ON A MIXED MONGE-AMPÈRE OPERATOR FOR QUASIPLURISUBHARMONIC FUNCTIONS WITH ANALYTIC SINGULARITIES

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Abstract. We consider mixed Monge-Ampère products of quasiplurisubharmonic functions with analytic singularities, and show that such products may be regularized as explicit one parameter limits of mixed Monge-Ampère products of smooth functions, generalizing results of Andersson, Błocki and the last author in the case of non-mixed Monge-Ampère products. Connections to the theory of residue currents, going back to Coleff-Herrera, Passare and others, play an important role in the proof. As a consequence we get an approximation of Chern and Segre currents of certain singular hermitian metrics on vector bundles by smooth forms in the corresponding Chern and Segre classes.

1. Introduction

Classical pluripotential theory, going back to Bedford-Taylor, [BT, BT2], gives a way of defining mixed Monge-Ampère products like

$$dd^c u_r \wedge \cdots \wedge dd^c u_1, \tag{1.1}$$

where $u_1, \ldots, u_r$ are locally bounded plurisubharmonic (psh) functions on a complex manifold X. Here and throughout $d^c = (\partial - \bar\partial)/(4\pi i)$. Let u be a locally bounded psh function and let T be a closed positive current on X. Then

$$dd^c u \wedge T := dd^c(uT) \tag{1.2}$$

is a well-defined closed positive current. In particular one can give meaning to mixed Monge-Ampère products like (1.1) by inductively applying (1.2). Theorem 2.1 in [BT2] asserts that (1.1) satisfies the following monotone continuity: If u_k^j are decreasing sequences of psh functions converging pointwise to u_k, then

$$dd^c u_r^j \wedge \cdots \wedge dd^c u_1^j \to dd^c u_r \wedge \cdots \wedge dd^c u_1. \tag{1.3}$$

Demailly later extended this construction to the situation where the unbounded loci of the u_i are small in a certain sense, [D2]. For general psh functions there is no such canonical (mixed) Monge-Ampère product as (1.1); e.g., one cannot expect (1.3) to hold in general.

Recall that a psh function u has *analytic singularities*[1] if locally

$$u = c \log |f|^2 + v, \tag{1.4}$$

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[1] See Remark 1.7.

where c is a positive constant, $f = (f_1, \dots, f_m)$ is a tuple of holomorphic functions, and v is smooth. In [LRSW], together with Raufi, we gave meaning to (1.1) for psh functions u_i with analytic singularities on X by inductively defining it as

$$dd^c u_k \wedge \cdots \wedge dd^c u_1 := dd^c\big(u_k \mathbf{1}_{X\setminus Z_k} dd^c u_{k-1} \wedge \cdots \wedge dd^c u_1\big), \tag{1.5}$$

where Z_k is the unbounded locus of u_k, for $k = 1, \dots, r$. Assuming that we have inductively defined $T := dd^c u_{k-1} \wedge \cdots \wedge dd^c u_1$, then for $u = u_k$ with unbounded locus Z we define

$$u\mathbf{1}_{X\setminus Z} T = \lim_{j\to\infty} u^j \mathbf{1}_{X\setminus Z} T, \tag{1.6}$$

where u^j is a sequence of smooth psh functions decreasing to u. Propositions 3.2 and 3.4 in [LRSW] assert that (1.6) has locally finite mass and is independent of the regularizing sequence u^j, and that

$$dd^c u \wedge \mathbf{1}_{X\setminus Z} T = dd^c(u\mathbf{1}_{X\setminus Z} T)$$

is closed and positive and coincides with the classical Bedford-Taylor-Demailly Monge-Ampère product when this is defined. The definition of the product (1.5) is a straightforward generalization of previous work [AW2] by Andersson and the last author, where the generalized Monge-Ampère product $(dd^c u)^m$ was defined for psh functions u with analytic singularities.

In [LRSW] the generalized mixed Monge-Ampère products (1.5) were used to construct Chern and Segre forms for certain singular metrics on vector bundles, and in [ASWY, AES$^+$] currents like these were used to understand nonproper intersection theory in terms of currents.

The main goal of this paper is to prove a one parameter regularization of the mixed Monge-Ampère products (1.5), similar to (1.3). In fact, we will work in a slightly more general setting: Recall that a function $\varphi : X \to \mathbf{R} \cup \{-\infty\}$ is *quasiplurisubharmonic (qpsh)* if it is locally given as $\varphi = u + a$, where u is psh and a is smooth. We say that φ has *analytic singularities* if u has. Moreover, we say that a closed current T that is locally given as a sum of currents (1.5) multiplied by smooth closed (p, p)-forms has *analytic singularities*, see Definition 2.1. In [LRSW, Lemma 3.5], we showed that $\varphi\mathbf{1}_{X\setminus Z} T := u\mathbf{1}_{X\setminus Z} T + a\mathbf{1}_{X\setminus Z} T$, where Z is the unbounded locus of φ, is independent of the decomposition $\varphi = u+a$. It follows that $dd^c\varphi\wedge T = dd^c(\varphi\mathbf{1}_{X\setminus Z} T)$ is a well-defined current with analytic singularities, and in particular we can inductively define products

$$dd^c\varphi_r \wedge \cdots \wedge dd^c\varphi_1, \tag{1.7}$$

if $\varphi_1, \dots, \varphi_r$ are qpsh functions with analytic singularities.

Since (1.3) does not hold in general one cannot expect convergence of any decreasing regularizing sequences φ_k^j. For example, one can find smooth decreasing sequences of psh functions u^j and v^j converging to the same psh function u with analytic singularities, but where $(dd^c u^j)^2$ and $(dd^c v^j)^2$ converge to different positive closed currents, see, e.g., Example 3.2 in [ABW].

Definition 1.1. Let $\rho\colon \mathbf{R} \to \mathbf{R}$ be a smooth, convex, increasing function such that $\rho(t)$ is constant for $t \ll 0$ and such that $\rho(t) = t$ for $t \gg 0$. Let $\rho_j(t) = \rho(t + j) - j$.

Note, that if φ is a qpsh function with analytic singularities, then $\rho_j \circ \varphi$ is a sequence of smooth functions decreasing to φ. In [ABW, Theorem 1.1] it was proved that if φ is a psh function with analytic singularities, then

$$\lim_{j\to\infty} \big(dd^c(\rho_j\circ\varphi)\big)^m = (dd^c\varphi)^m, \tag{1.8}$$

and in [B, Theorem 1] this was extended to the case when φ is qpsh. In [A] the product $(dd^c u)^m$ was defined in the case when φ is of the form $\log|f|^2$ and a version of (1.8) was proved in this case, see [A, Proposition 4.4].

It is not hard to see that (1.7) is not commutative in general, see, e.g., [LRSW, Example 3.1] and therefore it cannot hold in general that

$$dd^c(\rho_{j_2}\circ\varphi_2)\wedge dd^c(\rho_{j_1}\circ\varphi_1) \to dd^c\varphi_2\wedge dd^c\varphi_1$$

as $j_1\to\infty$ and $j_2\to\infty$ independently, cf. Remark 4.6. The following definition is inspired by the residue theory due to Coleff and Herrera, [CH].

Definition 1.2. We say that a sequence $(j_1,\dots,j_r)\colon \mathbf{N}\to\mathbf{R}^r$ *tends to* ∞ *along an admissible path*, if for any $q\geq 0$, and $k=1,\dots,r-1$,

$$j_k(\nu) - q\cdot j_{k+1}(\nu)\to\infty$$

and $j_r(\nu)\to\infty$ as $\nu\to\infty$.

Example 1.3. The sequence $(j_1,j_2,\dots,j_r) = (\nu^r,\nu^{r-1},\dots,\nu)$ tends to ∞ along an admissible path. □

Our main result is the following generalization of (1.8).

Theorem 1.4. *Assume that $\varphi_1,\dots,\varphi_r$ are qpsh functions with analytic singularities and let ρ_j be as in Definition 1.1. If the sequence $(j_1,\dots,j_r)\colon \mathbf{N}\to\mathbf{R}^r$ tends to ∞ along an admissible path, then*

$$\lim_{\nu\to\infty} \big(dd^c(\rho_{j_r(\nu)}\circ\varphi_r)\big)^{m_r}\wedge\cdots\wedge\big(dd^c(\rho_{j_1(\nu)}\circ\varphi_1)\big)^{m_1} = (dd^c\varphi_r)^{m_r}\wedge\cdots\wedge(dd^c\varphi_1)^{m_1}$$

for $m_1,\dots,m_r\geq 1$.

Indeed, in the case when $r=1$ we just get back (1.8). In fact, in [ABW, B] the results are slightly more general; a more general definition of analytic singularities is used, see Remark 1.7, and slightly more general sequences ρ_j are allowed, see Remark 3.3.

Inspired by [ABW, Theorem 1.2], in [LRSW] we introduced a formalism for global generalized mixed Monge-Ampère operators. If φ is a qpsh function with analytic singularities and unbounded locus Z, θ and η are closed $(1,1)$-forms, and T is a current with analytic singularities on X, we let

$$[\theta+dd^c\varphi]_\eta\wedge T := \theta\wedge\mathbf{1}_{X\setminus Z}T + dd^c\varphi\wedge\mathbf{1}_{X\setminus Z}T + \eta\wedge\mathbf{1}_Z T. \tag{1.9}$$

In fact, in [LRSW] we only allowed φ to be psh, but it is not hard to see that the definition extends to qpsh functions; Lemma 2.4 asserts that $[\theta+dd^c\varphi]_\eta\wedge T$ is a well-defined current with analytic singularities that is independent of the decomposition of the current $\theta+dd^c\varphi$ as the sum of θ and $dd^c\varphi$. In particular, if $\varphi_1,\dots,\varphi_r$ are qpsh functions with analytic singularities and $\theta_1,\dots,\theta_r$ and $\eta_1,\dots,\eta_r$ are closed $(1,1)$-forms, we can give meaning to the global mixed Monge-Ampère product

$$[\theta_r+dd^c\varphi_r]_{\eta_r}^{m_r}\wedge\cdots\wedge[\theta_1+dd^c\varphi_1]_{\eta_1}^{m_1} \tag{1.10}$$

by letting $[\theta_1+dd^c\varphi_1]_{\eta_1} = [\theta_1+dd^c\varphi_1]_{\eta_1}\wedge 1$ and inductively applying (1.9). We have the following mass formula:

Proposition 1.5. *Assume that X is compact. Moreover, assume that $\varphi_1, \dots, \varphi_r$ are qpsh functions with analytic singularities and that $\theta_1, \dots, \theta_r$ and $\eta_1, \dots, \eta_r$ are closed $(1,1)$-forms on X such that $\theta_k - \eta_k = d\alpha_k$ for some smooth forms α_k. Then,*

$$\int_X [\theta_r + dd^c\varphi_r]^{m_r}_{\eta_r} \wedge \cdots \wedge [\theta_1 + dd^c\varphi_1]^{m_1}_{\eta_1} = \int_X \theta_r^{m_r} \wedge \cdots \wedge \theta_1^{m_1},$$

where $m_1 + \cdots + m_r = \dim X$.

In the case when $r = 1$ (and φ_1 is psh and $\eta_1 = \theta_1$), this is just Theorem 1.2 in [ABW], see [LRSW, Remark 3.6] and Remark 2.7 below.

We have the following regularization result for the products (1.10) in the case when $\eta_k = \theta_k$.

Theorem 1.6. *Assume that $\varphi_1, \dots, \varphi_r$ are qpsh functions with analytic singularities, that $\theta_1, \dots, \theta_r$ are closed $(1,1)$-forms, and that ρ_j is as in Definition 1.1. If the sequence $(j_1, \dots, j_r) \colon \mathbf{N} \to \mathbf{R}^r$ tends to ∞ along an admissible path, then*

$$\lim_{\nu\to\infty} \big(\theta_r + dd^c(\rho_{j_r(\nu)} \circ \varphi_r)\big)^{m_r} \wedge \cdots \wedge \big(\theta_1 + dd^c(\rho_{j_1(\nu)} \circ \varphi_1)\big)^{m_1}$$
$$= [\theta_r + dd^c\varphi_r]^{m_r}_{\theta_r} \wedge \cdots \wedge [\theta_1 + dd^c\varphi_1]^{m_1}_{\theta_1}.$$

In Section 4 we present a regularization result, Theorem 4.1, for (1.10) in the general case. Theorems 1.4 and 1.6 are immediate consequences of Theorem 4.1 below. In fact, Theorem 1.4 also follows immediately from Theorem 1.6 by setting each $\theta_k = 0$.

When $r = 1$ Theorem 1.6 reads: if φ is a qpsh function with analytic singularities and θ is a closed $(1,1)$-form, then

$$\lim_{j\to\infty} \big(\theta + dd^c(\rho_j \circ \varphi)\big)^m = [\theta + dd^c\varphi]^m_\theta.$$

This is Theorem 1 in [B], except that the setting there is slightly more general, cf. the discussion after Theorem 1.4. Also in [B] the right hand side is denoted simply by $(\theta + dd^c\varphi)^m$, see Remark 2.7.

Mixed Monge-Ampère products of qpsh functions with analytic singularities are closely related to so-called residue currents in the sense of Coleff-Herrera, [CH], and the proofs of our results are based on regularization results for residue currents. In particular, we use a slightly modified result by the first author and Samuelsson Kalm [LS].

Remark 1.7. In the literature, sometimes a more general definition of psh functions with analytic singularities is used than here, namely that in (1.4), the function v is just required to be locally bounded. In the papers [AW2, ABW, LRSW, B] this more general definition of psh and qpsh functions with analytic singularities is considered. Also Proposition 1.5 and the results in Section 2 below work for this more general definition, while the smoothness of v appears to be essential in the proofs of Theorems 1.4 and 1.6. □

The paper is organized as follows. In Section 2 we discuss the construction of the generalized mixed Monge-Ampère operator from [LRSW]. In particular, we give a proof of Proposition 1.5. We also relate our products to mixed non-pluripolar Monge-Ampère products in the sense of [BT3, BEGZ] and rephrase Proposition 1.5 in terms of these. In Section 3 we give some background on (regularization of) residue currents and show how they are related to mixed Monge-Ampère products of (q)psh functions with analytic singularities. We also give a proof of a special case of Theorem 1.4. In Section 4 we prove Theorems 1.4 and 1.6 and more generally Theorem 4.1, and we also discuss some possible generalizations. Finally, in Section 5, we present an application of Theorem 1.6 to Chern and Segre currents for singular hermitian metrics with analytic singularities as defined in [LRSW]. Corollary 5.1 asserts that these Chern and Segre currents are given as one parameter limits of smooth forms in the corresponding Chern and Segre classes.

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2. Mixed Monge-Ampère products of qpsh functions with analytic singularities

In this section we give some further background on generalized mixed Monge-Ampère products of qpsh functions with analytic singularities. As pointed out in the introduction, within this section we allow psh and qpsh functions that have analytic singularities in the less restrictive way, i.e., where we only require v in the presentation (1.4) to be bounded, cf. Remark 1.7. Throughout the paper we will assume that X is a complex manifold. Recall that the *unbounded locus* of a psh function u on X is the set of points $x \in X$ such that u is unbounded in every neighborhood of x. The unbounded locus of a qpsh function φ, locally given as $\varphi = u + a$, is defined as the unbounded locus of u. Note that if u or φ has analytic singularities, then the unbounded locus is an analytic set, locally defined by $\{f = 0\}$ where u is given by (1.4).

The construction of mixed Monge-Ampère operators in [LRSW] is slightly more general than mentioned in the introduction. Assume that $u_1, \dots, u_r$ are psh functions with analytic singularities on X, with unbounded loci $Z_1, \dots, Z_r$, respectively. Moreover assume that $U_1, \dots, U_r \subset X$ are constructible sets contained in $X \setminus Z_1, \dots, X \setminus Z_r$, respectively. In [LRSW, Section 3] we gave meaning to the product

$$dd^c u_r \mathbf{1}_{U_r} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1}, \tag{2.1}$$

by defining it recursively as

$$dd^c u_k \mathbf{1}_{U_k} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1} := dd^c\big(u_k \mathbf{1}_{U_k} dd^c u_{k-1} \mathbf{1}_{U_{k-1}} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1}\big) \tag{2.2}$$

for $k = 1, \dots, r$. Here

$$u_k \mathbf{1}_{U_k} dd^c u_{k-1} \mathbf{1}_{U_{k-1}} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1} = \lim_{j \to \infty} u_k^j \mathbf{1}_{U_k} dd^c u_{k-1} \mathbf{1}_{U_{k-1}} \wedge \cdots \wedge dd^c u_1 \mathbf{1}_{U_1}, \tag{2.3}$$

where u_k^j is a sequence of smooth psh functions decreasing to u_k. Proposition 3.2 in [LRSW] asserts that (2.3) has locally finite mass and is independent of the regularizing sequence u_k^j, and that (2.2) is a closed positive current.

Definition 2.1. We say that a closed (p,p)-current has *analytic singularities* if it is locally of the form

$$T = \sum \beta_i \wedge \mathbf{1}_{U_i} T_i,$$

where the sum is finite, β_i are closed forms, $U_i \subset X$ are constructible sets, and T_i are currents of the form (2.1) or $T_i = 1$.

We should remark that this definition extends (in a non-essential way) the definition in [LRSW, Section 3]. There a current with analytic singularities refers to ($\mathbf{1}_U$ times) a current of the form (2.1).

Note, in particular, that if T is a current with analytic singularities, u is a psh function with analytic singularities with unbounded locus Z, and U is a constructible set contained in $X \setminus Z$, then $dd^c u \wedge \mathbf{1}_U T := dd^c(u\mathbf{1}_U T)$ is a well-defined current with analytic singularities, cf. Remark 3.3 in [LRSW].

In [LRSW, Lemma 3.5], it was proved that if T is a current with analytic singularities, $\varphi = u + a$ is a qpsh function with analytic singularities with unbounded locus Z, and $U \subset X \setminus Z$ is a constructible set, then

$$\varphi \mathbf{1}_U T := u\mathbf{1}_U T + a\mathbf{1}_U T, \tag{2.4}$$

is independent of the decomposition $\varphi = u + a$. It follows that $dd^c\varphi \wedge \mathbf{1}_U T := dd^c(\varphi\mathbf{1}_U T)$ is a well-defined current with analytic singularities. In particular, we can inductively define generalized mixed Monge-Ampère products

$$dd^c\varphi_r \mathbf{1}_{U_r} \wedge \cdots \wedge dd^c\varphi_1 \mathbf{1}_{U_1}, \tag{2.5}$$

if φ_i are qpsh functions with analytic singularities with unbounded loci Z_i and $U_i \subset X \setminus Z_i$ are constructible sets.

Remark 2.2. Assume that $\pi : X' \to X$ is a holomorphic modification and that $\varphi_1, \dots, \varphi_r$ are qpsh functions with analytic singularities on X. Then $\pi^*\varphi_1, \dots, \pi^*\varphi_r$ are qpsh functions with analytic singularities on X'. Moreover, using that $\alpha \wedge \pi_*\mu = \pi_*(\pi^*\alpha \wedge \mu)$ for any smooth form α on X and current μ on X', and that $\mathbf{1}_U \pi_*\mu = \pi_*(\mathbf{1}_{\pi^{-1}U}\mu)$ for any constructible set $U \subset X$ and any positive closed (or normal) current μ on X', it follows from the construction that, if $U_1, \dots, U_r \subset X$ are constructible sets contained in the complement of the unbounded loci of $\varphi_1, \dots, \varphi_r$, respectively, then

$$dd^c\varphi_r \mathbf{1}_{U_r} \wedge \cdots \wedge dd^c\varphi_1 \mathbf{1}_{U_1} = \pi_*\big(dd^c\pi^*\varphi_r \mathbf{1}_{\pi^{-1}U_r} \wedge \cdots \wedge dd^c\pi^*\varphi_1 \mathbf{1}_{\pi^{-1}U_1}\big)$$

More generally it follows that for any current T with analytic singularities on X there is a current T' with analytic singularities on X' such that $T = \pi_* T'$. □

Remark 2.3. Note that $dd^c\varphi \wedge \mathbf{1}_U T$ only depends on the current $dd^c\varphi$ and not on the particular choice of potential φ. Indeed, assume that $\varphi_1 = \varphi_2 + h$, where $dd^c h = 0$. Then h is smooth and thus

$$dd^c\varphi_1 \wedge \mathbf{1}_U T = dd^c(\varphi_2 + h) \wedge \mathbf{1}_U T = dd^c\varphi_2 \wedge \mathbf{1}_U T + dd^c h \wedge \mathbf{1}_U T = dd^c\varphi_2 \wedge \mathbf{1}_U T,$$

where the second equality follows since (2.4) is independent of the decomposition $\varphi = u + a$. □

As in the introduction we will use the shorthand notation

$$dd^c\varphi_r \wedge \cdots \wedge dd^c\varphi_1 = dd^c\varphi_r \mathbf{1}_{X\setminus Z_r} \wedge \cdots \wedge dd^c\varphi_1 \mathbf{1}_{X\setminus Z_1}, \tag{2.6}$$

where Z_k is the unbounded locus of φ_k. This product is neither commutative nor additive in any of the factors (except for the right-most one), cf. [LRSW, Example 3.1].

Let φ be a qpsh function with analytic singularities with unbounded locus Z, and let ρ_j be as in Definition 1.1. Since $\rho_j \circ \varphi$ is constant in a neighborhood of Z,

$$\lim_{j\to\infty} dd^c(\rho_j \circ \varphi) \wedge T = \lim_{j\to\infty} dd^c(\rho_j \circ \varphi) \wedge \mathbf{1}_{X\setminus Z} T = dd^c\varphi \wedge \mathbf{1}_{X\setminus Z} T. \tag{2.7}$$

In particular, with the shorthand notation (2.6), we get

$$\lim_{j_r\to\infty} \cdots \lim_{j_1\to\infty} dd^c(\rho_{j_r} \circ \varphi_r) \wedge \cdots \wedge dd^c(\rho_{j_1} \circ \varphi_1) = dd^c\varphi_r \wedge \cdots \wedge dd^c\varphi_1. \tag{2.8}$$

In fact, from this it follows that (2.6) coincides with the classical Bedford-Taylor-Demailly product when this is defined, cf. (the proof of) Proposition 3.4 in [LRSW].

In particular, (2.6) coincides with the classical product $dd^c\varphi_r \wedge \cdots \wedge dd^c\varphi_1$ outside $Z_1 \cup \cdots \cup Z_r$.

The following lemma shows that (1.9) is independent of the decomposition of the current $\theta + dd^c\varphi$ as the sum of θ and $dd^c\varphi$.

Lemma 2.4. *Let φ_1, φ_2 be qpsh functions with analytic singularities, let θ_1, θ_2, η be closed $(1,1)$-forms, and let T be a current with analytic singularities. Assume that $\theta_1 + dd^c\varphi_1 = \theta_2 + dd^c\varphi_2$. Then*

$$[\theta_1 + dd^c\varphi_1]_\eta \wedge T = [\theta_2 + dd^c\varphi_2]_\eta \wedge T. \tag{2.9}$$

Proof. It is enough to prove (2.9) locally in X and thus we may assume that the dd^c-lemma holds on X. Note that $\theta_1 - \theta_2 = dd^c(\varphi_2 - \varphi_1)$ is smooth and d-closed. Therefore, by the dd^c-lemma, there is a smooth function a such that $\theta_1 - \theta_2 = dd^c a$, i.e. $dd^c(\varphi_1 + a) = dd^c\varphi_2$. In particular, the difference of $\varphi_1 + a$ and φ_2 is pluriharmonic and thus smooth, so the unbounded loci of φ_1 and φ_2 coincide; let us denote this set by Z. Now

$$\begin{aligned}
&[\theta_1 + dd^c\varphi_1]_\eta \wedge T - [\theta_2 + dd^c\varphi_2]_\eta \wedge T = \\
&\quad \theta_1 \wedge \mathbf{1}_{X\setminus Z}T + dd^c\varphi_1 \wedge \mathbf{1}_{X\setminus Z}T - \theta_2 \wedge \mathbf{1}_{X\setminus Z}T - dd^c\varphi_2 \wedge \mathbf{1}_{X\setminus Z}T = \\
&\qquad dd^c a \wedge \mathbf{1}_{X\setminus Z}T + dd^c\varphi_1 \wedge \mathbf{1}_{X\setminus Z}T - dd^c\varphi_2 \wedge \mathbf{1}_{X\setminus Z}T = \\
&\qquad\qquad dd^c(a + \varphi_1) \wedge \mathbf{1}_{X\setminus Z}T - dd^c\varphi_2 \wedge \mathbf{1}_{X\setminus Z}T = 0,
\end{aligned}$$

where the third equality follows since (2.4) is independent of the decomposition $\varphi = u + a$, and the last equality follows in view of Remark 2.3 since $dd^c(\varphi_1 + a) = dd^c\varphi_2$. □

We obtain the following result regarding the d- and dd^c-cohomology for generalized Monge-Ampère products; a version of this appeared as Proposition 4.3 in [LRSW].

Proposition 2.5. *Assume that φ is a qpsh function with analytic singularities, that θ and η are closed $(1,1)$-forms, and that T is a current with analytic singularities. Moreover, assume that $\theta - \eta = d\alpha$, where α is a smooth form. Then, there is a current S such that*

$$[\theta + dd^c\varphi]_\eta \wedge T = \theta \wedge T + dS. \tag{2.10}$$

If moreover $\theta - \eta = dd^c a$, where a is a smooth function, then there is a current S' such that

$$[\theta + dd^c\varphi]_\eta \wedge T = \theta \wedge T + dd^c S'. \tag{2.11}$$

Proof. Since $\theta - \eta = d\alpha$,

$$\begin{aligned}
&[\theta + dd^c\varphi]_\eta \wedge T = \theta \wedge \mathbf{1}_{X\setminus Z}T + dd^c\varphi \wedge \mathbf{1}_{X\setminus Z}T + \eta \wedge \mathbf{1}_Z T = \\
&\quad \theta \wedge T + dd^c(\varphi \mathbf{1}_{X\setminus Z}T) + (\eta - \theta) \wedge \mathbf{1}_Z T = \theta \wedge T + d\big(d^c(\varphi \mathbf{1}_{X\setminus Z}T) - \alpha \wedge \mathbf{1}_Z T\big),
\end{aligned}$$

where we in the last equation used that $\mathbf{1}_Z T$ is closed by the Skoda-El Mir theorem. Thus (2.10) holds with $S = d^c(\varphi \mathbf{1}_{X\setminus Z}T) - \alpha \wedge \mathbf{1}_Z T$.

If $\theta - \eta = dd^c a$, then by the same arguments, (2.11) holds with $S' = \varphi \mathbf{1}_{X\setminus Z}T - a\mathbf{1}_Z T$. □

Now Proposition 1.5 follows immediately from Proposition 2.5.

Remark 2.6. Given psh functions $u_1, \dots, u_r$, the *mixed non-pluripolar Monge-Ampère product*

$$\big\langle (dd^c u_r)^{m_r} \wedge \cdots \wedge (dd^c u_1)^{m_1} \big\rangle = \lim_{j\to\infty} \mathbf{1}_{\bigcap_i \{u_i > -j\}} \big(dd^c \max(u_r, -j)\big)^{m_r} \wedge \cdots \wedge \big(dd^c \max(u_1, -j)\big)^{m_1}, \tag{2.12}$$

introduced in [BT3, BEGZ], is a closed positive current that does not charge any pluripolar set and that is well-defined if the unbounded loci of u_i are small in a certain sense, see [BEGZ, Section 1.2], in particular, if the u_i have analytic singularities.

Given closed $(1,1)$-forms θ_i and θ_i-psh functions φ_i, i.e., $\theta_i + dd^c\varphi_i \geq 0$ for $i = 1, \dots, r$ one can extend (2.12) to define the non-pluripolar product $\langle (\theta_r + dd^c\varphi_r)^{m_r} \wedge \cdots \wedge (\theta_1 + dd^c\varphi_1)^{m_1} \rangle$. If the φ_i have analytic singularities with unbounded loci Z_i and we let $Z = Z_1 \cup \cdots \cup Z_r$, it follows from the construction that

$$\big\langle (\theta_r + dd^c\varphi_r)^{m_r} \wedge \cdots \wedge (\theta_1 + dd^c\varphi_1)^{m_1} \big\rangle = \mathbf{1}_{X\setminus Z} [\theta_r + dd^c\varphi_r]^{m_r}_{\eta_r} \wedge \cdots \wedge [\theta_1 + dd^c\varphi_1]^{m_1}_{\eta_1},$$

if $\eta_1, \dots, \eta_r$ are closed $(1,1)$-forms, cohomologous to $\theta_1, \dots, \theta_r$, respectively. Thus the mass formula Proposition 1.5 can be rephrased as

$$\int_X \big\langle (\theta_r + dd^c\varphi_r)^{m_r} \wedge \cdots \wedge (\theta_1 + dd^c\varphi_1)^{m_1} \big\rangle = \int_X \theta_r^{m_r} \wedge \cdots \wedge \theta_1^{m_1} - \int_X \mathbf{1}_Z [\theta_r + dd^c\varphi_r]^{m_r}_{\eta_r} \wedge \cdots \wedge [\theta_1 + dd^c\varphi_1]^{m_1}_{\eta_1},$$

cf. [ABW, Equation (5.5)]. □

Remark 2.7. Note that $[\theta + dd^c\varphi]_\theta \wedge T = \theta \wedge T + dd^c\varphi \wedge \mathbf{1}_{X\setminus Z} T$. In particular,

$$[\theta + dd^c\varphi]^m_\theta = \big(\theta + dd^c\varphi \mathbf{1}_{X\setminus Z}\big)^m = \sum_{\ell=0}^{m} \binom{m}{\ell} \theta^{m-\ell} \wedge (dd^c\varphi)^\ell,$$

where we use the shorthand notation (2.6) in the rightmost expression. In [B] this global Monge-Ampère product was just denoted by $(\theta + dd^c\varphi)^m$. We prefer to use the notation $[\theta + dd^c\varphi]^m_\theta$ to emphasize that it depends not only on the current $\theta + dd^c\varphi$ but also on the decomposition as the sum of θ and $dd^c\varphi$, cf. Theorem 3 in [B] and the following discussion.

Alternatively,

$$[\theta + dd^c\varphi]^m_\theta = \big((\theta + dd^c\varphi)\mathbf{1}_{X\setminus Z} + \theta \mathbf{1}_Z\big)^m = (\theta + dd^c\varphi)^m + \sum_{\ell=0}^{m-1} \theta^{m-\ell} \wedge \mathbf{1}_Z (\theta + dd^c\varphi)^\ell. \tag{2.13}$$

In particular, it follows that $[\theta + dd^c\varphi]^m_\theta$ equals the ordinary Monge-Ampère product $(\theta + dd^c\varphi)^m$, if φ is locally bounded. In [ABW] the mass formula Theorem 1.2 was formulated in terms of the right-hand side of (2.13), see [LRSW, Remark 3.6]. □

Remark 2.8. Assume that $L \to X$ is a holomorphic line bundle. We say that a positive hermitian singular (in the sense of Demailly [D]) metric $e^{-\phi}$ on L has *analytic singularities* if the local weights ϕ are psh functions with analytic singularities. Since two local weights differ by a pluriharmonic function the first Chern form $dd^c\phi$ is a well-defined closed positive current on X.

Let $e^{-\psi}$ be a smooth metric on L with first Chern form $\theta = dd^c\psi$. Then $\varphi := \phi - \psi$ is a well-defined qpsh function on X and $dd^c\varphi = dd^c\phi - \theta$, and thus if T is a current with analytic singularities on X, we can write

$$[dd^c\phi]_\theta \wedge T := [\theta + dd^c\varphi]_\theta \wedge T. \tag{2.14}$$

In particular, if $\phi_1, \dots, \phi_r$ are positive hermitian metrics with analytic singularities on L and $\theta_1, \dots, \theta_r$ are Chern forms of smooth metrics $e^{-\psi_1}, \dots, e^{-\psi_r}$ on L, we can write

$$[dd^c\phi_r]_{\theta_r}^{m_r} \wedge \cdots \wedge [dd^c\phi_1]_{\theta_1}^{m_1} = [\theta_r + dd^c\varphi_r]_{\theta_r}^{m_r} \wedge \cdots \wedge [\theta_1 + dd^c\varphi_1]_{\theta_1}^{m_1}, \tag{2.15}$$

where $\varphi_i = \phi_i - \psi_i$, cf. [LRSW, Section 4]. □

3. Residue currents

In this section we give some background on (regularizations of) residue currents and relate them to certain mixed Monge-Ampère products. In particular we prove a special case of Theorem 1.4.

Throughout this paper, by a *cut-off function* we mean a function $\chi : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ which is smooth, increasing and such that $\chi(t) \equiv 0$ for $t \ll 1$ and $\chi(t) \equiv 1$ for $t \gg 1$.

In [AW] was introduced a class of so-called *pseudomeromorphic* currents that includes all smooth forms, is closed under multiplication with smooth forms and the following operations: If f is a holomorphic function, $Z = \{f = 0\}$, χ is a cut-off function, $\chi_\epsilon := \chi(|f|^2/\epsilon)$, and T is a pseudomeromorphic current on X, then the following are well-defined pseudomeromorphic currents:

$$\frac{1}{f}\,T := \lim_{\epsilon\to 0} \frac{\chi_\epsilon}{f}\,T, \quad \bar\partial\frac{1}{f} \wedge T := \lim_{\epsilon\to 0} \frac{\bar\partial\chi_\epsilon}{f} \wedge T \text{ and } \quad \mathbf{1}_{X\setminus Z}T := \lim_{\epsilon\to 0} \chi_\epsilon T, \tag{3.1}$$

see also [AW3]. Since $\mathbf{1}_{X\setminus Z}T = T$ outside of Z, and $\bar\partial\chi_\epsilon$ has its support outside of Z, it follows that

$$\frac{1}{f} fT = \mathbf{1}_{X\setminus Z}T \quad \text{and} \quad \bar\partial\frac{1}{f} \wedge \mathbf{1}_{X\setminus Z}T = \bar\partial\frac{1}{f} \wedge T. \tag{3.2}$$

In particular, if $f_1, \dots, f_r$ are holomorphic functions, then

$$\bar\partial\frac{1}{f_r} \wedge \cdots \wedge \bar\partial\frac{1}{f_1} := \lim_{\epsilon_r\to 0} \cdots \lim_{\epsilon_1\to 0} P_\epsilon, \tag{3.3}$$

where

$$P_\epsilon = \frac{\bar\partial\chi_{r,\epsilon_r}}{f_r} \wedge \cdots \wedge \frac{\bar\partial\chi_{1,\epsilon_1}}{f_1}, \tag{3.4}$$

$\epsilon = (\epsilon_1, \dots, \epsilon_r)$, and

$$\chi_{k,\epsilon} = \chi(|f_k|^2/\epsilon), \tag{3.5}$$

is a well-defined pseudomeromorphic current. Products like these were first defined by Coleff and Herrera, [CH], and therefore, (3.3) is often referred to as the *Coleff-Herrera product* of $f_1, \dots, f_r$. The products in [CH] were defined in a slightly different way, taking one parameter limits along certain so-called admissible paths instead of iterated limits like in (3.3).

Definition 3.1. We say that a sequence $(\epsilon_1, \dots, \epsilon_r)\colon \mathbf{N} \to \mathbf{R}^r_{>0}$ *tends to* 0 *along an admissible path*, if for any $q \geq 0$, and $k = 1, \dots, r-1$,

$$\epsilon_k(\nu)/\epsilon^q_{k+1}(\nu) \to 0$$

and $\epsilon_r(\nu) \to 0$ as $\nu \to \infty$.

Given a sequence $(j_1,\dots,j_r)\colon \mathbf{N}\to\mathbf{R}^r$, let $(\epsilon_1,\dots,\epsilon_r)\colon \mathbf{N}\to\mathbf{R}^r_{>0}$ be the sequence defined by $\epsilon_k := e^{-j_k}$ for $k=1,\dots,r$. Then note that $(\epsilon_1,\dots,\epsilon_r)$ tends to 0 along an admissible path if and only if $(j_1,\dots,j_r)$ tends to ∞ along an admissible path, see Definition 1.2. If $(\epsilon_1,\dots,\epsilon_r)$ tends to 0 along an admissible path, then it follows by [LS, Theorem 2] that

$$\lim_{\nu\to\infty} P_{(\epsilon_1(\nu),\dots,\epsilon_r(\nu))} = \lim_{\epsilon'_r\to 0}\cdots\lim_{\epsilon'_1\to 0} P_{(\epsilon'_1,\dots,\epsilon'_r)}, \tag{3.6}$$

where P_ϵ is defined by (3.4). The left-hand side thus provides a regularization of (3.3) as a one parameter limit of smooth forms.

To be precise, in [CH], the product $\bar\partial(1/f_r)\wedge\cdots\wedge\bar\partial(1/f_1)$ is defined as the limit of P_ϵ along admissible paths, but where $\chi=\chi_{[1,\infty)}$ is the characteristic function of $[1,\infty)$ and the factor $\bar\partial\chi_{r,\epsilon_r}\wedge\cdots\wedge\bar\partial\chi_{1,\epsilon_1}$ in P_ϵ then should be interpreted as the current of integration along $\cap\{|f_k|^2=\epsilon_k\}$. By combining ideas from [CH] and [P] one can show that (3.3) in fact coincides with Coleff-Herrera's original definition, see [LS, Section 1]; in particular, this follows from Theorem 11 in [LS].

Let $\varphi_k=\log|f_k|^2$, where f_k is a holomorphic function, and let $Z_k=\{f_k=0\}$. Then the mixed Monge-Ampère product (1.7) is closely related to the Coleff-Herrera product (3.3). Formally, if T is a pseudomeromorphic current, in view of (3.1),

$$dd^c\varphi_k\wedge\mathbf{1}_{X\setminus Z_k}T=\frac{1}{2\pi i}\,\bar\partial\partial\varphi_k\wedge\mathbf{1}_{X\setminus Z_k}T=\frac{1}{2\pi i}\,\bar\partial\frac{1}{f_k}\wedge\partial f_k\wedge T \tag{3.7}$$

and so, formally,

$$dd^c\varphi_r\wedge\cdots\wedge dd^c\varphi_1=\bar\partial\frac{1}{f_r}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge\Theta,\quad\text{where }\Theta=\frac{1}{(2\pi i)^r}\partial f_1\wedge\cdots\wedge\partial f_r. \tag{3.8}$$

To give a rigorous proof of (3.8), let ρ and ρ_j be as in Definition 1.1 and let $\chi=\rho'\circ\log$. Then note that χ is a cut-off function and $(\rho'_j\circ\log)(t)=\chi(te^j)$. Then

$$\rho'_j\circ\varphi_k=\rho'_j(\log|f_k|^2)=\chi(|f_k|^2e^j)=\chi_{k,e^{-j}} \tag{3.9}$$

see (3.5). Thus

$$\partial(\rho_j\circ\varphi_k)=\rho'_j\circ\varphi_k\;\partial\varphi_k=\chi_{k,e^{-j}}\frac{\partial f_k}{f_k}. \tag{3.10}$$

Since $\partial f_k/f_k$ is holomorphic on the support of $\chi_{k,e^{-j}}$ it follows that

$$dd^c(\rho_j\circ\varphi_k)=\frac{1}{2\pi i}\frac{\bar\partial\chi_{k,e^{-j}}}{f_k}\wedge\partial f_k. \tag{3.11}$$

Now, let $(\epsilon_1,\dots,\epsilon_r)$ be defined by $\epsilon_k=e^{-j_k}$. Then

$$dd^c(\rho_{j_r}\circ\varphi_r)\wedge\cdots\wedge dd^c(\rho_{j_1}\circ\varphi_1)=P_\epsilon\wedge\Theta, \tag{3.12}$$

cf. (3.3) and (3.8). Taking iterated limits $\lim_{j_r\to\infty}\cdots\lim_{j_1\to\infty}$ of both sides of (3.12), in view of (2.8) and (3.3), we get (3.8).

Remark 3.2. If χ is a cut-off function, then note that $\rho(t):=\int_0^t\chi(e^s)ds+c$ is as in Definition 1.1 for an appropriate choice of constant c and that $\rho'(\log t)=\chi(t)$. □

Note that Theorem 1.4 in this case, when $\varphi_k=\log|f_k|^2$ and $m_k=1$ for $k=1,\dots,r$, follows directly from (3.12), (3.6), and (2.8), using that $(j_1,\dots,j_r)$ tends to ∞ along an admissible path if and only if $(\epsilon_1,\dots,\epsilon_r)$ tends to 0 along an admissible path.

Remark 3.3. The reason that we require ρ and ρ_j in Definition 1.1 to be slightly more restrictive than in [ABW, B] is that then χ defined above is a cut-off function, which is used in, e.g., [P, LS]. Possibly the results (we need) in [LS] could be extended to more general χ that would correspond to more general ρ. □

Next, let us consider functions of the form $\varphi_k = c_k \log|f_k|^2 + v_k$, where $c_k > 0$, f_k is a single holomorphic function, and v_k is smooth. In fact, after a principalization and resolution of singularities, any qpsh function with analytic singularities is of this form. Then formally, using (3.2),

$$dd^c\varphi_k \wedge \mathbf{1}_{X\setminus Z_k} T = \left(\frac{c_k}{2\pi i}\,\bar\partial\frac{1}{f_k}\wedge\partial f_k + \frac{1}{f_k}\cdot f_k dd^c v_k\right)\wedge T,$$

cf. (3.7), so that, formally,

$$dd^c\varphi_r \wedge\cdots\wedge dd^c\varphi_1 = \left(\frac{c_r}{2\pi i}\,\bar\partial\frac{1}{f_r}\wedge\partial f_r + \frac{1}{f_r}\cdot f_r dd^c v_r\right)\wedge\cdots\wedge\left(\frac{c_1}{2\pi i}\,\bar\partial\frac{1}{f_1}\wedge\partial f_1 + \frac{1}{f_1}\cdot f_1 dd^c v_1\right) \tag{3.13}$$

The right-hand side of (3.13) may be approximated in a similar way as above, cf. (4.6) below, and Theorem 1.4 in this situation may then be proved using Proposition 4.4, which is a generalization of (3.6) that allows for products of factors which are either $\bar\partial(1/f_k)$ or $1/f_k$.

4. Regularizations of mixed Monge-Ampère products

In this section we prove Theorems 1.4 and 1.6. In fact, we prove the following more general result.

Theorem 4.1. *Assume that $\varphi_1,\dots,\varphi_r$ are qpsh functions with analytic singularities, that $\theta_1,\dots,\theta_r$ and $\eta_1,\dots,\eta_r$ are closed $(1,1)$-forms, that $m_1,\dots,m_r\geq 1$, and that ρ_j is as in Definition 1.1. Let*

$$\alpha_j^{(k)} = \big(\eta_k + \rho_j'\circ\varphi_k\cdot(\theta_k-\eta_k) + dd^c(\rho_j\circ\varphi_k)\big)^{m_k}. \tag{4.1}$$

Assume that the sequence $(j_1,\dots,j_r)\colon \mathbf{N}\to\mathbf{R}^r$ tends to ∞ along an admissible path. Then

$$\lim_{\nu\to\infty}\alpha_{j_r(\nu)}^{(r)}\wedge\cdots\wedge\alpha_{j_1(\nu)}^{(1)} = [\theta_r + dd^c\varphi_r]_{\eta_r}^{m_r}\wedge\cdots\wedge[\theta_1+dd^c\varphi_1]_{\eta_1}^{m_1}. \tag{4.2}$$

Theorem 1.4 then corresponds to $\theta_i=\eta_i=0$ and Theorem 1.6 corresponds to $\theta_i=\eta_i$ for $i=1,\dots,r$.

The proof is essentially an elaboration of the proof of the special case of Theorem 1.4 in the previous section. Before giving the proof we need some preparatory results. First, let us assume that φ_k is of the form

$$\varphi_k = c_k\log|f_k|^2 + v_k, \tag{4.3}$$

where c_k is a positive constant, f_k is a tuple of holomorphic functions, and v_k is smooth. Let ρ and ρ_j be as in Definition 1.1. Let χ be the cut-off function $\chi := \rho\circ\log$, let $\epsilon_j := e^{-j}$, and let

$$\chi_{k,\epsilon} := \chi(|f_k|^{2c_k}e^{v_k}/\epsilon), \tag{4.4}$$

cf. (3.5)[2]. Then,

$$\rho_j'\circ\varphi_k = \rho_j'(c_k\log|f_k|^2+v_k) = (\rho_j'\circ\log)(|f_k|^{2c_k}e^{v_k}) = \chi(|f_k|^{2c_k}e^{v_k}e^j) = \chi_{k,\epsilon_j}, \tag{4.5}$$

[2]Note that (3.5) corresponds to $c_k=1$ and $v_k=0$ in (4.3).

cf. (3.9), and thus

$$\alpha_j^{(k)} = \big(\eta_k + \chi_{k,\epsilon_j} \cdot (\theta_k - \eta_k) + dd^c(\rho_j \circ \varphi_k)\big)^{m_k}.$$

Next, assume that φ_k is a qpsh function of the form (4.3), but where f_k is a single holomorphic function. Also, let us drop the index k and assume that φ is a function of the form

$$\varphi = c \log |f|^2 + v,$$

where f is a holomorphic function, $c > 0$, and v is smooth, and write $\chi_\epsilon = \chi(|f|^{2c} e^v/\epsilon)$. Moreover, let $Z = \{f = 0\}$ denote the unbounded locus of φ. Then it follows from (4.5) that

$$\partial(\rho_j \circ \varphi) = \rho_j' \circ \varphi \, \partial\varphi = \chi_{\epsilon_j} \cdot \left(c\frac{\partial f}{f} + \partial v\right),$$

cf. (3.10). Since $\partial f/f$ is holomorphic on the support of $\bar\partial\chi_{\epsilon_j}$ it follows that

$$dd^c(\rho_j \circ \varphi) = \frac{1}{2\pi i}\bar\partial\left(\chi_{\epsilon_j}\left(c\frac{\partial f}{f} + \partial v\right)\right) = \bar\partial\chi_{\epsilon_j} \wedge \frac{1}{2\pi i}\left(c\frac{\partial f}{f} + \partial v\right) + \chi_{\epsilon_j} dd^c v, \tag{4.6}$$

cf. (3.11).

Lemma 4.2. *Assume that T is a current with analytic singularities. Then,*

$$\lim_{j\to\infty} \bar\partial\chi_{\epsilon_j} \wedge \frac{1}{2\pi i}\left(c\frac{\partial f}{f} + \partial v\right) \wedge T \to dd^c(c \log|f|^2) \wedge \mathbf{1}_{X\setminus Z} T. \tag{4.7}$$

Proof. Using (4.6) and (2.7), we get

$$\lim_{j\to\infty} \bar\partial\chi_{\epsilon_j} \wedge \frac{1}{2\pi i}\left(c\frac{\partial f}{f} + \partial v\right) \wedge T = \lim_{j\to\infty} dd^c(\rho_j \circ \varphi) \wedge T - \lim_{j\to\infty} \chi_{\epsilon_j} dd^c v \wedge T =$$
$$dd^c\varphi \wedge \mathbf{1}_{X\setminus Z} T - dd^c v \wedge \mathbf{1}_{X\setminus Z} T.$$

Since (2.4) is independent of the decomposition $\varphi = u + a$, it follows that $dd^c\varphi \wedge \mathbf{1}_{X\setminus Z}T - dd^c v \wedge \mathbf{1}_{X\setminus Z}T = dd^c(c\log|f|^2) \wedge \mathbf{1}_{X\setminus Z}T$, and thus (4.7) follows. □

Lemma 4.3. *Assume that θ and η are closed $(1,1)$-forms and let*

$$\alpha_j = \big(\eta + \chi_{\epsilon_j} \cdot (\theta - \eta) + dd^c(\rho_j \circ \varphi)\big)^m.$$

Then there exist smooth forms $\Theta_{\ell,1}$ and $\Theta_{\ell,2}$, $\ell = 1, \dots, m$, independent of j, such that

$$\alpha_j = \eta^m + \sum_{\ell=1}^m \left(\frac{\bar\partial\chi_{\epsilon_j}^\ell}{f} \wedge \Theta_{\ell,1} + \frac{\chi_{\epsilon_j}^\ell}{f} \cdot \Theta_{\ell,2}\right).$$

Furthermore, if T is a current with analytic singularities, then

$$\lim_{j\to\infty} \alpha_j \wedge T = [\theta + dd^c\varphi]_\eta^m \wedge T.$$

Proof. First note that

$$\alpha_j = \sum_{\ell=0}^m \binom{m}{\ell} \eta^{m-\ell} \wedge \big(\chi_{\epsilon_j} \cdot (\theta - \eta) + dd^c(\rho_j \circ \varphi)\big)^\ell.$$

Set $\beta = \theta - \eta + dd^c v$. Then by (4.6)

$$\chi_{\epsilon_j} \cdot (\theta - \eta) + dd^c(\rho_j \circ \varphi) = \chi_{\epsilon_j}\beta + \bar\partial\chi_{\epsilon_j} \wedge \frac{1}{2\pi i}\left(c\frac{\partial f}{f} + \partial v\right)$$

and using that $(\bar\partial\chi_{\epsilon_j})^2=0$ and $\ell\chi_{\epsilon_j}^{\ell-1}\bar\partial\chi_{\epsilon_j}=\bar\partial\chi_{\epsilon_j}^{\ell}$ we get

$$\big(\chi_{\epsilon_j}\cdot(\theta-\eta)+dd^c(\rho_j\circ\varphi)\big)^\ell=\left(\chi_{\epsilon_j}^\ell\beta+\bar\partial\chi_{\epsilon_j}^\ell\wedge\frac{1}{2\pi i}\left(c\frac{\partial f}{f}+\partial v\right)\right)\wedge\beta^{\ell-1}.$$

Thus

$$\alpha_j=\eta^m+\sum_{\ell=1}^m\binom{m}{\ell}\eta^{m-\ell}\wedge\left(\chi_{\epsilon_j}^\ell\beta+\bar\partial\chi_{\epsilon_j}^\ell\wedge\frac{1}{2\pi i}\left(c\frac{\partial f}{f}+\partial v\right)\right)\wedge\beta^{\ell-1}, \tag{4.8}$$

so that α_j is of the desired form with $\Theta_{\ell,1}:=\binom{m}{\ell}\eta^{m-\ell}\wedge\beta^{\ell-1}\wedge\frac{1}{2\pi i}(c\partial f+f\partial v)$ and $\Theta_{\ell,2}:=\binom{m}{\ell}\eta^{m-\ell}\wedge\beta^\ell\cdot f$.

Next, by (3.1) and Lemma 4.2, since χ^ℓ is a cut-off function whenever χ is,

$$\lim_{j\to\infty}\left(\chi_{\epsilon_j}^\ell\beta+\bar\partial\chi_{\epsilon_j}^\ell\wedge\frac{1}{2\pi i}\left(c\frac{\partial f}{f}+\partial v\right)\right)\wedge T=(\beta+c\,dd^c\log|f|^2)\wedge\mathbf{1}_{X\setminus Z}T. \tag{4.9}$$

Since $\mathbf{1}_{X\setminus Z}dd^c\log|f|^2\wedge T=0$ and β is smooth, by induction over ℓ we get that

$$\big((\beta+c\,dd^c\log|f|^2)\wedge\mathbf{1}_{X\setminus Z}\big)^\ell\wedge T=(\beta+c\,dd^c\log|f|^2)\wedge\mathbf{1}_{X\setminus Z}\beta^{\ell-1}\wedge T. \tag{4.10}$$

Combining (4.8), (4.9) and (4.10) we get

$$\lim_{j\to\infty}\alpha_j\wedge T=\eta^m\wedge T+\sum_{\ell=1}^m\binom{m}{\ell}\eta^{m-\ell}\wedge(\beta+c\,dd^c\log|f|^2)\wedge\mathbf{1}_{X\setminus Z}\beta^{\ell-1}T=$$

$$\eta^m\wedge T+\sum_{\ell=1}^m\binom{m}{\ell}\eta^{m-\ell}\wedge\big((\beta+c\,dd^c\log|f|^2)\wedge\mathbf{1}_{X\setminus Z}\big)^\ell T=$$

$$\big(\eta+(\beta+c\,dd^c\log|f|^2)\wedge\mathbf{1}_{X\setminus Z}\big)^m\wedge T=\big(\eta\mathbf{1}_Z+(\theta+dd^c\varphi)\mathbf{1}_{X\setminus Z}\big)^m\wedge T=$$

$$[\theta+dd^c\varphi]_\eta^m\wedge T.$$

□

To prove Theorem 4.1 we need the following more general version of (3.6), which essentially follows from the proof of Theorem 11 in [LS].

Proposition 4.4. *Let P_k^ϵ be either $\tilde\chi_{k,\epsilon}/f_k$ or $\bar\partial\tilde\chi_{k,\epsilon}/f_k$, where $\epsilon>0$, f_k is a holomorphic function and $\tilde\chi_{k,\epsilon}=\chi_k(|f_k|^{2c_k}e^{v_k}/\epsilon)$, where χ_k is a cut-off function, $c_k>0$, and v_k is smooth, for $k=1,\dots,r$. For any $(\epsilon_1,\dots,\epsilon_r)$ that tends to 0 along an admissible path,*

$$\lim_{\nu\to\infty}P_r^{\epsilon_r(\nu)}\wedge\cdots\wedge P_1^{\epsilon_1(\nu)}=\lim_{\epsilon_r'\to0}\cdots\lim_{\epsilon_1'\to0}P_r^{\epsilon_r'}\wedge\cdots\wedge P_1^{\epsilon_1'}.$$

Note that the difference between $\tilde\chi_{k,\epsilon}$ and $\chi_{k,\epsilon}$ in (4.4) is that we allow different cut-off functions χ_k in $\tilde\chi_{k,\epsilon}$.

For this result, it is crucial that the v_k are smooth. Indeed, the proof below uses a change of variables involving e^{v_k}, and this would not be possible if v_k was just assumed to be a locally bounded psh function.

Proof. If $\tilde\chi_{k,\epsilon}=\chi(|f_k|^2/\epsilon)$, i.e., when $\chi_k=\chi$ for some cut-off function χ, $c_k=1$ and $v_k=0$, then this indeed follows from [LS, Theorem 11]. To reduce to the case $c_k=1$, one lets $\hat\chi_k(t)=\chi_k(t^{c_k})$, which is also a cut-off function, $\hat v_k=v_k/c_k$ and $\hat\epsilon_k=\epsilon_k^{1/c_k}$, so that $\tilde\chi_{k,\epsilon_k}=\hat\chi_k(|f|^2e^{\hat v_k}/\hat\epsilon_k)$. To allow for general v_k and χ_k, one just has to observe that the proof goes through in the same way in that situation. Indeed, to allow for the case that $v_k\not\equiv0$, one just notices that in the beginning of the proof of [LS, Theorem 11], one may simply replace ξ by ξ times (the pullback to a resolution of singularities of) e^{v_k}. To allow for different χ_k, in the proof, where $\chi_j^\epsilon = \chi(|x^{\tilde{\alpha}_j}|^2\xi_j/\epsilon_{\nu(j)})$ or $\chi_j^\epsilon = \chi(|y^{\tilde{\alpha}_j}|^2/\epsilon_{\nu(j)})$ appears, one just replaces χ in the right-hand side by χ_j and the proof will proceed in exactly the same way. □

Proof of Theorem 4.1. As above, let $\chi = \rho' \circ \log$, cf. Remark 3.2. Since (4.2) is a local statement, we may assume that each φ_k is of the form (4.3). Moreover, after a principalization and a resolution of singularities we may assume that each f_k is a single holomorphic function, cf. [ABW, Section 4] and Remark 2.2. By recursively applying the second part of Lemma 4.3 we have

$$\lim_{j'_r\to\infty}\cdots\lim_{j'_1\to\infty}\alpha^{(r)}_{j'_r}\wedge\cdots\wedge\alpha^{(1)}_{j'_1} = [\theta_r + dd^c\varphi_r]^{m_r}_{\eta_r}\wedge\cdots\wedge[\theta_1+dd^c\varphi_1]^{m_1}_{\eta_1}$$

and thus it suffices to prove

$$\lim_{\nu\to\infty}\alpha^{(r)}_{j_r(\nu)}\wedge\cdots\wedge\alpha^{(1)}_{j_1(\nu)} = \lim_{j'_r\to\infty}\cdots\lim_{j'_1\to\infty}\alpha^{(r)}_{j'_r}\wedge\cdots\wedge\alpha^{(1)}_{j'_1}. \tag{4.11}$$

As above let $\epsilon_j = e^{-j}$, and let

$$P^j_{k,\ell,1} = \bar\partial(\chi_{k,\epsilon_j})^\ell/f_k \text{ and } P^j_{k,\ell,2} = (\chi_{k,\epsilon_j})^\ell/f_k$$

for $k = 1,\dots,r$, $\ell = 1,\dots,m_k$. Since χ^ℓ is a cut-off function whenever χ is, it follows that $P^j_{k,\ell,i}$ are as in Proposition 4.4. By the first part of Lemma 4.3 there exist smooth forms $\Theta_{K,L,I}$ such that

$$\alpha^{(r)}_{j_r}\wedge\cdots\wedge\alpha^{(1)}_{j_1} = \eta_r^{m_r}\wedge\cdots\wedge\eta_1^{m_1} + \sum_{s=1}^{r}\sum_{K,L,I}\Theta_{K,L,I}\wedge P^{j_{k_s}}_{k_s,\ell_s,i_s}\wedge\cdots\wedge P^{j_{k_1}}_{k_1,\ell_1,i_1} \tag{4.12}$$

where the inner sum is taken over all integer tuples $K = (k_1,\dots,k_s)$ with $1\le k_1 < \cdots < k_s \le r$, all integer tuples $L = (\ell_1,\dots,\ell_s)$ with $1 \le \ell_\kappa \le m_{k_\kappa}$, $\kappa = 1,\dots,s$, and all tuples $I = (i_1,\dots,i_s)$ with $i_\kappa \in \{1,2\}$, $\kappa = 1,\dots,s$.

Since $(j_1,\dots,j_r)\colon \mathbf{N}\to\mathbf{R}^r$ tends to ∞ along an admissible path, then so does $(j_{k_1},\dots,j_{k_s})\colon\mathbf{N}\to\mathbf{R}^s$, if $K = (k_1,\dots,k_s)$ is as above, and so $(\epsilon_{k_1},\dots,\epsilon_{k_s})$ tends to 0 along an admissible path. Thus, by Proposition 4.4

$$\lim_{\nu\to\infty}P^{j_{k_s}(\nu)}_{k_s,\ell_s,i_s}\wedge\cdots\wedge P^{j_{k_1}(\nu)}_{k_1,\ell_1,i_1} = \lim_{j'_s\to\infty}\cdots\lim_{j'_1\to\infty}P^{j'_s}_{k_s,\ell_s,i_s}\wedge\cdots\wedge P^{j'_1}_{k_1,\ell_1,i_1},$$

and hence (4.11) follows in view of (4.12). □

Remark 4.5. With simple adaptations to the above proof, we get regularizations also of the more general mixed Monge-Ampère products (2.5). For instance, let $\varphi_1,\varphi_2,\varphi_3$ be qpsh functions with analytic singularities, let Z_2 be the unbounded locus of φ_2, and let $(j_1,j_2,j_3)\colon\mathbf{N}\to\mathbf{R}^3$ be a sequence tending to ∞ along an admissible path. Then,

$$\lim_{\nu\to\infty}\big(dd^c(\rho_{j_3(\nu)}\circ\varphi_3)\big)^{m_3}\wedge(\rho'_{j_2(\nu)}\circ\varphi_2)\cdot\big(dd^c(\rho_{j_1(\nu)}\circ\varphi_1)\big)^{m_1} = (dd^c\varphi_3)^{m_3}\wedge\mathbf{1}_{X\setminus Z_2}(dd^c\varphi_1)^{m_1}.$$

□

Remark 4.6. It could appear natural in the situation of Theorem 1.4 to consider one parameter limits like

$$\lim_{j\to\infty}\big(dd^c(\rho_j\circ\varphi_r)\big)^{m_r}\wedge\cdots\wedge\big(dd^c(\rho_j\circ\varphi_1)\big)^{m_1}, \tag{4.13}$$

i.e., where all the j_k are all equal to a single j. This would correspond to letting all the ϵ_k in Proposition 4.4 be equal to a single ϵ. If P_k^ϵ are as in Proposition 4.4, then limits of expressions like $P_r^{\epsilon_r} \wedge \cdots \wedge P_1^{\epsilon_1}$ are very sensitive to how $(\epsilon_1, \ldots, \epsilon_r)$ tends to 0. In fact, if we let

$$I(\mathbf{s}) := \lim_{\delta \to 0} P_r^{\delta s_r} \wedge \cdots \wedge P_1^{\delta s_1},$$

where $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbf{R}^r_{>0}$, then by [P, Proposition 1], there exist finitely many vectors $\mathbf{n}_i \in \mathbf{Q}^r$, $i = 1, \ldots, N$, such that $I(\mathbf{s})$ is well-defined and locally constant on $\mathbf{R}^r_{>0} \setminus \cup\{\mathbf{n}_i \cdot \mathbf{s} = 0\}$. The case above with all ϵ_k equal to ϵ corresponds to when $\mathbf{s} = (1, \ldots, 1)$, and it could very well happen that $\mathbf{s}$ lies in one of the hyperplanes $\{\mathbf{n}_i \cdot \mathbf{s} = 0\}$, in which case we would not know whether $I(1, \ldots, 1)$ is well-defined. Hence, we do not know in general if the limit (4.13) exists. □

Remark 4.7. Assume that we are in the situation of Theorem 1.4, or more generally Theorem 4.1. By choosing $1 < r_1 < r_2 < \cdots < r_p = r$, we may divide $\{1, \ldots, r\}$ into p blocks

$$\{1, \ldots, r_1\}, \{r_1 + 1, \ldots, r_2\}, \ldots, \{r_{p-1} + 1, \ldots, r_p\}.$$

It could be natural to consider limits that tend to ∞ along admissible paths iteratively in each block, so that the left-hand side in Theorem 1.4 corresponds to the iterated limit when there is just a single block $\{1, \ldots, r\}$, while the right-hand side corresponds to the limit when we have r blocks $\{1\}, \ldots, \{r\}$.

In fact, in [LS] certain generalized admissible paths are considered that give regularization results like this for residue currents. By small adaptations of our proofs to this situation we would get results like

$$\lim_{\nu_p \to \infty} \cdots \lim_{\nu_1 \to \infty} \big(dd^c(\rho_{j_{r_p}(\nu_p)} \circ \varphi_{r_p})\big)^{m_{r_p}} \wedge \cdots \wedge \big(dd^c(\rho_{j_{r_{p-1}+1}(\nu_p)} \circ \varphi_{r_{p-1}+1})\big)^{m_{r_{p-1}+1}} \wedge$$
$$\cdots \wedge \big(dd^c(\rho_{j_{r_1}(\nu_1)} \circ \varphi_{r_1})\big)^{m_{r_1}} \wedge \cdots \wedge \big(dd^c(\rho_{j_1(\nu_1)} \circ \varphi_1)\big)^{m_1}\Big) = (dd^c\varphi_r)^{m_r} \wedge \cdots \wedge (dd^c\varphi_1)^{m_1}.$$

if each $(j_{r_k+1}, \ldots, j_{r_{k+1}})$ tends to infinity along an admissible path. □

5. Chern and Segre forms of metrics with analytic singularities

In [LRSW], we use generalized mixed Monge-Ampère products to construct Chern and Segre forms, or rather currents, for hermitian metrics on holomorphic vector bundles that have analytic singularities in a certain sense. In this section, we apply the results presented above to get an approximation of these Chern and Segre currents by smooth forms in the corresponding Chern and Segre classes.

Let us briefly recall the construction in [LRSW]; for details and references we refer to that paper. Assume that $E \to X$ is a holomorphic vector bundle of rank r. Let us first consider the classical setting and assume that h is a smooth hermitian metric on E. Let $\pi : \mathbf{P}(E) \to X$ be the projective bundle of lines in E^*. Then h^* induces a metric on the tautological line bundle $\mathcal{O}_{\mathbf{P}(E)}(-1) \subset \pi^*E^*$; let $e^{-\phi}$ be the dual metric on $L := \mathcal{O}_{\mathbf{P}(E)}(1)$. If h is Griffiths semipositive, then $e^{-\phi}$ is a semipositive metric, i.e., the local weights ϕ are psh. The kth *Segre form* can be defined as

$$s_k(E, h) := (-1)^k \pi_*(dd^c\phi)^{k+r-1}. \tag{5.1}$$

This definition coincides with the classical definition of Segre forms, which means that the total Segre form $s(E, h) = 1 + s_1(E, h) + s_2(E, h) + \cdots$ is the multiplicative inverse of the total Chern form $c(E, h) = 1 + c_1(E, h) + c_2(E, h) + \cdots$.

In [LRSW] we considered Griffiths semipositive singular metrics h on E in the sense of Berndtsson-Păun, [BP], such that the corresponding singular metrics $e^{-\phi}$ on L satisfy that the local weights ϕ are psh with analytic singularities[3]; we say that such h have *analytic singularities.* For these metrics we constructed Chern and Segre forms by mimicking the smooth setting. Let θ be a first Chern form of a smooth metric $e^{-\psi}$ on L, and let

$$s_k(E,h,\theta) := (-1)^k \pi_*[dd^c\phi]_\theta^{k+r-1}, \tag{5.2}$$

see Remark 2.8; this is a closed normal (k,k)-current. Since $[dd^c\phi]_\theta^m = (dd^c\phi)^m$ where h is smooth, cf. Remark 2.7, it follows that $s_k(E,h,\theta)$ coincides with the classical Segre form $s_k(E,h)$ where h is smooth. Moreover, by Proposition 2.5, $[dd^c\phi]_\theta^m$ is cohomologous to θ^m, and thus $s_k(E,h,\theta)$ is in the kth Segre class $s_k(E)$ of E, i.e., the class of the kth Segre form of a smooth metric.

To construct Chern forms we defined products of the Segre forms (5.2). Let $E_1,\dots,E_t$ be t disjoint copies of E and let $\varpi : Y \to X$ be the fiber product $Y = \mathbf{P}(E_t)\times_X \cdots \times_X \mathbf{P}(E_1)$. Let ϕ_i and θ_i denote the pullbacks to Y of the metric and form on $\mathbf{P}(E_i)$ corresponding to ϕ and θ, respectively. Now, for $k_1,...,k_t \geq 1$, we define

$$s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta) := (-1)^k \varpi_*\big([dd^c\phi_t]_{\theta_t}^{k_t+r-1}\wedge\cdots\wedge[dd^c\phi_1]_{\theta_1}^{k_1+r-1}\big),$$

where $k = k_1+\cdots+k_t$, see Remark 2.8, and

$$c_k(E,h,\theta) := \sum_{k_1+\cdots+k_t=k} (-1)^t s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta), \tag{5.3}$$

so that the total Chern form $1 + c_1(E,h,\theta) + \cdots$ times the total Segre form $1 + s_1(E,h,\theta)+\cdots$ equals 1. As above, it follows from the construction that $c_k(E,h,\theta)$ coincides with the classical Chern form $c_k(E,h)$ where h is smooth and that it is in the kth Chern class $c_k(E)$ of E, see [LRSW, Theorem 1.1]. We also show that $s_k(E,h,\theta)$ and $c_k(E,h,\theta)$ coincide with the Chern and Segre forms for singular metrics defined by the first two authors and Raufi and Ruppenthal in [LRRS] when these are defined. Moreover, we show that although the currents $s_k(E,h,\theta)$ and $c_k(E,h,\theta)$ depend on the choice of θ in general, the Lelong numbers at each point $x\in X$ are independent of θ.

We want to use our regularization results to regularize these currents. Let ρ_j be as in Definition 1.1 and let

$$\alpha_{k,j} = \big(\theta + dd^c(\rho_j\circ\varphi)\big)^{k+r-1} \quad \text{and} \quad \beta_{k,j} = (-1)^k\pi_*\alpha_{k,j},$$

where φ is the qpsh function $\varphi = \phi-\psi$, cf. (4.1) and Remark 2.8. Then $\beta_{k,j}$ is a smooth form since it is the direct image of a smooth form under a submersion. Moreover, clearly $\alpha_{k,j}$ is cohomologous to θ^{k+r-1} and thus $\beta_{k,j}\in s_k(E)$, cf. (5.1). From Theorem 1.6 we get the following regularization result.

Corollary 5.1. *Assume that we are in the situation above. If $(j_1,\dots,j_t) : \mathbf{N}\to\mathbf{R}^t$ tends to ∞ along an admissible path, then*

$$\lim_{\nu\to\infty} \beta_{k_t,j_t(\nu)}\wedge\cdots\wedge\beta_{k_1,j_1(\nu)} = s_{k_t}(E,h,\theta)\wedge\cdots\wedge s_{k_1}(E,h,\theta).$$

In particular, in view of (5.3), it follows that $s_k(E,h,\theta)$ and $c_k(E,h,\theta)$ are given as limits of smooth forms in the classes $s_k(E)$ and $c_k(E)$, respectively.

[3]Recall that in [LRSW] we use the less restrictive definition of analytic singularities, cf. Remark 1.7 above.

Proof. Following the notation in [LRSW], let $\tilde{\alpha}_{k,j,i}$ be the form on $\mathbf{P}(E_i)$ corresponding to $\alpha_{k,j}$. Moreover let $\alpha_{k,j,i} = \varpi_i^* \tilde{\alpha}_{k,j,i}$, where ϖ_i is the projection $Y \to \mathbf{P}(E_i)$. By Theorem 1.6, in view of (2.15),

$$(5.4) \quad [dd^c\phi_t]_{\theta_t}^{k_t+r-1} \wedge \cdots \wedge [dd^c\phi_1]_{\theta_1}^{k_1+r-1} = \lim_{\nu\to\infty} \alpha_{k_t,j_t(\nu),t} \wedge \cdots \wedge \alpha_{k_1,j_1(\nu),1} = \\ = \lim_{\nu\to\infty} \varpi_t^* \tilde{\alpha}_{k_t,j_t(\nu),t} \wedge \cdots \wedge \varpi_1^* \alpha_{k_1,j_1(\nu),1}.$$

Let π_i be the projection $\mathbf{P}(E_i) \to X$. Applying ϖ_* to (5.4), using that $\beta_{k,j} = (-1)^k (\pi_i)_* \tilde{\alpha}_{k,j,i}$ and that

$$\varpi_*(\varpi_t^*\gamma_t \wedge \cdots \wedge \varpi_1^*\gamma_1) = (\pi_t)_*\gamma_t \wedge \cdots \wedge (\pi_1)_*\gamma_1$$

for all smooth forms $\gamma_1, \ldots, \gamma_t$ on $\mathbf{P}(E_1), \ldots, \mathbf{P}(E_t)$, respectively, see, e.g., [LRSW, Lemma 6.3], we obtain

$$s_{k_t}(E,h,\theta) \wedge \cdots \wedge s_{k_1}(E,h,\theta) = \lim_{\nu\to\infty} \varpi_*\big(\varpi_t^* \tilde{\alpha}_{k_t,j_t(\nu),t} \wedge \cdots \wedge \varpi_1^* \tilde{\alpha}_{k_1,j_1(\nu),1}\big) = \\ \lim_{\nu\to\infty} (\pi_t)_* \tilde{\alpha}_{k_t,j_t(\nu),t} \wedge \cdots \wedge (\pi_1)_* \tilde{\alpha}_{k_1,j_1(\nu),1} = \lim_{\nu\to\infty} \beta_{k_t,j_t(\nu)} \wedge \cdots \wedge \beta_{k_1,j_1(\nu)}.$$

□

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