\{1, \dots, n\}$ associated with $\mathcal{I}$ as in Section 3.1, and set $\tau = \eta \circ \sigma$.*

*Then $S_{\sigma,\alpha}$ is a cuboid with side lengths*

$$a^{i_{\tau(1)}}_{\sigma(1)}, \big(a^{i_{\tau(1)}} \vee a^{i_{\tau(2)}} - a^{i_{\tau(1)}}\big)_{\sigma(2)}, \dots, \big(a^{i_{\tau(1)}} \vee \cdots \vee a^{i_{\tau(n)}} - a^{i_{\tau(1)}} \vee \cdots \vee a^{i_{\tau(n-1)}}\big)_{\sigma(n)}.$$

In particular,

$$\operatorname{Vol}(S_{\sigma,\alpha}) = a^{i_{\tau(1)}}_{\sigma(1)} \times \big(a^{i_{\tau(1)}} \vee a^{i_{\tau(2)}} - a^{i_{\tau(1)}}\big)_{\sigma(2)} \times \cdots \times \big(a^{i_{\tau(1)}} \vee \cdots \vee a^{i_{\tau(n)}} - a^{i_{\tau(1)}} \vee \cdots \vee a^{i_{\tau(n-1)}}\big)_{\sigma(n)}.$$

4.1. **Proof of Lemma 4.1.** To prove the lemma, let us first assume that $\sigma$ is the identity permutation and write $S_\alpha = S_{\sigma,\alpha}$, $\Delta_{a^1,\dots,a^k} = \Delta_{\sigma,a^1,\dots,a^k}$ (so that $\Delta_{a^1,\dots,a^k}$ is the subcomplex of $\Delta$ whose top-dimensional simplices $\alpha$ satisfy $\alpha_\ell = a^\ell_\ell$ for $\ell = 1,\dots,k$), and $\Delta^*_{a^1,\dots,a^k} = \Delta^*_{\sigma,a^1,\dots,a^k}$.

We will decompose $S$ in a seemingly different way. First we will construct certain lower-dimensional staircases with corners in $\Delta$. Given an inner corner $a = (a_1,\dots,a_n)$ of $S$ let

$$T_a = \{x \in S \mid x_1 = a_1, x_\ell > a_\ell, \ell = 2,\dots,n\}. \tag{4.1}$$

Note that $T_a$ is contained in the boundary $\partial S$ of $S$. Let $H_a$ be the hyperplane $\{x_1 = a_1\}$, and let $U_a$ be defined as in Section 2. Then note that $T_a = U_a \cap S \subset H_a \cap S$.

**Claim 4.2.** *Let $a$ be an inner corner of $S$. Then $T_a = \emptyset$ if and only if $a_1 = 0$. If $T_a$ is non-empty, then it is a staircase in the hyperplane $H_a$ with origin $a$. The outer corners of $T_a$ are the top-dimensional faces of $\Delta_a$. The inner corners are the lattice points $a \vee b$, where $b$ is a vertex in $\Delta^*_a$.*

*Proof.* If $a_1 = 0$, then $H_a \cap S = \emptyset$, and thus $T_a$ is empty. In general, note that $T_a$ is empty exactly if $a + (0,1,\dots,1) \notin S$. If $a_1 > 0$, so that $a + (0,1,\dots,1) \in \mathbf{R}^n_{>0}$, this means that there is an inner corner $c$ of $S$ such that $c \prec a + (0,1,\dots,1)$. In particular, $c \le a$, which contradicts that $a$ is an inner corner. Thus $T_a \neq \emptyset$ if $a_1 > 0$.

Assume that $T_a$ is non-empty and that $\beta = (\beta_1,\dots,\beta_n)$ is maximal in $T_a$. Since $T_a \subset H_a$, $\beta_1 = a_1$. Moreover, since $S$ is of the form (2.3), there is a maximal $\gamma \in S$ such that $\gamma \ge \beta$. By the definition of $T_a$, $\gamma_\ell \ge \beta_\ell > a_\ell$ for $\ell = 2,\dots,n$ and thus if $\gamma_1 > a_1$, then $\gamma \succ a$, which contradicts that $\gamma \in S$. Hence $\gamma_1 = a_1$ and, since $\gamma_\ell > a_\ell$ for $\ell = 2,\dots,n$, $\gamma \in T_a$. Since $\beta$ is maximal in $T_a$, $\gamma = \beta$, which means that, in fact, $\beta$ is maximal in $S$ and thus $\beta \in \Delta(n)$. Since $\beta_1 = a_1$, $\beta \in \Delta_a(n)$ by the definition of $\Delta_a$.

On the other hand, if $\beta \in \Delta_a(n)$, then $\beta$ is maximal in $S$ and contained in $T_a \subset S$, and thus it is maximal in $T_a$. We conclude that the maximal elements in $T_a$ are the top-dimensional faces of $\Delta_a$.

Since $S$ is of the form (2.3),

$$T_a = U_a \cap \bigcup_{\beta \in \Delta_a(n)} \{x \mid x \le \beta\},$$

which in light of (2.4) means that $T_a$ is a staircase in $H_a$ with origin $a$ and outer corners $\Delta_a(n)$.

It remains to describe the inner corners of $T_a$. Assume that $\beta = (a_1,\beta_2,\dots,\beta_n)$ is an inner corner of $T_a$, i.e., $\beta$ is minimal in $\overline{U_a \setminus S}$. This means that any $\tilde\beta$, such that $\tilde\beta_\ell > \beta_\ell$ for $\ell = 2,\dots,n$, is not contained in $S$, which implies that there is an inner corner $b$ of $S$ such that $b \prec \tilde\beta$ for any such $\tilde\beta$. In particular, $b \le \beta$ and $b_1 < a_1$. To conclude, there is an inner corner $b \neq a$ of $S$ such that $b \le \beta$. Now $a \vee b \in \overline{U_a \setminus S}$, since $a \vee b \ge a$, $(a \vee b)_1 = a_1$, and $z^{a\vee b} \in M$. Moreover, $a \vee b \le a \vee \beta = \beta$, since $b \le \beta$, and, since $\beta$ by assumption is minimal in $\overline{U_a \setminus S}$, it follows that $\beta = a \vee b$. Now, since $T_a$ is a staircase there is a maximal $\alpha \in \Delta_a(n)$ such that $b \le \beta \le \alpha$. It follows that $b$ is a vertex of $\alpha$, i.e., $b \in \Delta^*_a(1)$.

Conversely, pick $b \in \Delta^*_a(1)$ and let $\beta = a \vee b$. Since $a$ and $b$ are inner corners of $S$, $z^\beta \in M$ and thus $\beta \in \overline{\mathbf{R}^n_{>0} \setminus S}$. Moreover, since $b \in \Delta_a$, $b_1 \le a_1$, so that $\beta_1 = (a \vee b)_1 = a_1$, and thus $\beta \in \overline{U_a \setminus S}$. Assume that there is a $\gamma \in \overline{U_a \setminus S}$ such that $\gamma \le \beta$. Then, as above, there is an inner corner $c \neq a$ of $S$ such that $c \prec \gamma + (0,1,\dots,1)$

and $\gamma = a \vee c$. Now the inner corners $a, b, c$ of $S$ satisfy $a \vee b \vee c = a \vee b$. Since $b \in \Delta_a$, $\{a, b\}$ is an edge of $\Delta$, which means that $\{z^a, z^b\}$ is the unique set of minimal generators with least common multiple $z^{a \vee b}$. It follows that $c \in \{a, b\}$ and since $c \neq a$, we have that $c = b$, and thus $\gamma = \beta$. Hence $\beta$ is minimal in $\overline{U_a \setminus S}$. We conclude that the inner corners of $T_a$ are exactly the lattice points $a \vee b$, where $b \in \Delta^*_a(1)$.

□

Let $\pi : \mathbf{R}^n_{x_1,\ldots,x_n} \to \mathbf{R}^{n-1}_{x_2,\ldots,x_n}$ be the projection $\pi : (x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n)$ and let $\rho : \mathbf{R}^n_{x_1,\ldots,x_n} \to \mathbf{R}^{n-1}_{x_2,\ldots,x_n}$ be the affine mapping defined by $\rho(x) = \pi(x - a)$. Let $M_a$ be the monomial ideal in $A_{n-1}$ (with variables $z_2, \ldots, z_n$) defined by $T_a$ and $\rho$ as in Section 2.

**Claim 4.3.** *$M_a$ is a generic Artinian monomial ideal. The Scarf complex $\Delta_{M_a}$ consists of faces of the form $\{\rho(a \vee b^1), \ldots, \rho(a \vee b^j)\}$, where $\{b^1, \ldots, b^j\}$ is a face of $\Delta^*_a$.*

*Proof.* Since $S$ is bounded, $T_a \subset S$ is bounded, and thus $M_a$ is Artinian.

To show that $M_a$ is generic, assume that there are two minimal generators $z^\beta$ and $z^\gamma$ that have the same positive degree in some variable, i.e., $\beta_\ell = \gamma_\ell$ for some $2 \leq \ell \leq n$. Assume that $\beta = \rho(a \vee b)$ and $\gamma = \rho(a \vee c)$, where $b, c \in \Delta^*_a(1)$. Then $\beta_\ell = (a \vee b)_\ell - a_\ell$ and $\gamma_\ell = (a \vee c)_\ell - a_\ell$, and thus $\beta_\ell = \gamma_\ell > 0$ implies that $(a \vee b)_\ell = (a \vee c)_\ell > a_\ell$, which in turn implies that $b_\ell = c_\ell$. Since $M$ is generic there is a minimal generator $z^d$ of $M$ that strictly divides $\mathrm{lcm}(z^b, z^c)$, i.e., $d_k < (b \vee c)_k$ for all $k$ such that $(b \vee c)_k > 0$. In particular, $d_1 < (b \vee c)_1 = a_1$. Set $\delta = \rho(a \vee d)$ and take $k$ such that

$$0 < (\beta \vee \gamma)_k = \big((a \vee b - a) \vee (a \vee c - a)\big)_k = (a \vee b \vee c - a)_k.$$

Then $(b \vee c)_k > a_k \geq 0$ and thus $d_k < (b \vee c)_k$. This implies that

$$\delta_k = (d \vee a - a)_k < (a \vee b \vee c - a)_k = (\beta \vee \gamma)_k.$$

It follows that $z^\delta$ strictly divides $\mathrm{lcm}(z^\beta, z^\gamma) = z^{\beta \vee \gamma}$. Now $a \vee d \in \overline{U_a \setminus S}$, since $a \vee d \geq a$, $(a \vee d)_1 = a_1$ and $z^{a \vee d} \in M$. Hence $z^\delta \in M_a$ by definition. To conclude, $M_a$ is a generic monomial ideal.

Since $M_a$ is generic, the Scarf complex $\Delta_{M_a}$ is simplicial and the top-dimensional faces are precisely the outer corners of the staircase of $M_a$, i.e., simplices of the form $\{\rho(a \vee b^1), \ldots, \rho(a \vee b^{n-1})\}$, where $\{b^1, \ldots, b^{n-1}\}$ is in $\Delta^*_a(n-1)$. Since $\Delta_{M_a}$ and $\Delta_a$ are simplicial, it follows that the faces of $\Delta_{M_a}$ are of the desired form. □

**Claim 4.4.** *Assume that $a \neq b$ are inner corners of $S$. Then $\pi(T_a) \cap \pi(T_b) = \emptyset$.*

In particular, it follows that

$$T_a \cap T_b = \emptyset \text{ if } a \neq b. \tag{4.2}$$

*Proof.* Let us first assume that $a_1 \neq b_1$ and that $\pi(T_a) \cap \pi(T_b) \neq \emptyset$; without loss of generality we may assume that $a_1 > b_1$. Pick $\beta = (\beta_2, \ldots, \beta_n) \in \pi(T_a) \cap \pi(T_b)$ and let $\gamma = (a_1, \beta_2, \ldots, \beta_n)$. Since $\beta \in \pi(T_b)$, $\beta_j > b_j$ for $j = 2, \ldots, n$ and since $a_1 > b_1$ it follows that $\gamma \succ b$. Hence $\gamma$ is in the interior of $\mathbf{R}^n_{>0} \setminus S$. On the other hand, $\beta \in \pi(T_a)$ implies that $\gamma \in T_a \subset \partial S$, which contradicts that $\gamma$ is in the interior of $\mathbf{R}^n_{>0} \setminus S$. Thus $\pi(T_a) \cap \pi(T_b) = \emptyset$ if $a_1 \neq b_1$.

Next, assume that $a_1 = b_1 > 0$ and that $\pi(T_a) \cap \pi(T_b) \neq \emptyset$; if $a_1 = 0$ or $b_1 = 0$, the claim is trivially true by Claim 4.2. Since $T_a$ and $T_b$ are both contained in the hyperplane $H_a$, $\pi(T_a) \cap \pi(T_b) \neq \emptyset$ is equivalent to that $T_a \cap T_b \neq \emptyset$. Assume that $\beta \in T_a \cap T_b$. Then $\beta_\ell > (a \vee b)_\ell$ for $\ell = 2, \ldots, n$. Since $M$ is generic and $a_1 = b_1$, there is a minimal generator $z^c$ that strictly divides $\mathrm{lcm}(z^a, z^b)$; in particular $c_1 < (a \vee b)_1 = \beta_1$. It follows that $\beta \succ c$, and thus $\beta$ is contained in the interior of $\mathbf{R}^n_{>0} \setminus S$, which contradicts that $\beta \in T_a \cap T_b \subset \partial S$. Thus we have proved that $\pi(T_a) \cap \pi(T_b) = \emptyset$ when $a \neq b$. □

**Claim 4.5.** *For each $x \in S$, there is an inner corner $a$ of $S$ such that $\pi(x) \in \pi(T_a)$.*

*Proof.* Consider $x \in \mathbf{R}^n_{>0}$. Then there is at least one inner corner $a^0$ of $S$ such that $\pi(x) \succ \pi(a^0)$. Indeed, since $M$ is Artinian, there is a generator of the form $z_1^{a_1}$, whose exponent $(a_1, 0, \dots, 0)$ is mapped to the origin in $\mathbf{R}^{n-1}$, and thus we can choose $a^0 = (a_1, 0, \dots, 0)$.

Given such an $a^0$, either $\pi(x) \in \pi(T_{a^0})$ or $\pi(x) \succ \pi(\beta^1)$ for some inner corner $\beta^1$ of $T_{a^0}$. In the latter case, by Claim 4.2, $\beta^1 = a^0 \vee a^1$, where $a^1 \in \Delta^*_{a^0}(1)$; in particular, $a^1_1 < a^0_1$. Now $\pi(x) \succ \pi(a^0 \vee a^1) \geq \pi(a^1)$, which implies that either $\pi(x) \in \pi(T_{a^1})$ or $\pi(x) \succ \pi(\beta^2)$ for some $\beta^2 = a^1 \vee a^2$, where $a^2 \in \Delta^*_{a^1}(1)$; in particular, $a^2_1 < a^1_1$.

By repeating this argument we get a sequence of inner corners $a^0, \dots, a^k$, such that $\pi(x) \succ \pi(a^j)$ for $j = 0, \dots, k$ and either $\pi(x) \in \pi(T_{a^k})$ or $a^k_1 = 0$. If $a^k_1 = 0$, then $\pi(x) \succ \pi(a^k)$ implies that $x \succ a^k$, which means that $x \notin S$. Hence, either $\pi(x) \in \pi(T_a)$ for some inner corner $a$ of $S$ or $\pi \notin S$.

□

Next, we will use the staircases $T_a$ to construct a partition of $S$. For each inner corner $a$ of $S$, let

$$P_a = \{x \in S \mid \pi(x) \in \pi(T_a)\}.$$

In other words, $P_a$ consists of everything in $S$ "below" the staircase $T_a$. By a slight abuse of notation, $P_a = ]0, a_1] \times T_a$.

*Remark* 4.6. By Claim 4.5, each $x \in S$ is contained in a $P_a$ for some inner corner $a$, and by Claim 4.4 the intersection $P_a \cap P_b$ is empty if $a$ and $b$ are different inner corners. Thus the set of (non-empty) $P_a$ gives a partition of $S$. □

*Remark* 4.7. Note that $P_a$ is a staircase itself with the same outer corners as $T_a$, i.e., $\alpha \in \Delta_a(n)$. □

Next, we will see that each $S_\alpha$ is contained in a $P_a$.

**Claim 4.8.** *For each $\alpha \in \Delta(n)$, $S_\alpha$ is contained in a $P_a$. More precisely, if $\alpha \in \Delta_a(n)$, then $S_\alpha \subset P_a$.*

*Proof.* Let us fix $\alpha \in \Delta(n)$. Recall from Section 3.2 that there is a unique $a$ such that $\alpha \in \Delta_a(n)$. We need to show that $S_\alpha \cap P_b = \emptyset$ for all $b \neq a$.

We first consider the case when $b$ is such that $b_1 > \alpha_1$. Take $x \in P_b$. By Remark 4.7, $P_b$ is a staircase with outer corners $\Delta_b(n)$. It follows that $x \leq \beta$ for some $\beta \in \Delta_b(n)$, which, by the definition of the $S_\gamma$, implies that $x \in \bigcup_{\gamma \geq_\sigma \beta} S_\gamma$. Since $\beta_1 = b_1 > \alpha_1$, $\beta >_\sigma \alpha$, and thus, since the $S_\gamma$ are disjoint, $x \notin S_\alpha$. We conclude that $S_\alpha \cap P_b = \emptyset$ in this case.

Next we consider the case when $b_1 = \alpha_1$. Assume that $x \in S_\alpha \cap P_b$. Then $\alpha_\ell \geq x_\ell > b_\ell$ for $\ell = 2, \dots, n$. Since $\alpha_1 = b_1$ it follows that $\alpha \in T_b$. On the other hand, by Claim 4.2, $\alpha \in \Delta_a(n)$ implies that $\alpha \in T_a$, which, by (4.2), contradicts that $\alpha \in T_b$. It follows that $S_\alpha \cap P_b = \emptyset$.

Finally we consider the case when $b_1 < \alpha_1$. Assume that $x \in S_\alpha \cap P_b$. Then, as above, $\alpha_\ell \geq x_\ell > b_\ell$ for $\ell = 2, \dots, n$. Since also $\alpha_1 > b_1$, it follows that $\alpha \succ b$, which however contradicts that $b$ is an inner corner of $S$. Hence $S_\alpha \cap P_b = \emptyset$ also in this case, which concludes the proof.

□

Next, we will inductively define staircases and partitions of $S$ associated with faces of $\Delta$ of higher dimension. Given vertices $a^1, \dots, a^{k-1}$ of $\Delta$, such that $\Delta_{a^1,\dots,a^{k-1}}$ is non-empty (in particular, $a^j$ is in $\Delta_{a^1,\dots,a^{j-1}}$ for $j = 2, \dots, k-1$) and an inner corner $a^k$ of $\Delta^*_{a^1,\dots,a^{k-1}}$, assuming that $T_{a^1,\dots,a^{k-1}}$ is defined, we let

$$T_{a^1,\dots,a^k} := \{x \in T_{a^1,\dots,a^{k-1}} \mid x_k = (a^1 \vee \cdots \vee a^k)_k, x_j > (a^1 \vee \cdots \vee a^k)_j, j = k+1, \dots, n\}.$$

Recall that by the definition of the sequence $a^1, \dots, a^k$, in fact, $(a^1 \vee \cdots \vee a^k)_j = a^j_j$ for $j = 1, \dots, k$, see Section 3.2. Moreover, note that $T_{a^1,\dots,a^k}$ is contained in the codimension $k$-plane

$$H_{a^1,\dots,a^k} := \{x_1 = (a^1 \vee \cdots \vee a^k)_1, \dots, x_k = (a^1 \vee \cdots \vee a^k)_k\} = \{x_1 = a^1_1, \dots, x_k = a^k_k\}.$$

Let $\pi_k : \mathbf{R}^n_{x_1,\dots,x_n} \to \mathbf{R}^{n-k}_{x_{k+1},\dots,x_n}$ be the projection $\pi_k : (x_1, \dots, x_n) \mapsto (x_{k+1}, \dots, x_n)$, and let $\rho_k : \mathbf{R}^n_{x_1,\dots,x_n} \to \mathbf{R}^{n-k}_{x_{k+1},\dots,x_n}$ be the affine mapping defined by $\rho_k : x \mapsto \pi_k(x - a^1 \vee \cdots \vee a^k)$.

**Claim 4.9.** *Assume that $T_{a^1,\dots,a^{k-1}}$ is non-empty and that $a_k \in \Delta^*_{a^1,\dots,a^{k-1}}(1)$. Then $T_{a^1,\dots,a^k} = \emptyset$ if and only if $a^k_k = (a^1 \vee \cdots \vee a^{k-1})_k$. If $T_{a^1,\dots,a^k}$ is non-empty, then it is a staircase in $H_{a^1,\dots,a^k}$. The origin of $T_{a^1,\dots,a^k}$ is $a^1 \vee \cdots \vee a^k$, the outer corners are the top-dimensional faces of $\Delta_{a^1,\dots,a^k}$ and the inner corners are the lattice points of the form $a^1 \vee \ldots \vee a^k \vee b$, where $b$ is a vertex of $\Delta^*_{a^1,\dots,a^k}$.*

*The monomial ideal $M_{a^1,\dots,a^k}$ defined by $T_{a^1,\dots,a^k}$ and $\rho_k$ as in Section 2, i.e., it has staircase $\rho_k(T_{a^1,\dots,a^k})$, is an Artinian generic monomial ideal. The Scarf complex $\Delta_{M_{a^1,\dots,a^k}}$ consists of faces of the form $\{\rho_k(a^1 \vee \cdots \vee a^k \vee b^1), \dots, \rho_k(a^1 \vee \cdots \vee a^k \vee b^j)\}$, where $\{b^1, \dots, b^j\}$ is a face of $\Delta^*_{a^1,\dots,a^k}$.*

*Proof.* By Claims 4.2 and 4.3 the claim holds for $k = 1$. Assume that it holds for $k = \kappa - 1$; we then need to prove that it holds for $k = \kappa$.

First, it is clear from the definition that $T_{a^1,\dots,a^\kappa}$ is contained in $H_{a^1,\dots,a^\kappa}$. If $a^\kappa_\kappa = (a^1 \vee \cdots \vee a^{\kappa-1})_\kappa$ then $H_{a^1,\dots,a^\kappa} \cap T_{a^1,\dots,a^{\kappa-1}} = \emptyset$ and thus $T_{a^1,\dots,a^\kappa}$ is empty. In general, note that $T_{a^1,\dots,a^\kappa}$ is empty exactly if $a^1 \vee \cdots \vee a^\kappa + (0, \dots, 0, 1, \dots, 1) \notin S$; here $(0, \dots, 0, 1, \dots, 1)$ means that the first $\kappa$ entries are 0 and the rest are 1. Assume that $a^\kappa_\kappa \neq (a^1 \vee \cdots \vee a^{\kappa-1})_\kappa$. Then, by the definition of the $a^j_j$, in fact, $a^\kappa_\kappa > (a^1 \vee \cdots \vee a^{\kappa-1})_\kappa \geq 0$. Since $T_{a^1,\dots,a^{\kappa-1}} \neq \emptyset$ and the claim holds for $k = \kappa - 1$ by assumption, $a^j_j > 0$ for $j = 1, \dots, \kappa - 1$, and thus $a + (0, \dots, 0, 1, \dots, 1) \in \mathbf{R}^n_{>0}$. Then the condition $a^1 \vee \cdots \vee a^\kappa + (0, \dots, 0, 1, \dots, 1) \notin S$ implies that there is an inner corner $c$ of $S$ such that $c \prec a + (0, \dots, 0, 1, \dots, 1)$. In particular, $c \leq a$, which contradicts that $a$ is an inner corner. Thus $T_{a^1,\dots,a^\kappa} \neq \emptyset$ if $a^\kappa_\kappa \neq (a^1 \vee \cdots \vee a^{\kappa-1})_\kappa$.

Let us now assume that $a^\kappa_\kappa > (a^1 \vee \cdots \vee a^{\kappa-1})_\kappa$. We will use Claim 4.2 to show that $T_{a^1,\dots,a^\kappa}$ is a staircase of the desired form. Let $\widetilde{S}$ be the staircase $\rho_{\kappa-1}(T_{a^1,\dots,a^{\kappa-1}}) \subset \mathbf{R}^{n-\kappa+1}_{x_\kappa,\dots,x_n}$ of $M_{a^1,\dots,a^{\kappa-1}}$ and choose an inner corner $\tilde{a} := \rho_{\kappa-1}(a^1 \vee \cdots \vee a^\kappa)$ of $\widetilde{S}$. Then note that

$$(4.3)\quad T_{\tilde a} = \{x \in \widetilde S \mid x_\kappa = \tilde a_\kappa, x_\ell > \tilde a_\ell, \ell = \kappa+1,\dots,n\} =$$
$$\{x \in \rho_{\kappa-1}(T_{a^1,\dots,a^{\kappa-1}}) \mid x_\kappa = \rho_{\kappa-1}(a^1\vee\cdots\vee a^\kappa)_\kappa, x_\ell > \rho_{\kappa-1}(a^1\vee\cdots\vee a^\kappa)_\ell, \ell = \kappa+1,\dots,n\} =$$
$$\rho_{\kappa-1}(T_{a^1,\dots,a^\kappa}).$$

Now, by Claim 4.2, $T_{\tilde a}$ is a staircase in the hyperplane $\{x_\kappa = \tilde a_\kappa\} \subset \mathbf{R}^{n-\kappa+1}_{x_\kappa,\dots,x_n}$ with origin $\tilde a$. The outer corners are the top-dimensional faces $\tilde\alpha$ of $\Delta_{\tilde a}$, i.e., the top-dimensional faces $\tilde\alpha$ of $\Delta_{M_{a^1,\dots,a^{\kappa-1}}}$ such that

$$\tilde\alpha_\kappa = \tilde a_\kappa = \rho_{\kappa-1}(a^1 \vee\cdots\vee a^\kappa)_\kappa = (a^1\vee\cdots\vee a^\kappa)_\kappa = a^\kappa_\kappa. \tag{4.4}$$

Since the claim holds for $k = \kappa-1$, that $\tilde\alpha$ is a top-dimensional face of $\Delta_{M_{a^1,\dots,a^{\kappa-1}}}$ means that $\tilde\alpha = \rho_{\kappa-1}(a^1\vee\cdots\vee a^{\kappa-1}\vee\beta)$, where $\beta$ is a top-dimensional face of $\Delta^*_{a^1,\dots,a^{\kappa-1}}$. In other words, $\tilde\alpha = \rho_{\kappa-1}(\alpha)$, where $\alpha$ is a top-dimensional face of $\Delta$ such that $\alpha_j = a^j_j$ for $j = 1,\dots,\kappa-1$. By (4.4) we also have that $\alpha_\kappa = \rho_{\kappa-1}(\alpha)_\kappa = a^\kappa_\kappa$, so that $\alpha \in \Delta_{a^1,\dots,a^\kappa}(n)$. To conclude, the outer corners of $T_{\tilde a}$ are of the form $\rho_{\kappa-1}(\alpha)$ where $\alpha \in \Delta_{a^1,\dots,a^\kappa}(n)$.

Moreover, by Claim 4.2 the inner corners of $T_{\tilde a}$ are the lattice points $\tilde a \vee \tilde b$, where $\tilde b$ is a vertex of $\Delta^*_{\tilde a}$. Since the lemma holds for $k = \kappa-1$, this means that $\tilde b = \rho_{\kappa-1}(a^1 \vee \cdots\vee a^{\kappa-1}\vee b)$, where $b \in \Delta^*_{a^1,\dots,a^{\kappa-1}}(1)$. Since $\tilde b \neq \tilde a$, $b \neq a^\kappa$, and thus $\tilde a \vee \tilde b = \rho_{\kappa-1}(a^1\vee\cdots\vee a^\kappa\vee b)$, where $b \in \Delta^*_{a^1,\dots,a^\kappa}(1)$.

Since the restriction $\rho_{\kappa-1} : H_{a^1,\dots,a^{\kappa-1}} \to \mathbf{R}^{n-\kappa+1}$ is a just a translation of the plane $H_{a^1,\dots,a^{\kappa-1}}$ (if we consider $\mathbf{R}^{n-\kappa+1}$ as embedded in $\mathbf{R}^n$) it follows that $T_{a^1,\dots,a^\kappa}$ is a staircase in $H_{a^1,\dots,a^\kappa}$ with origin $a^1\vee\cdots\vee a^\kappa$, where the outer corners are the top-dimensional faces of $\Delta_{a^1,\dots,a^\kappa}$ and the inner corners are of the form $a^1\vee\cdots\vee a^\kappa\vee b$, where $b \in \Delta^*_{a^1,\dots,a^\kappa}(1)$. This proves the first part of the claim.

Next, we will use Claim 4.3 to prove the second part of the claim. Let $\tilde\rho : \mathbf{R}^{n-\kappa+1}_{x_\kappa,\dots,x_n} \to \mathbf{R}^{n-\kappa}_{x_{\kappa+1},\dots,x_n}$ be the affine map $\tilde\rho : (x_\kappa,\dots,x_n) \mapsto (x_{\kappa+1} - \tilde a_{\kappa+1},\dots,x_n - \tilde a_n)$. Note that $\rho_\kappa = \tilde\rho\rho_{\kappa-1}$. It follows that the ideal $M_{a^1,\dots,a^\kappa}$ has staircase

$$\rho_\kappa(T_{a^1,\dots,a^\kappa}) = \tilde\rho\rho_{\kappa-1}(T_{a^1,\dots,a^\kappa}) = \tilde\rho(T_{\tilde a}),$$

where we have used (4.3) for the second equality. In other words, $M_{a^1,\dots,a^\kappa}$ is the ideal defined by $T_{\tilde a}$ and $\tilde\rho$ as in Section 2. Thus by Claim 4.3, it is an Artinian generic monomial ideal.

Moreover, by Claim 4.3, the Scarf complex $\Delta_{M_{a^1,\dots,a^\kappa}}$ consists of faces of the form

$$\{\tilde\rho(\tilde a \vee \tilde b^1),\dots,\tilde\rho(\tilde a\vee\tilde b^j)\}, \tag{4.5}$$

where $\{\tilde b^1,\dots,\tilde b^j\}$ is a face of $\Delta^*_{\tilde a}$. As above, $\tilde b^\ell \in \Delta^*_{\tilde a}(1)$ implies that $\tilde a\vee\tilde b^\ell = \rho_{\kappa-1}(a^1\vee\cdots\vee a^{\kappa-1}\vee b^\ell)$, where $b \in \Delta^*_{a^1,\dots,a^\kappa}(1)$. Hence,

$$\tilde\rho(\tilde a\vee\tilde b^\ell) = \tilde\rho\rho_{\kappa-1}(a^1\vee\cdots\vee a^\kappa\vee b^\ell) = \rho_\kappa(a^1\vee\cdots\vee a^\kappa\vee b^\ell)$$

where $b \in \Delta^*_{a^1,\dots,a^\kappa}(1)$. Thus the faces (4.5) are of the desired form, and we have proved the second part of the claim.

□

To construct the partitions associated with the staircases $T_{a^1,\dots,a^k}$, we define inductively

$$P_{a^1,\dots,a^k} = \{x \in P_{a^1,\dots,a^{k-1}} \mid \pi_k(x) \in \pi_k(T_{a^1,\dots,a^k})\}.$$

Then $P_{a^1,\dots,a^k}$ is a $k$-dimensional cuboid times the $(n-k)$-dimensional staircase $T_{a^1,\dots,a^k}$. The $\ell$th side length is given as the "height" of $T_{a^1,\dots,a^\ell}$ in $T_{a^1,\dots,a^{\ell-1}}$, which equals $(a^1 \vee \dots \vee a^\ell - a^1 \vee \dots \vee a^{\ell-1})_\ell$. By a slight abuse of notation

$$P_{a^1,\dots,a^k} = ]0, a_1^1] \times ]a_2^1, (a^1 \vee a^2)_2] \times \dots \times ](a^1 \vee \dots \vee a^{k-1})_k, (a^1 \vee \dots \vee a^k)_k] \times T_{a^1,\dots,a^k}.$$

In particular,

$$P_{a^1,\dots,a^n} = ]0, a_1^1] \times ]a_2^1, (a^1 \vee a^2)_2] \times \dots \times ](a^1 \vee \dots \vee a^{n-1})_n, (a^1 \vee \dots \vee a^n)_n]. \tag{4.6}$$

*Remark* 4.10. Note that $P_{a^1,\dots,a^k}$ is, in fact, a staircase with outer corners $\alpha \in \Delta_{a^1,\dots,a^k}(n)$. □

**Claim 4.11.** *For each $k$, the set of non-empty $P_{a^1,\dots,a^k}$ gives a partition of $S$.*

In particular, since $T_{a^1,\dots,a^k} \subset P_{a^1,\dots,a^k}$,

$$T_{a^1,\dots,a^k} \cap T_{b^1,\dots,b^k} = \emptyset \text{ if } \{a^1,\dots,a^k\} \neq \{b^1,\dots,b^k\}. \tag{4.7}$$

*Proof.* By Remark 4.6, the claim holds for $k = 1$. Assume that the claim holds for $k = \kappa - 1$. To prove that it holds for $k = \kappa$ it suffices to show that, given a sequence $a^1,\dots,a^{\kappa-1}$ of inner corners, the set of $P_{a^1,\dots,a^\kappa}$, where $a^\kappa \in \Delta^*_{a^1,\dots,a^{\kappa-1}}(1)$, gives a partition of $P_{a^1,\dots,a^{\kappa-1}}$. Take $x \in P_{a^1,\dots,a^{\kappa-1}}$. We then need to show that there is exactly one choice of $a^\kappa \in \Delta^*_{a^1,\dots,a^{\kappa-1}}(1)$ such that $\pi_\kappa(x) \in \pi_\kappa(T_{a^1,\dots,a^\kappa})$.

Let $\widetilde{S}$ be the staircase $\pi_{\kappa-1}(T_{a^1,\dots,a^{\kappa-1}}) \subset \mathbf{R}^{n-\kappa+1}_{x_\kappa,\dots,x_n}$. By Claim 4.9, $\widetilde{S}$ is a staircase with (origin $\pi_{\kappa-1}(a^1 \vee \dots \vee a^{\kappa-1})$ and) inner corners $\pi_{\kappa-1}(a^1 \vee \dots \vee a^\kappa)$, where $a^\kappa \in \Delta^*_{a^1,\dots,a^{\kappa-1}}(1)$. Given an inner corner $\tilde{a} := \pi_{\kappa-1}(a^1 \vee \dots \vee a^\kappa)$ of $\widetilde{S}$, note that

$$T_{\tilde{a}} = \{x \in \widetilde{S} \mid x_\kappa = \tilde{a}_\kappa, x_\ell > \tilde{a}_\ell \text{ for } \ell = \kappa+1,\dots,n\} = \tag{4.8}$$

$$\{x \in \pi_{\kappa-1}(T_{a^1,\dots,a^{\kappa-1}}) \mid x_\kappa = \pi_{\kappa-1}(a^1 \vee \dots \vee a^\kappa)_\kappa, x_\ell > \pi_{\kappa-1}(a^1 \vee \dots \vee a^\kappa)_\ell \text{ for } \ell = \kappa+1,\dots,n\} =$$

$$\pi_{\kappa-1}(T_{a^1,\dots,a^\kappa})$$

Let $\tilde{\pi} : \mathbf{R}^{n-\kappa+1}_{x_\kappa,\dots,x_n} \to \mathbf{R}^{n-\kappa}_{x_{\kappa+1},\dots,x_n}$ be the projection $(x_\kappa,\dots,x_n) \mapsto (x_{\kappa+1},\dots,x_n)$. Then, clearly, $\tilde{\pi}\pi_{\kappa-1} = \pi_\kappa$. By a slight modification of the proofs, Claims 4.4 and 4.5 hold also for staircases with origin different from 0; it follows that for $\tilde{x} \in \widetilde{S}$ there is exactly one inner corner $\tilde{a} = \pi_{\kappa-1}(a^1 \vee \dots \vee a^\kappa)$ of $\widetilde{S}$ such that

$$\tilde{\pi}(\tilde{x}) \in \tilde{\pi}(T_{\tilde{a}}) = \tilde{\pi}\pi_{\kappa-1}(T_{a^1,\dots,a^\kappa}) = \pi_\kappa(T_{a^1,\dots,a^\kappa}),$$

where we have used (4.8) for the second equality. Now take $x \in P_{a^1,\dots,a^{\kappa-1}}$. Then $\pi_{\kappa-1}(x) \in \pi_{\kappa-1}(T_{a^1,\dots,a^{\kappa-1}})$, and thus there is exactly one choice of $a^\kappa \in \Delta^*_{a^1,\dots,a^{\kappa-1}}(1)$ such that $\tilde{\pi}\big(\pi_{\kappa-1}(x)\big) = \pi_\kappa(x) \in \pi_\kappa(T_{a^1,\dots,a^\kappa})$. This concludes the proof. □

**Claim 4.12.** *For each $k$ and each $\alpha \in \Delta(n)$, there is a unique sequence of $k$ inner corners $a^1,\dots,a^k$ such that $S_\alpha$ is contained in $P_{a^1,\dots,a^k}$. More precisely, if $\alpha \in \Delta_{a^1,\dots,a^k}$, then $S_\alpha \subset P_{a^1,\dots,a^k}$.*

*Proof.* Let us fix $k$. Recall from Section 3.2 that given $\alpha \in \Delta(n)$, there is a unique sequence of $k$ inner corners $a^1,\dots,a^k$ such that $\alpha \in \Delta_{a^1,\dots,a^k}(n)$. Also, recall from the definition of $\Delta_{a^1,\dots,a^k}$ that $(a^1 \vee \dots \vee a^k)_j = a^j_j$ for $j = 1,\dots,k$. To prove the claim we need to show that $S_\alpha \cap P_{b^1,\dots,b^k} = \emptyset$ for all sequences of $k$ inner corners $b^1,\dots,b^k$ different from $a^1,\dots,a^k$.

We first consider the case when there is an $\ell \leq k$, such that $b^j_j = a^j_j$ for $j < \ell$ and $b^\ell_\ell > a^\ell_\ell$. Pick $x \in P_{b^1,\ldots,b^k}$. Since $P_{b^1,\ldots,b^k}$ is a staircase with outer corners in $\Delta_{b^1,\ldots,b^k}(n)$, see Remark 4.10, it follows that $x \leq \beta$ for some $\beta \in \Delta_{b^1,\ldots,b^k}(n)$. By the definition of the $S_\gamma$, then $x \in \bigcup_{\gamma \geq_\sigma \beta} S_\gamma$. Since $\beta^j_j = b^j_j = a^j_j$ for $j = 1, \ldots, \ell - 1$ and $\beta^\ell_\ell = b^\ell_\ell > a^\ell_\ell$, $\beta >_\sigma \alpha$, and thus, since the $S_\gamma$ are disjoint, $x \notin S_\alpha$. We conclude that $S_\alpha \cap P_{b^1,\ldots,b^k} = \emptyset$ in this case.

Next, we consider the case when $b^j_j = a^j_j$ for $j = 1, \ldots, k$. Assume that $x \in S_\alpha \cap P_{b^1,\ldots,b^k}$. Then $\alpha_j = a^j_j = b^j_j$ for $j = 1, \ldots, k$ and $\alpha_j \geq x_j > (b^1 \vee \cdots \vee b^k)_j$ for $j = k+1, \ldots, n$. Thus by definition $\alpha \in T_{b^1,\ldots,b^k}$. On the other hand, by Claim 4.9 $\alpha \in T_{a^1,\ldots,a^k}$, which by (4.7) contradicts that $\alpha \in T_{b^1,\ldots,b^k}$. It follows that $S_\alpha \cap P_{b^1,\ldots,b^k} = \emptyset$.

Finally we consider the case when there is an $\ell \leq k$, such that $b^j_j = a^j_j$ for $j < \ell$ and $b^\ell_\ell < a^\ell_\ell$. If $\ell = 1$ we know from (the proof of) Claim 4.8 that $S_\alpha \cap P_{b^1} = \emptyset$ and thus $S_\alpha \cap P_{b^1,\ldots,b^k} \subset S_\alpha \cap P_{b^1} = \emptyset$. Assume that $\ell \geq 2$ and that

$$x \in S_\alpha \cap P_{b^1,\ldots,b^k} \subset S_\alpha \cap P_{b^1,\ldots,b^\ell}.$$

Then $\alpha_j = a^j_j = b^j_j$ for $j = 1, \ldots, \ell - 1$, $\alpha_\ell = a^\ell_\ell > b^\ell_\ell$, and $\alpha_j \geq x_j > (b^1 \vee \cdots \vee b^\ell)_j$ for $j = \ell + 1, \ldots, n$. It follows that $\alpha \in T_{b^1,\ldots,b^{\ell-1}}$, but, from Claim 4.9 we know that $\alpha \in T_{a^1,\ldots,a^{\ell-1}}$, which leads to a contradiction by (4.7). Hence $S_\alpha \cap P_{b^1,\ldots,b^k} = \emptyset$ also in this case.

□

Recall from Section 3.2 that $\Delta_{a^1,\ldots,a^n}$ is just the simplex $\alpha \in \Delta(n)$ with vertices $a^1, \ldots, a^n$. On the other hand, each outer corner $\alpha$ gives rise to a non-empty $P_\alpha := P_{a^1,\ldots,a^n}$ by choosing $a^\ell$ as the $x_\ell$-vertex of $\alpha$. By Claim 4.12, $S_\alpha \subset P_\alpha$, and since both $\{P_\alpha\}$ and $\{S_\alpha\}$ give partitions of $S$, we conclude that $S_\alpha = P_\alpha$.

Now, given $\mathcal{I} = \{i_1, \ldots, i_n\} \in \Delta(n)$, we choose $a^\ell$ as the $x_\ell$-vertex $a^{i_{\eta(\ell)}}$. Then Lemma 4.1 follows in light of (4.6).

For a general choice of $\sigma$ the above proof works verbatim, with the coordinates $x_\ell$ and the variables $z_\ell$ replaced by $x_{\sigma(\ell)}$ and $z_{\sigma(\ell)}$, respectively, and $\eta$ replaced by $\tau$.

4.2. **Computing** $d\varphi$. Let us now compute the $e^*_{\mathcal{I}}$-entry of $d_\sigma\varphi$ for a given $\mathcal{I} = \{i_1, \ldots, i_n\} \in \Delta(n)$. Recall from (3.1) that

$$\varphi_k = \sum_{\mathcal{J}=\{j_1,\ldots,j_k\}\subset\mathcal{I}} \sum_{\ell=1}^{k} (-1)^{\ell-1} z^{a^{j_1}\vee\cdots\vee a^{j_k} - a^{j_1}\vee\cdots\widehat{a^{j_\ell}}\cdots\vee a^{j_k}} e^*_{\mathcal{J}} \otimes e_{\mathcal{J}_\ell} + \varphi'_k \tag{4.9}$$

where $\varphi'_k$ are the remaining terms that will not contribute to the $e^*_{\mathcal{I}}$-entry. It follows that the coefficient of $e^*_{\mathcal{I}}$ in $d_\sigma\varphi$ equals

$$\begin{aligned}(4.10)\quad \operatorname{sgn}\big((n,\ldots,1)\big) \sum_\tau \operatorname{sgn}(\tau) \frac{\partial}{\partial z_{\sigma(1)}} z^{a^{i_{\tau(1)}}} dz_{\sigma(1)} \wedge \frac{\partial}{\partial z_{\sigma(2)}} z^{a^{i_{\tau(1)}}\vee a^{i_{\tau(2)}} - a^{i_{\tau(1)}}} dz_{\sigma(2)} \wedge \\ \cdots \wedge \frac{\partial}{\partial z_{\sigma(n-1)}} z^{a^{i_{\tau(1)}}\vee\cdots\vee a^{i_{\tau(n-1)}} - a^{i_{\tau(1)}}\vee\cdots\vee a^{i_{\tau(n-2)}}} dz_{\sigma(n-1)} \wedge \\ \frac{\partial}{\partial z_{\sigma(n)}} z^{a^{i_{\tau(1)}}\vee\cdots\vee a^{i_{\tau(n)}} - a^{i_{\tau(1)}}\vee\cdots\vee a^{i_{\tau(n-1)}}} dz_{\sigma(n)} =: \sum_\tau F_\tau,\end{aligned}$$

where the sum is over all permutations $\tau$ of $\{1, \ldots, n\}$ and $\operatorname{sgn}(\tau)$ denotes the sign of the permutation $\tau$.

Let $\eta$ be the permutation of $\{1,\dots,n\}$ associated with $\mathcal{I}$ as in Section 3.1 and let $\alpha$ be the label of $\mathcal{I}$. Then, by the definition of $\eta$, $\big(a^{i_{\tau(1)}} \vee \cdots \vee a^{i_{\tau(\kappa)}}\big)_\ell = \alpha_\ell$ precisely for $\ell = \eta^{-1}\big(\tau(1)\big),\dots,\eta^{-1}\big(\tau(\kappa)\big)$. It follows that

$$z^{a^{i_{\tau(1)}}\vee\cdots\vee a^{i_{\tau(k)}} - a^{i_{\tau(1)}}\vee\cdots\vee a^{i_{\tau(k-1)}}}$$

is a monomial in the variables $z_{\eta^{-1}(\tau(k))},\dots,z_{\eta^{-1}(\tau(n))}$. Therefore the last factor in $F_\tau$ vanishes unless $\tau(n) = \eta\big(\sigma(n)\big)$. Given, $\tau(n) = \eta\big(\sigma(n)\big)$, the next to last factor vanishes unless $\tau(n-1) = \eta\big(\sigma(n-1)\big)$, etc. To conclude, $F_\tau$, where $\tau = \eta\circ\sigma$, is the only non-vanishing term in (4.10).

Now with $\tau = \eta\circ\sigma$,

$$(4.11)\quad F_\tau = \operatorname{sgn}(\tau)\times a^{i_{\tau(1)}}_{\sigma(1)} \times \Big(a^{i_{\tau(1)}} \vee a^{i_{\tau(2)}} - a^{i_{\tau(1)}}\Big)_{\sigma(2)} \times\cdots\times$$
$$\Big(a^{i_{\tau(1)}} \vee\cdots\vee a^{i_{\tau(n)}} - a^{i_{\tau(1)}}\vee\cdots\vee a^{i_{\tau(n-1)}}\Big)_{\sigma(n)} z^{a^{i_{\tau(1)}}\vee\cdots\vee a^{i_{\tau(n)}}} \frac{dz_{\sigma(n)}}{z_{\sigma(n)}}\wedge\cdots\wedge\frac{dz_{\sigma(1)}}{z_{\sigma(1)}} =$$
$$\operatorname{sgn}(\eta)\operatorname{Vol}(S_{\sigma,\alpha}) z^{\alpha-\mathbf{1}} dz_n\wedge\cdots\wedge dz_1,$$

where the last equality follows from Lemma 4.1. This concludes the proof of Theorem 1.1, since, by definition $\operatorname{sgn}(\alpha) = \operatorname{sgn}(\eta)$, see Section 3.1.

## 5. Examples

Let us illustrate (the proof of) Theorem 1.1 by some examples.

*Example* 5.1. Assume that $n = 1$. Then each monomial ideal $M$ is a principal ideal generated by a monomial $z^a$. The staircase of $M$ is just the line segment $]0,a] \subset \mathbf{R}_{>0}$ with one outer corner $\alpha = a$ so that $S_\alpha = S$. Moreover, the Scarf complex is just a point with label $z^a$, and thus the Scarf resolution is just $0\to A \xrightarrow{z^a} A$. Thus in this case Theorem 1.1 just reads

$$d\varphi = d(z^a) = \operatorname{Vol}(]0,a])z^{a-1}dz = az^{a-1}dz.$$

□

*Example* 5.2. Assume that $n = 2$. Then each Artinian monomial ideal $M \subset A_2$ is of the form $M = (z_1^{a_1}z_2^{b_1},\dots,z_1^{a_r}z_2^{b_r})$ for some integers $a_1 > \ldots > a_r = 0$ and $0 = b_1 < \ldots < b_r$. Since no two minimal monomial generators have the same positive degree in any variable, $M$ is trivially generic. In this case the staircase of $M$ looks like an actual staircase with $r$ inner corners $(a_j, b_j)$ and $r-1$ outer corners $\alpha^j := (a_j, b_{j+1})$, see Figure 5.1. In particular, $M = \bigcap_{j=1}^{r-1}(z_1^{a_j}, z_2^{b_{j+1}})$. With $\sigma$ as the identity, $T_{(a_j,b_j)}$ is just the line segment $\{x_1 = a_j, b_j < x_2 \le b_{j+1}\}$.

Note that $\sigma = (1,2)$ corresponds to the ordering $\alpha^1 \ge_\sigma \ldots \ge_\sigma \alpha^{r-1}$ of the outer corners and

$$S_{(1,2),\alpha^j} = \{x\in\mathbf{R}^2_{>0} \mid 0\le x_1 < a_j, b_j \le x_2 < b_{j+1}\},$$

whereas $\sigma = (2,1)$ corresponds to the reverse ordering of the outer corners and so

$$S_{(2,1),\alpha^j} = \{x\in\mathbf{R}^2_{>0} \mid a_{j+1}\le x_1 < a_j, 0 \le x_2 < b_{j+1}\},$$

see Figure 5.1. Thus, the partitions just correspond to vertical and horisontal, respectively, slicing of $S$.

In this case the Scarf complex is just a triangulation of the one-dimensional simplex, and it is not very hard to directly compute $d\varphi$, cf. [LW, Section 7]. □

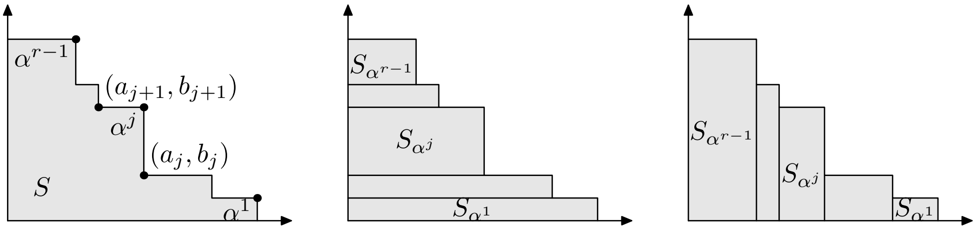


FIGURE 5.1. The staircase $S$ of $M$ in Example 5.2 and the partitions $\{S_{\alpha^j}\}_j = \{S_{\sigma,\alpha^j}\}_j$ of $S$ corresponding to the permutations $\sigma = (1,2)$ and $\sigma = (2,1)$, respectively.

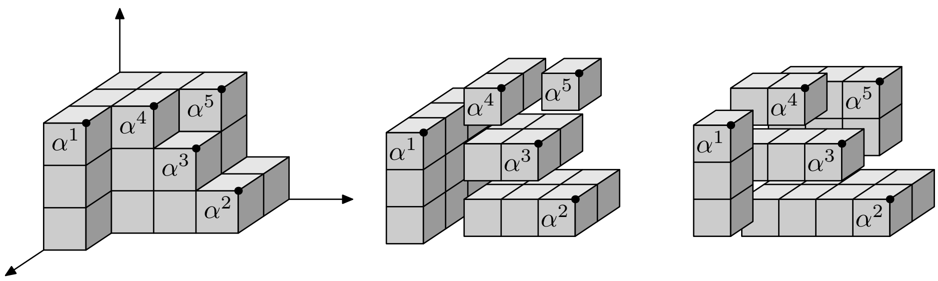


FIGURE 5.2. The staircase of $M$ in Example 5.3 and the partitions $\{S_{\alpha^j}\}_j$ corresponding to the orderings $\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5$ and $\alpha^2, \alpha^5, \alpha^3, \alpha^4, \alpha^1$, respectively.

*Example* 5.3. Let $M$ be the generic monomial ideal $M = (z_1^3, z_1^2 z_2, z_1 z_2^2 z_3^2, z_2^4, z_2^3 z_3, z_3^3) \subset A_3$. The staircase $S$ of $M$, depicted in Figure 5.2, has six inner corners, $a^1 = (3,0,0)$, $a^2 = (2,1,0)$, $a^3 = (1,2,2)$, $a^4 = (0,4,0)$, $a^5 = (0,3,1)$, and $a^6 = (0,0,3)$, and five outer corners, $\alpha^1 = (3,1,3)$, $\alpha^2 = (2,4,1)$, $\alpha^3 = (2,3,2)$, $\alpha^4 = (2,2,3)$, and $\alpha^5 = (1,3,3)$. By Claim 4.2, $T_{a^j}$ is non-empty for $j = 1,2,3$. These two-dimensional staircases are the light grey regions facing the reader in the first figure in Figure 5.3.

The six different permutations $\sigma$ of $\{1,2,3\}$ give rise to six different orderings of the $\alpha^j$: for example $\sigma^1 := (1,2,3)$ and $\sigma^2 := (2,3,1)$ correspond to the orderings $\alpha^1 \geq_\sigma \alpha^2 \geq_\sigma \alpha^3 \geq_\sigma \alpha^4 \geq_\sigma \alpha^5$ and $\alpha^2 \geq_\sigma \alpha^5 \geq_\sigma \alpha^3 \geq_\sigma \alpha^4 \geq_\sigma \alpha^1$, respectively. In the first case $S_{\sigma^1,\alpha^1}$ is the cuboid $]0,3]\times]0,1]\times]0,3]$, $S_{\sigma^1,\alpha^2} = ]0,2]\times]1,4]\times]0,1]$, $S_{\sigma^1,\alpha^3} = ]0,2]\times]1,3]\times]1,2]$, $S_{\sigma^1,\alpha^4} = ]0,2]\times]1,2]\times]2,3]$, and $S_{\sigma^1,\alpha^5} = ]0,1]\times]2,3]\times]2,3]$, see Figure 5.2, where also the $S_{\sigma^2,\alpha^j}$ are depicted.

□

## 6. General (monomial) ideals

The Scarf resolution is an instance of a more general construction of so-called *cellular resolutions* of monomial ideals, introduced by Bayer-Sturmfels [BS]. The Scarf complex is then replaced by a more general oriented polyhedral cell complex $X$, with vertices corresponding to and labeled by the generators of the monomial ideal $M$; as above a face $\gamma$ of $X$ is labeled by the least common multiple $m_\gamma$ of the vertices. Analogously to the Scarf complex, $X$ encodes a graded complex of free $A$-modules: for $k = 0, \ldots, \dim X + 1$, let $E_k$ be a free $A$-module of rank equal to the number of $(k-1)$-dimensional faces of $X$ and let $\varphi_k : E_k \to E_{k-1}$ be defined by $\varphi_k : e_\gamma \mapsto \sum_{\delta \subset \gamma} \operatorname{sgn}(\delta, \gamma) \frac{m_\gamma}{m_\delta} e_\delta$, where $\gamma$ and $\delta$ are

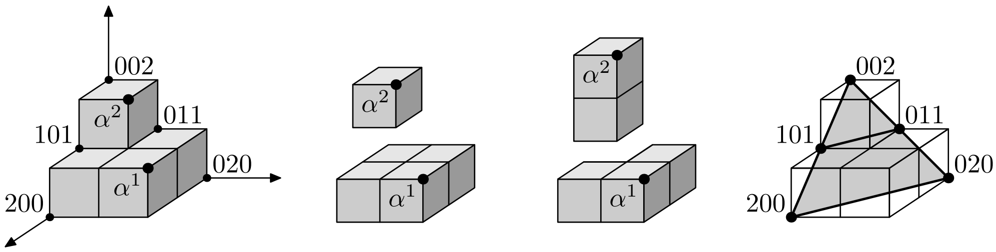


FIGURE 6.1. The staircase of $M$ in Example 6.1, the partitions $S_{\alpha^1}$ and $S_{\alpha^2}$ corresponding to the orderings $\alpha^1, \alpha^2$ and $\alpha^2, \alpha^1$, respectively, and the hull complex of $M$.

faces of $X$ of dimension $k-1$ and $k-2$, respectively, and where $\operatorname{sgn}(\delta, \gamma) = \pm 1$ comes from the orientation of $X$. The complex $E_\bullet, \varphi_\bullet$ is exact if $X$ satisfies a certain acyclicity condition see, e.g., [MS, Proposition 4.5], and thus gives a resolution - a so-called *cellular resolution* - of the cokernel of $\varphi_0$, which, with the identification $E_0 = A$, equals $A/M$. For more details we refer to [BS] or [MS].

In [BS] was also introduced a certain canonical choice of $X$. Given $t \in \mathbf{R}$, let $\mathcal{P}_t = \mathcal{P}_t(M)$ be the convex hull in $\mathbf{R}^n$ of $\{(t^{\alpha_1}, \dots, t^{\alpha_n}) \mid z^\alpha \in M\}$. Then $\mathcal{P}_t$ is an unbounded polyhedron in $\mathbf{R}^n$ of dimension $n$ and the face poset of bounded faces of $\mathcal{P}_t$ (i.e., the set of bounded faces partially ordered by inclusion) is independent of $t$ if $t \gg 0$. The *hull complex* of $M$ is the polyhedral cell complex of all bounded faces of $\mathcal{P}_t$ for $t \gg 0$. The corresponding complex $E_\bullet, \varphi_\bullet$ is exact and thus gives a resolution, the *hull resolution*, of $A/M$. It is in general not minimal, but it has length at most $n$. If $M$ is generic, however, the Hull complex coincides with the Scarf complex; in particular, it is minimal.

In [LW] together with Lärkäng we computed the residue current $R$ associated with the hull resolution, or, more generally, any cellular resolution where the underlying polyhedral complex $X$ is a polyhedral subdivision of the $(n-1)$-simplex, of an Artinian monomial ideal. Theorem 5.1 in [LW] states that the entries of $R$ are of the form (1.11), where the sum is now over all top-dimensional faces (with label $\alpha$) of $X$ and $\operatorname{sgn}(\alpha)$ comes from the orientation of $X$.

Note that the definition of $S_{\sigma,\alpha}$ still makes sense when $M$ is a general Artinian monomial ideal. However, in general the $S_{\sigma,\alpha}$ will not be cuboids as the following example shows.

*Example* 6.1. Let $M = (z_1^2, z_1z_2, z_1z_3, z_2^2, z_3^2) \subset A_3$. Then $M$ is not generic, since there is no generator that strictly divides $\operatorname{lcm}(z_1z_2, z_1z_3) = z_1z_2z_3$. The staircase $S$ of $M$ is depicted in Figure 6.1. Note that $S$ has two outer corners $\alpha^1 = (2,2,1)$ and $\alpha^2 = (1,1,2)$.

Assume that $\sigma$ is a permutation of $\{1,2,3\}$ such that $\sigma(1)$ equals 1 or 2. Then the lexicographical order of the outer corners is $\alpha^1 \geq_\sigma \alpha^2$. Otherwise, if $\sigma(1) = 3$, the lexicographical order is reversed. In the first case $S_{\alpha^1}$ is the cuboid $]0,2]\times]0,2]\times]0,1]$, and $S_{\alpha^2}$ is the cuboid $]0,1]\times]0,1]\times]1,2]$, see Figure 6.1. In the second case $S_{\alpha^2}$ is the cuboid $]0,1]\times]0,1]\times]0,2]$, whereas $S_{\alpha^1}$ is the set $]0,2]\times]0,2]\times]0,1]\setminus]0,1]\times]0,1]\times]0,1]$; in particular $S_{\alpha^1}$ is not a cuboid.

In this case, the hull resolution is a minimal resolution of $A/M$. There are two top-dimensional faces in the hull complex, with vertices $\{z_1^2, z_1z_2, z_1z_3, z_2^2\}$ and $\{z_1z_2, z_1z_3, z_3^2\}$ and thus labels $z^{\alpha^1}$ and $z^{\alpha^2}$, respectively, see Figure 6.1. A computation yields that if $\sigma(1) = 3$, then the coefficient of $\operatorname{sgn}(\alpha^1)z^{\alpha^1-\mathbf{1}}dze^*_{\alpha^1}$ is $3 = \operatorname{Vol}(S_{\sigma,\alpha^1})$ and the coefficient

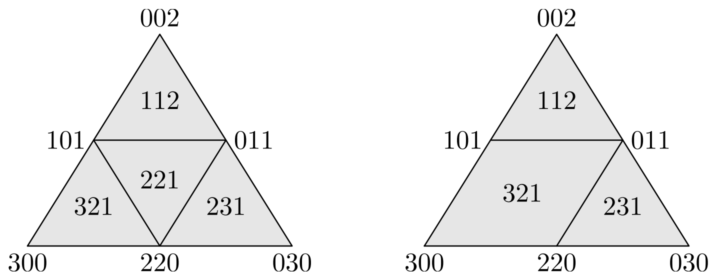


FIGURE 6.2. The hull complex of the ideal $M$ in Example 6.2 (labels on vertices and 2-faces) (left) and the minimal free resolution (right).

of $\operatorname{sgn}(\alpha^2)z^{\alpha^2-\mathbf{1}}dze^*_{\alpha^2}$ is $2 = \operatorname{Vol}(S_{\sigma,\alpha^2})$. Otherwise the coefficients are $4 = \operatorname{Vol}(S_{\sigma,\alpha^1})$ and $1 = \operatorname{Vol}(S_{\sigma,\alpha^2})$, respectively. Thus in this case Theorem 1.1 holds. □

Example 6.1 suggests that Theorem 1.1 might hold when $E_\bullet, \varphi_\bullet$ is the hull resolution of an Artinian monomial ideal and this resolution is minimal. However, we do not know how to prove it in general. The proof in Section 4 does not extend to this situation. For example the staircases $T_a$ constructed in Section 4.1 are not disjoint in general, cf. (4.2). Choose $\sigma$ such that $\sigma(1) = 3$ and consider the inner corners $a = (1,1,0)$ and $b = (1,0,1)$ of the staircase $S$ in Example 6.1. Then

$$T_a \cap T_b = \{x \in \mathbf{R}^3 | x_3 = 1, 1 < x_j \leq 2, j = 1,2\}.$$

Also, the computation of $d\varphi$ is more involved in this case. Indeed, in general it is not true that the coefficient of $e^*_{\mathcal{I}}$ in $R$ just consists of one non-vanishing term as in (4.10).

Example 6.2 below shows that Theorem 1.1 does not hold for the hull resolution in general if it is not minimal, and also that it does not hold for arbitrary minimal resolutions of monomial ideals. It would be interesting to look for an alternative description of the coefficients of $d\varphi$ that extends to general (monomial) resolutions.

*Example* 6.2. Let $M = (z_1^3, z_1^2z_2^2, z_1z_3, z_2^3, z_2z_3, z_3^2)$. Then $M$ is not generic; for example, as in Example 6.1, there is no generator of $M$ that strictly divides $\operatorname{lcm}(z_1z_2, z_1z_3) = z_1z_2z_3$. The staircase of $M$ has three outer corners, $\alpha^1 = (3,2,1)$, $\alpha^2 = (2,3,1)$, and $\alpha^3 = (1,1,2)$.

In this case the hull resolution is not minimal. The hull complex consists of four triangles; one triangle $\alpha^j$ for each outer corner and one extra triangle $\beta$ with vertices $\{z_1^2z_2^2, z_1z_3, z_2z_3\}$ and thus label $z^\beta = z_1^2z_2^2z_3$, see Figure 6.2. A computation yields that the coefficient of $\operatorname{sgn}(\alpha^3)z^{\alpha^3-\mathbf{1}}dze^*_{\alpha^3}$ in $d_\sigma\varphi$ equals $\operatorname{Vol}(S_{\sigma,\alpha^3})$ for all permutations $\sigma$. The coefficient of $\operatorname{sgn}(\alpha^1)z^{\alpha^1-\mathbf{1}}dze^*_{\alpha^1}$ in $d_{(3,1,2)}\varphi$, however, equals 4, whereas $\operatorname{Vol}(S_{(3,1,2),\alpha^1}) = 5$. Thus Theorem 1.1 does not hold in this case.

One can create a minimal cellular resolution from the hull complex, e.g., by removing the edge between $z_1^2z_2^2$ and $z_1z_3$. The polyhedral cell complex $X$ so obtained has one top-dimensional face for each outer corner $\alpha^j$ in $S$. The face corresponding to $\alpha^1$ is the union of the two triangles $\alpha^1$ and $\beta$ in the hull complex, see Figure 6.2. It turns out that the coefficient of $\operatorname{sgn}(\alpha^1)z^{\alpha^1-\mathbf{1}}dze^*_{\alpha^1}$ in $d_\sigma\varphi$ is the sum of the coefficients of $\operatorname{sgn}(\alpha^1)z^{\alpha^1-\mathbf{1}}dze^*_{\alpha^1}$ and $\operatorname{sgn}(\beta)z^{\beta-\mathbf{1}}dze^*_\beta$ for each $\sigma$, whereas the coefficients of $\operatorname{sgn}(\alpha^2)z^{\alpha^2-\mathbf{1}}dze^*_{\alpha^2}$ and $\operatorname{sgn}(\alpha^3)z^{\alpha^3-\mathbf{1}}dze^*_{\alpha^3}$ are the same as above. Thus, as above, the coefficient of $\operatorname{sgn}(\alpha^2)z^{\alpha^2-\mathbf{1}}dze^*_{\alpha^2}$ in $d_{(3,2,1)}\varphi$ is different from $\operatorname{Vol}(S_{(3,1,2),\alpha^2})$, and so Theorem 1.1 fails to hold also in this case.

□

Although Theorem 1.1 fails to hold in Example 6.2, Corollary 1.2 still holds for both resolutions. In fact, the left hand side of (1.12) is independent of $\sigma$ for all free resolutions of Artinian ideals. We will present an argument of this communicated to us by Jan Stevens, [S].

Assume that

$$0 \to E_n \xrightarrow{\varphi_n} \dots \xrightarrow{\varphi_2} E_1 \xrightarrow{\varphi_1} E_0 \cong \mathcal{O}_0 \tag{6.1}$$

is a resolution of minimal length of an Artinian ideal $\mathfrak{a} \subset \mathcal{O}_0$ and let $R$ be the associated residue current as constructed in [AW]. Since $\mathfrak{a}$ is Artinian, it follows from the construction that

$$\varphi_n R = 0, \tag{6.2}$$

see [AW, Proposition 2.2]. Moreover, $R$ satisfies that if $\psi$ is (a germ of ) a holomorphic function, then $\psi R = 0$ if and only if $\psi \in \mathfrak{a}$, see [AW, Theorem 1.1]. In particular,

$$\varphi_1 \xi R = 0 \tag{6.3}$$

for any $\mathrm{End}(E_n, E_2)$-valued section $\xi$.

**Proposition 6.3.** *Assume that* (6.1) *is a resolution of an Artinian ideal* $\mathfrak{a} \subset \mathcal{O}_0$ *and that* $z_1, \dots, z_n$ *are holomorphic coordinates at* $0 \in \mathbf{C}^n$. *Let* $\sigma$ *be a permutation of* $\{1, \dots, n\}$. *Then*

$$\frac{\partial \varphi_1}{\partial z_{\sigma(1)}} dz_{\sigma(1)} \wedge \dots \wedge \frac{\partial \varphi_n}{\partial z_{\sigma(n)}} dz_{\sigma(n)} \wedge R$$

*is independent of* $\sigma$.

*Proof.* Since each permutation of $\{1, \dots, n\}$ can be obtained as a composition of permutations $\sigma$ of the form

$$\sigma_j : \{1, \dots, n\} \mapsto \{1, \dots, j-1, j+1, j, j+2, \dots, n\}, \tag{6.4}$$

it suffices to prove that

$$\frac{\partial \varphi_1}{\partial z_1} dz_1 \wedge \dots \wedge \frac{\partial \varphi_n}{\partial z_n} dz_n \wedge R = \frac{\partial \varphi_1}{\partial z_{\sigma_j(1)}} dz_{\sigma_j(1)} \wedge \dots \wedge \frac{\partial \varphi_n}{\partial z_{\sigma_j(n)}} dz_{\sigma_j(n)} \wedge R. \tag{6.5}$$

Since $\varphi_j \varphi_{j+1} = 0$,

$$0 = \frac{\partial^2 (\varphi_j \varphi_{j+1})}{\partial z_j \partial z_{j+1}} = \frac{\partial^2 \varphi_j}{\partial z_j \partial z_{j+1}} \varphi_{j+1} + \frac{\partial \varphi_j}{\partial z_j} \frac{\partial \varphi_{j+1}}{\partial z_{j+1}} + \frac{\partial \varphi_j}{\partial z_{j+1}} \frac{\partial \varphi_{j+1}}{\partial z_j} + \varphi_j \frac{\partial^2 \varphi_{j+1}}{\partial z_j \partial z_{j+1}}. \tag{6.6}$$

Let us compose (6.6) from the left and the right by

$$\frac{\partial \varphi_1}{\partial z_1} \cdots \frac{\partial \varphi_{j-1}}{\partial z_{j-1}} \text{ and } \frac{\partial \varphi_{j+2}}{\partial z_{j+2}} \cdots \frac{\partial \varphi_n}{\partial z_n},$$

respectively. Since $\varphi_k \varphi_{k+1} = 0$ for each $k$, Leibniz's rule gives that

$$\frac{\partial \varphi_k}{\partial z_\ell} \varphi_{k+1} = -\varphi_k \frac{\varphi_{k+1}}{\partial z_\ell},$$

cf. (6.6). Using this repeatedly for $k = j+1, \dots, n-1$ we get that the term corresponding to the first term in the left hand side of (6.6) equals

$$\pm \frac{\partial \varphi_1}{\partial z_1} \cdots \frac{\partial \varphi_{j-1}}{\partial z_{j-1}} \frac{\partial^2 \varphi_j}{\partial z_j \partial z_{j+1}} \frac{\partial \varphi_{j+1}}{\partial z_{j+2}} \cdots \frac{\partial \varphi_{n-1}}{\partial z_n} \varphi_n.$$

Similarly the term corresponding to the last term in the left hand side of (6.6) equals

$$\pm\varphi_1 \frac{\partial\varphi_2}{\partial z_1} \cdots \frac{\partial\varphi_j}{\partial z_{j-1}} \frac{\partial^2\varphi_{j+1}}{\partial z_j \partial z_{j+1}} \frac{\partial\varphi_{j+2}}{\partial z_{j+2}} \cdots \frac{\partial\varphi_n}{\partial z_n}.$$

Next let us compose from the right by $R$. Using (6.2) and (6.3) we get

$$0 = \frac{\partial\varphi_1}{\partial z_1} \cdots \frac{\partial\varphi_n}{\partial z_n} R + \frac{\partial\varphi_1}{\partial z_1} \cdots \frac{\partial\varphi_j}{\partial z_{j+1}} \frac{\partial\varphi_{j+1}}{\partial z_j} \cdots \frac{\partial\varphi_n}{\partial z_n} R = \frac{\partial\varphi_1}{\partial z_1} \cdots \frac{\partial\varphi_n}{\partial z_n} R + \frac{\partial\varphi_1}{\partial z_{\sigma_j(1)}} \cdots \frac{\partial\varphi_n}{\partial z_{\sigma_j(n)}} R.$$

Combining this with $dz_1 \wedge \cdots \wedge dz_n = -dz_{\sigma_j(1)} \wedge \cdots \wedge dz_{\sigma_j(n)}$ we obtain (6.5). $\square$

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