RESIDUE CURRENTS AND CYCLES OF COMPLEXES OF VECTOR BUNDLES

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ABSTRACT. We give a factorization of the cycle of a bounded complex of vector bundles in terms of certain associated differential forms and residue currents. This is a generalization of previous results in the case when the complex is a locally free resolution of the structure sheaf of an analytic space and it can be seen as a generalization of the classical Poincaré-Lelong formula.

1. INTRODUCTION

Given a holomorphic function f on a complex manifold X, recall that the classical *Poincaré-Lelong formula* asserts that $\bar{\partial}\partial \log |f|^2 = 2\pi i[Z]$, where $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}(f)$, $\mathcal{J}(f)$ is the ideal generated by f, and $[Z]$ is the current of integration along Z, counted with multiplicities or, more precisely, the (fundamental) cycle of Z. Formally we can rewrite the Poincaré-Lelong formula as

$$\frac{1}{2\pi i}\bar{\partial}\frac{1}{f} \wedge df = [Z]. \tag{1.1}$$

This factorization of $[Z]$ can be made rigorous if we construe $\bar{\partial}(1/f)$ as the *residue current* of $1/f$, where $1/f$ is the principal value distribution as introduced by Dolbeault, [9], and Herrera and Lieberman, [14]. The current $\bar{\partial}(1/f)$ satisfies that a holomorphic function g on X is 0 in $\mathcal{O}_Z$ if and only if $g\bar{\partial}(1/f) = 0$. This is referred to as the *duality principle* and it is central to many applications of residue currents; in a way $\bar{\partial}(1/f)$ can be thought of as a current representation of the structure sheaf $\mathcal{O}_Z$.

In this article we give a similar analytic formula for the cycle of any bounded complex of vector bundles. The *cycle* of the coherent sheaf $\mathcal{F}$ on X is the cycle

$$[\mathcal{F}] = \sum_i m_i[Z_i],$$

where Z_i are the irreducible components of $\operatorname{supp}\mathcal{F}$, and m_i is the *geometric multiplicity* of Z_i in $\mathcal{F}$. For generic $z \in Z_i$, $\mathcal{F}$ can locally be given the structure of a free $\mathcal{O}_{Z_i}$-module of constant rank, and m_i is this rank. Alternatively, expressed in an algebraic manner, $m_i = \operatorname{length}_{\mathcal{O}_{Z,Z_i}}(\mathcal{F}_{Z_i})$, see, e.g., [15, Section 2]. If $\mathcal{F} = \mathcal{O}_Z$, then $[\mathcal{F}]$ coincides with the cycle of Z, cf., e.g., [13, Chapter 1.5].

Next, let

$$0 \to E_N \xrightarrow{\varphi_N} E_{N-1} \to \cdots \to E_1 \xrightarrow{\varphi_1} E_0 \to 0 \tag{1.2}$$

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be a generically exact complex of vector bundles on X. We let the *cycle* of (E,φ) be the cycle

$$[E] := \sum (-1)^\ell [\mathcal{H}_\ell(E)], \tag{1.3}$$

where $\mathcal{H}_\ell(E)$ is the homology group of (E,φ) at level ℓ. We have not found the definition of such a cycle in this setting in the literature. However, when all the homology groups have support at a single point, our cycle simply corresponds to the Euler characteristic of the complex, and (1.3) appears to be a natural generalization for general complexes. Note that if (E,φ) is a locally free resolution of a coherent sheaf $\mathcal{F}$, i.e., it is exact at all levels > 0 and $\mathcal{H}_0(E) \cong \mathcal{F}$, then $[E] = [\mathcal{F}]$.

If the E_ℓ are equipped with hermitian metrics, we say that (E,φ) is a *hermitian* complex. Given a hermitian complex (E,φ) that is exact outside a subvariety $Z \subset X$, in [4] Andersson and the second author introduced an associated residue current $R = R^E$ with support on Z, that takes values in $\operatorname{End} E$, where $E = \oplus E_k$, and that in some sense measures the exactness of (E,φ). In particular, if (E,φ) is a locally free resolution of $\mathcal{F}$, then the component R_k^ℓ that takes values in $\operatorname{Hom}(E_\ell, E_k)$ vanishes if $\ell > 0$ and R satisfies a duality principle for $\mathcal{F}$.

If f is a holomorphic function on X and $E_0 \cong \mathcal{O}_X$ and $E_1 \cong \mathcal{O}_X$ are trivial line bundles, then

$$0 \to \mathcal{O}_X \xrightarrow{\varphi_1} \mathcal{O}_X \to 0,$$

where φ_1 is the 1×1-matrix $[f]$, gives a locally free resolution of $\mathcal{O}_Z = \mathcal{O}/\mathcal{J}(f)$. In this case (the coefficient of) $R = R_1^0$ is just $\bar\partial(1/f)$, and the Poincaré-Lelong formula (1.1) can be written as[1]

$$\frac{1}{2\pi i} d\varphi_1 R_1^0 = [E]. \tag{1.4}$$

Our main result is the following generalization of (1.4). Recall that a coherent sheaf $\mathcal{F}$ has pure dimension d if $\operatorname{supp} \mathcal{F}$ has pure dimension d. Given an $\operatorname{End} E_\ell$-valued current α let $\operatorname{tr} \alpha$ denotes the trace of α.

Theorem 1.1. *Let (E,φ) be a hermitian complex of vector bundles (1.2) such that all its homology groups $\mathcal{H}_\ell(E)$ have pure codimension $p > 0$ or vanish, and let D be the connection on $\operatorname{End} E$ induced by arbitrary $(1,0)$-connections[2] on $E_0, \dots, E_N$. Then*

$$\frac{1}{(2\pi i)^p p!} \sum_{\ell=0}^{N-p} (-1)^\ell \operatorname{tr} D\varphi_{\ell+1} \cdots D\varphi_{\ell+p} R_{\ell+p}^\ell = [E]. \tag{1.5}$$

Note that the endomorphisms $D\varphi_{\ell+1} \cdots D\varphi_{\ell+p}$ depend on the choice of connections on $E_0, \dots, E_N$ and the currents $R_{\ell+p}^\ell$ in general depend on the choice of hermitian metrics on $E_0, \dots, E_N$. There is no assumption of any relation between the connections and the hermitian metrics.

The proof of Theorem 1.1, which occupies Section 4, is by induction over the number of nonvanishing homology groups $\mathcal{H}_\ell(E)$. The basic case is the special case when (E,φ) has nonvanishing homology only at level 0.

[1] For an explanation of the relation between the signs in (1.1) and (1.4), see [17, Section 2.5], cf. Section 2.1.

[2] See (2.3) for how this connection is defined.

Theorem 1.2. *Let $\mathcal{F}$ be a coherent sheaf of pure codimension p, let (E,φ) be a hermitian locally free resolution of $\mathcal{F}$, and let D be the connection on $\operatorname{End} E$ induced by arbitrary $(1,0)$-connections on $E_0,\dots,E_N$. Then*

$$\frac{1}{(2\pi i)^p p!}\operatorname{tr} D\varphi_1\cdots D\varphi_p R_p^0 = [\mathcal{F}]. \tag{1.6}$$

In [17] we gave a proof of Theorem 1.2 when $\mathcal{F}$ is the structure sheaf $\mathcal{O}_Z$ of an analytic subspace $Z\subset X$ by comparing (E,φ) to a certain universal free resolution due to Scheja and Storch, [21], and Eisenbud, Riemenschneider, and Schreyer, [12]. That proof should be possible to modify to the setting of a general $\mathcal{F}$. However, we give a simpler proof using induction over a filtration of $\mathcal{F}$, see Section 3. Note that in [17], the assumption that the connections on $E_0,\dots,E_N$ should be $(1,0)$ is missing, see the comment before Lemma 2.4 below.

If (E,φ) is the Koszul complex of a tuple of holomorphic functions $f_1,\dots,f_m$, then the coefficients of R are the so-called *Bochner-Martinelli residue currents* introduced by Passare, Tsikh, and Yger [19], and further developed by Andersson, [1], see Section 2.3. In particular, if $m=p:=\operatorname{codim} Z(f)$, where $Z(f)=\{f_1=\cdots=f_m=0\}$, (the coefficient of) the only nonvanishing component R_p^0 coincides with the classical *Coleff-Herrera product* $\bar\partial(1/f_p)\wedge\cdots\wedge\bar\partial(1/f_1)$, introduced by Coleff and Herrera in [8], see [19, Theorem 4.1] and [3, Corollary 3.2]. In this case (1.5) reads

$$\frac{1}{(2\pi i)^p}\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge df_1\wedge\cdots\wedge df_p = [Z]. \tag{1.7}$$

This generalization of the Poincaré-Lelong formula (1.1) was proved by Coleff and Herrera [8, Section 3.6]. If $m>p$, then $[E]=0$ and we can give an alternative proof of Theorem 1.1 by explicitly computing the left-hand side of (1.5), see Section 5. Since both sides in (1.5) are alternating sums, it would be a natural guess that the terms at respective levels in the sums coincide. However, this in not true in general and the Koszul complex provides a counterexample, see Example 5.1.

There are various other special cases of Theorem 1.2 and related results in the literature, see, e.g., the introduction in [17]. There are also related cohomological results by Lejeune-Jalabert and Lejeune-Jalabert-Angéniol, [6,18]. Given a free resolution (E,φ) of $\mathcal{O}_{Z,z}$, where Z is a Cohen-Macaulay analytic space, Lejeune-Jalabert, [18], constructed a generalization of the Grothendieck residue pairing, which can be seen as a cohomological version of R^E, and proved that the fundamental class of Z at z then is represented by $D\varphi_1\cdots D\varphi_p$. In [6] this construction was extended to a residue pairing associated with a more general complex of free $\mathcal{O}_z$-modules and a cohomological version, [6, Theorem I.8.2.2.3], of Theorem 1.1 was given.

In Section 6 we discuss possible extensions of our results to the case when the homology groups $\mathcal{H}_\ell(E)$ do not have pure dimension or are not of the same dimension. In particular, we present a version of Theorem 1.2 for a general, not necessarily pure dimensional, coherent sheaf $\mathcal{F}$, generalizing [17, Theorem 1.5].

2. Preliminaries

Throughout this article, (E,φ) will be a complex (1.2), where the E_k are either vector bundles on X or germs of free $\mathcal{O}$-modules, where $\mathcal{O}=\mathcal{O}_x=\mathcal{O}_{X,x}$ is the ring of germs of holomorphic functions at some $x\in X$. We will always assume that $E_k=0$ for $k<0$ and $k>N$. Since a complex (E,φ) of $\mathcal{O}$-modules can be extended to a vector bundle complex in a neighborhood of x it makes sense to equip it with hermitian metrics, and thus to talk about a hermitian complex of $\mathcal{O}$-modules.

We let $\mathcal{E}$ and $\mathcal{E}^\bullet$ be the sheaves of smooth functions and forms, respectively, on X. Given a vector bundle $E \to X$ we let $\mathcal{E}^\bullet(E) = \mathcal{E}^\bullet \otimes \mathcal{E}(E)$ denote the sheaf of form-valued sections.

2.1. **Signs and superstructure.** As in [4], we will consider the complex (E, φ) to be equipped with a so-called superstructure, i.e., a $\mathbb{Z}_2$-grading, which splits $E = \oplus E_k$ into odd and even elements E^+ and E^-, where $E^+ = \oplus E_{2k}$ and $E^- = \oplus E_{2k+1}$. Also $\operatorname{End} E$ gets a superstructure by letting the even elements be the endomorphisms preserving the degree, and the odd elements the endomorphisms switching degrees.

This superstructure affects how form- and current-valued endomorphisms act. Assume that $\alpha = \omega \otimes \gamma$ is a section of $\mathcal{E}^\bullet(\operatorname{End} E)$, where γ is a holomorphic section of $\operatorname{Hom}(E_\ell, E_k)$, and ω is a smooth form of degree m. Then we let $\deg_f \alpha = m$ and $\deg_e \alpha = k - \ell$ denote the *form* and *endomorphism* degrees, respectively, of α. The *total* degree is $\deg \alpha = \deg_f \alpha + \deg_e \alpha$. The following formulas, which can be found in [17], will be important to get the signs right in the proofs of the main results. Assume that $\alpha = \omega \otimes \gamma$ and $\alpha' = \omega' \otimes \gamma'$ are sections of $\mathcal{E}^\bullet(\operatorname{End} E)$, where ω, ω' are sections of $\mathcal{E}^\bullet$ and γ, γ' are sections of $\operatorname{End} E$. Due to how form-valued endomorphisms are defined to act on form-valued sections, one obtains the following composition of form-valued endomorphisms, [17, equation (2.2)],

$$\alpha\alpha' = (-1)^{(\deg_e \alpha)(\deg_f \alpha')} \omega \wedge \omega' \otimes \gamma\gamma'. \tag{2.1}$$

We have the following formula for the trace, see [17, equation (2.14)],

$$\operatorname{tr}(\alpha\alpha') = (-1)^{(\deg \alpha)(\deg \alpha') - (\deg_e \alpha)(\deg_e \alpha')} \operatorname{tr}(\alpha'\alpha). \tag{2.2}$$

If the bundles $E_0, \dots, E_N$ are equipped with connections D_{E_i}, there is an induced connection $D_E := \oplus D_{E_i}$ on E, which in turn induces a connection D_{End} on $\operatorname{End} E$, that takes the superstructure into account, through

$$D_{\mathrm{End}}\, \alpha := D_E \circ \alpha - (-1)^{\deg \alpha} \alpha \circ D_E. \tag{2.3}$$

This connection satisfies the following Leibniz rule, [17, equation (2.4)],

$$D_{\mathrm{End}}\, (\alpha\alpha') = D_{\mathrm{End}}\, \alpha\alpha' + (-1)^{\deg \alpha} \alpha D_{\mathrm{End}}\, \alpha'. \tag{2.4}$$

To simplify notation, we will drop the subscript End and simply denote this connection by D. All of the above formulas hold also when α and α' are current-valued instead of form-valued, as long as the involved products of currents are well-defined.

Since $\varphi_m \varphi_{m+1} = 0$ and the φ_j have odd degree, by the Leibniz rule, $\varphi_m D\varphi_{m+1} = D\varphi_m \varphi_{m+1}$, and using this repeatedly, we get that

$$D\varphi_\ell \cdots D\varphi_{k-1} \varphi_k = \varphi_\ell D\varphi_{\ell+1} \cdots D\varphi_k \tag{2.5}$$

for all $\ell < k$.

The following result is a slight generalization of [17, Lemma 4.4], that follows by the same arguments.

Lemma 2.1. *Let p be fixed. Assume that (E, φ) and (G, η) are complexes of vector bundles and that $b : (E, \varphi) \to (G, \eta)$ is a morphism of complexes. Let D be the* *connection on* $End(E \oplus G)$ *induced by arbitrary connections on* $E_\ell, \dots, E_{\ell+p}$ *and* $G_\ell, \dots, G_{\ell+p}$*, and let*[3]

$$\delta_\ell := \sum_{j=\ell}^{\ell+p-1} D\eta_{\ell+1} \cdots D\eta_j D b_j D\varphi_{j+1} \cdots D\varphi_{\ell+p-1},$$

$$\alpha_\ell := \eta_{\ell+1}\delta_{\ell+1}, \quad \beta_\ell := \delta_\ell \varphi_{\ell+p}, \text{ and } \gamma_\ell := b_\ell D\varphi_{\ell+1} \cdots D\varphi_{\ell+p}.$$

Then

$$D\eta_{\ell+1} D\eta_{\ell+2} \cdots D\eta_{\ell+p} b_{\ell+p} = \alpha_\ell + \beta_\ell + \gamma_\ell.$$

Given a complex (E, φ), let $(\widetilde{E}, \tilde{\varphi})$ be the complex where the signs are reversed, i.e., let $\widetilde{E}_k$ be E_k but with opposite sign and let $\tilde{\varphi}_k$ be the mapping $\widetilde{E}_k \to \widetilde{E}_{k-1}$ induced by φ_k. Note that $\tilde{\varphi}_k$ is odd. More generally, for any section α of $\operatorname{End} E$ or $\mathcal{E}^\bullet(\operatorname{End} E)$, let $\tilde{\alpha}$ denote the corresponding section of $\operatorname{End} \widetilde{E}$ or $\mathcal{E}^\bullet(\operatorname{End} \widetilde{E})$, respectively. Note that if $\alpha = \omega \otimes \gamma$ is a section of $\mathcal{E}^\bullet(\operatorname{End} E)$, then $\tilde{\alpha} = \omega \otimes \tilde{\gamma}$.

Next, let $\varepsilon : E_k \to \widetilde{E}_k$ be the map induced by the identity on E_k. Note that ε is an odd mapping. If γ is a section of $\operatorname{Hom}(E_k, E_{k-1})$, then

$$\varepsilon\gamma = \tilde{\gamma}\varepsilon. \tag{2.6}$$

If $\alpha = \omega \otimes \gamma$ is a section of $\mathcal{E}^\bullet(\operatorname{Hom}(E_k, E_{k-1}))$, then

$$\varepsilon\alpha = (-1)^{(\deg_e \varepsilon)(\deg_f \alpha)} \omega \otimes \varepsilon\gamma = (-1)^{\deg_f \alpha} \omega \otimes \varepsilon\gamma = (-1)^{\deg_f \alpha} \omega \otimes \tilde{\gamma}\varepsilon;$$

here we have used (2.1) for the first equality, that $\deg_e \varepsilon = 1$ for the second equality, and (2.6) for the third equality. Moreover, by (2.1),

$$\tilde{\alpha}\varepsilon = (-1)^{(\deg_e \tilde{\alpha})(\deg_f \varepsilon)} \omega \otimes \tilde{\gamma}\varepsilon = \omega \otimes \tilde{\gamma}\varepsilon,$$

since $\deg_f \varepsilon = 0$. To conclude

$$\varepsilon\alpha = (-1)^{\deg_f \alpha} \tilde{\alpha}\varepsilon. \tag{2.7}$$

2.2. **Residue currents and the comparison formula.** We will recall some properties from [4] of the residue current $R = R^E$ associated with a hermitian complex (E, φ), cf. the introduction. The part $R_k^\ell = (R^E)_k^\ell$ that takes values in $\operatorname{Hom}(E_\ell, E_k)$ is a $(0, k-\ell)$-current when $\ell < k$ and $R_k^\ell = 0$ otherwise. For us, a key property of the current R^E is that it is ∇_{End}-closed, which means that

$$\varphi_{k+1} R_{k+1}^\ell - R_k^{\ell-1} \varphi_\ell - \bar{\partial} R_k^\ell = 0, \tag{2.8}$$

for each ℓ, k, see [4, Section 2].

The residue currents R^E are examples of so-called *pseudomeromorphic currents*, introduced in [5]. Another important example is currents of integration along subvarieties $Z \subset X$, as follows, e.g., from [2, Theorem 1.1]. The sheaf of pseudomeromorphic currents is closed under multiplication by smooth forms. Moreover pseudomeromorphic currents share some properties with normal currents, and in particular they satisfy the following *dimension principle*, [5, Corollary 2.4]:

Proposition 2.2. *Let* T *be a pseudomeromorphic* $(*, p)$*-current on* X*, and assume that* T *has support on a subvariety* $Z \subset X$ *of* $\operatorname{codim} Z > p$*. Then* $T = 0$*.*

[3]Here $D\eta_{\ell+1} \cdots D\eta_j$ and $D\varphi_{j+1} \cdots D\varphi_{\ell+p-1}$ are to be interpreted as 1 if $j = \ell$ and $j = \ell + p - 1$, respectively.

If (E,φ) is pointwise exact outside a subvariety Z of codimension p and $k-\ell<p$, since R has support on Z, it follows from the dimension principle that $R_k^\ell=0$. Then (2.8) becomes

$$\varphi_{k+1}R_{k+1}^\ell = R_k^{\ell-1}\varphi_\ell. \tag{2.9}$$

In the special case when $\ell=0$ and $k=p-1$, since $E_{-1}=0$, (2.9) gives that

$$\varphi_p R_p^0 = 0. \tag{2.10}$$

If the sheaf complex (E,φ) is exact except at E_0, then $R_k^\ell=0$ for $\ell\geq 1$, see [4, Theorem 3.1]. We can then write without ambiguity $R_k=R_k^E$ for $R_k^0=(R^E)_k^0$. In this case, for $\ell=1$ and $k=p$, (2.9) reads

$$R_p\varphi_1 = 0. \tag{2.11}$$

Given a morphism $a:(F,\psi)\to(E,\varphi)$ of complexes of free $\mathcal{O}$-modules or vector bundles, the *comparison formula* from [16] relates the associated residue currents R^E and R^F. We begin by recalling an important situation when one can construct such a morphism, see for example [16, Proposition 3.1]. In this result, it is crucial that (F,ψ) and (E,φ) are complexes of free $\mathcal{O}$-modules; the corresponding statement would not necessarily be true globally if they were instead complexes of vector bundles over X.

Proposition 2.3. *Let $\alpha: A'\to A$ be a homomorphism of $\mathcal{O}$-modules, let (F,ψ) be a complex of free $\mathcal{O}$-modules with* $\operatorname{coker}\psi_1 \xrightarrow{\cong} A'$*, and let (E,φ) be a free resolution of A. Then, there exists a morphism $a:(F,\psi)\to(E,\varphi)$ of complexes which extends α.*

Here, we say that *a extends α* if the induced map $A'\xrightarrow{\cong}\operatorname{coker}\psi_1\xrightarrow{(a_0)_*}\operatorname{coker}\varphi_1\xrightarrow{\cong}A$ equals α. The comparison formula in its most general form, [16, equation (3.4)], states that for $k>\ell$, there exist pseudomeromorphic $(0,k-\ell-1)$-currents M_k^ℓ with values in $\operatorname{Hom}(F_\ell,E_k)$ and support on the union Z of the sets where (E,φ) and (F,ψ) are not pointwise exact, such that

$$(R^E)_k^\ell a_\ell = a_k(R^F)_k^\ell + \varphi_{k+1}M_{k+1}^\ell + M_k^{\ell-1}\psi_\ell - \bar\partial M_k^\ell.$$

Here, $M_k^{\ell-1}$ is to be interpreted as 0 if $\ell=0$. In all the cases we consider in this article, we have that $k-\ell\leq\operatorname{codim}Z$. Then it follows from the dimension principle that M_k^ℓ vanishes, since it is a $(0,k-\ell-1)$-current with support on Z and the comparison formula becomes

$$(R^E)_k^\ell a_\ell = a_k(R^F)_k^\ell + \varphi_{k+1}M_{k+1}^\ell + M_k^{\ell-1}\psi_\ell. \tag{2.12}$$

Since M_k^ℓ takes values in $\operatorname{Hom}(F_\ell,E_k)$ it follows that if $F_{\ell-1}=0$, then $M_k^{\ell-1}=0$ and (2.12) reads

$$(R^E)_k^\ell a_\ell = a_k(R^F)_k^\ell + \varphi_{k+1}M_{k+1}^\ell, \tag{2.13}$$

and if in addition $E_{k+1}=0$, then

$$(R^E)_k^\ell a_\ell = a_k(R^F)_k^\ell. \tag{2.14}$$

The following result is a corrected version of [17, Lemma 4.1]. In the proof in [17, Lemma 4.1] it was used that the connections are $(1,0)$-connections, i.e., that the $(0,1)$-part of the connections is $\bar\partial$, but we missed adding this assumption in the statement of the lemma, and then consequently in all other results relying on this, i.e., Theorem 1.1, 1.2, 1.5, 6.1 and Lemma 4.2.

Lemma 2.4. *Let M be a finitely generated $\mathcal{O}$-module of codimension p and let (E,φ) and (F,ψ) be hermitian free resolutions of M. Then,*

$$\operatorname{tr} D\varphi_1 \cdots D\varphi_p R^E_p = \operatorname{tr} D\psi_1 \cdots D\psi_p R^F_p,$$

where D is the connection on $End(E \oplus F)$ induced by arbitrary $(1,0)$-connections on $E_0, \dots, E_p$ and $F_0, \dots, F_p$.

2.3. **The Koszul complex, Coleff-Herrera products, and Bochner-Martinelli residue currents.** Let $f = (f_1, \dots, f_m)$ be a tuple of holomorphic functions on X and let (E,φ) be the Koszul complex of f, i.e., consider f as a section $f = \sum f_j e^*_j$ of a trivial rank m bundle F^* with a frame $e^*_1, \dots, e^*_m$, let $E_j = \bigwedge^j F$, where F is the dual bundle of F^*, and let $\varphi_k = \delta_f$ be contraction with f.

The residue current R associated with the Koszul complex (E,φ) equipped with hermitian metrics induced by a hermitian metric on F^*, was introduced and studied by Andersson in [1]. For E_k, we have the frame $\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_k} \mid I = (i_1, \dots, i_k), 1 \le i_1 < \cdots < i_k \le m\}$, where $e_1, \dots, e_m$ is the dual frame of $e^*_1, \dots, e^*_m$, and in particular, $e_\emptyset$ is a frame for E_0. In this frame we can write $R^0_p = \sum R_I \wedge e_I \wedge e^*_\emptyset$. To get the superstructure right, in [1] it is convenient to consider the endomorphism-valued currents and forms that appear in the construction of R as sections of the exterior algebra of $F^* \oplus T^*_{0,1}$, i.e., with the convention that $dz_j \wedge e_k = -e_k \wedge dz_j$ etc. If the metric on F^* is trivial, then the coefficients R_I coincide with the Bochner-Martinelli residue currents from [19].

In the case when $m = p = \operatorname{codim} Z(f)$, so that the ideal $\mathcal{J}(f)$ generated by f is a complete intersection, then the Koszul complex is a locally free resolution of $\mathcal{O}_Z := \mathcal{O}/\mathcal{J}(f)$. Since $R^\ell_k = 0$ for $k - \ell < p$ by the dimension principle, in this case $R = R^0_p$ and R^0_p consists of only one component, $R_{\{1,\dots,p\}} \wedge e_1 \wedge \cdots \wedge e_p$, where $R_{\{1,\dots,p\}} = \bar\partial(1/f_p) \wedge \cdots \wedge \bar\partial(1/f_1)$, cf. the introduction. Moreover, $D\varphi_j$ is contraction with $\sum df_j \wedge e^*_j$ and it follows that

$$D\varphi_1 \cdots D\varphi_p = p! df_1 \wedge \cdots \wedge df_p \wedge e^*_p \wedge \cdots \wedge e^*_1. \tag{2.15}$$

Therefore, the generalized Poincaré-Lelong formula (1.7) by Coleff and Herrera can be rewritten as

$$\frac{1}{(2\pi i)^p p!} D\varphi_1 \cdots D\varphi_p R^E_p = [Z]. \tag{2.16}$$

For an explanation of the signs when going from endomorphism-valued currents to scalar-valued currents, see [17, Section 2.5] and also Section 2.1 above.

2.4. **The mapping cone of a morphism of complexes.** Let $c : (L,\lambda) \to (K,\kappa)$ be a morphism of complexes. The *mapping cone* of c is the complex (C,μ) given by $C_k = K_k \oplus \widetilde{L}_{k-1}$ for $k \ge 1$ and $C_0 = K_0$, with

$$\mu_k = \begin{bmatrix} -\kappa_k & c_{k-1}\varepsilon^{-1} \\ 0 & \tilde\lambda_{k-1} \end{bmatrix} \text{ for } k \ge 2 \text{ and } \mu_1 = \begin{bmatrix} -\kappa_1 & c_0\varepsilon^{-1} \end{bmatrix}.$$

Here, the bundles and morphisms take into account the signs and superstructure from Section 2.1. Let

$$\theta_k : K_k \to C_k,\ \theta_k = \begin{bmatrix} (-1)^k \operatorname{Id}_{K_k} \\ 0 \end{bmatrix} \text{ for } k \ge 1,\ \theta_0 = \begin{bmatrix} \operatorname{Id}_{K_0} \end{bmatrix} \tag{2.17}$$

and

$$\vartheta_k : C_{k+1} \to L_k,\ \vartheta_k = \begin{bmatrix} 0 & \varepsilon^{-1} \end{bmatrix} \text{ for } k \geq 0. \tag{2.18}$$

Then $\theta : (K, \kappa) \to (C, \mu)$ and $\vartheta : (C, \mu) \to (L, \lambda)$ are morphisms of complexes (the latter of degree -1). From this construction one obtains, cf., e.g., [23, Chapter 1.5], an induced long exact sequence

$$\cdots \to H_{k+1}(C) \xrightarrow{\vartheta_k} H_k(L) \xrightarrow{c_k} H_k(K) \xrightarrow{\theta_k} H_k(C) \xrightarrow{\vartheta_{k-1}} H_{k-1}(L) \to \cdots . \tag{2.19}$$

Proposition 2.5. *Let*

$$0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$$

be a short exact sequence of $\mathcal{O}$-modules. Assume that (E, φ) and (F, ψ) are free resolutions of A and A', respectively, and that $a : (F, \psi) \to (E, \varphi)$ is a morphism of complexes extending α. Let (G, η) be the mapping cone of a and let $b : (E, \varphi) \to (G, \eta)$ be the morphism θ as defined by (2.17). Then there is an isomorphism $H_0(G) \xrightarrow{\cong} A''$, which makes (G, η) a free resolution of A'' and such that b extends β.

Proof. By (2.19), we obtain that $H_\ell(G) = 0$ for $\ell \neq 0, 1$ and the exact sequence

$$0 \to H_1(G) \to H_0(F) \to H_0(E) \to H_0(G) \to 0.$$

The morphism $H_0(F) \to H_0(E)$ equals the morphism $A' \xrightarrow{\alpha} A$ which is injective, so $H_1(G) = 0$. Thus (G, η) is a free resolution of $A/(\operatorname{im} \alpha)$. Since β gives an isomorphism $A/(\operatorname{im} \alpha) = A/(\ker \beta) \xrightarrow{\cong} \operatorname{im} \beta = A''$, (G, η) is a free resolution of A''. By construction, b extends the morphism $A \to A/(\operatorname{im} \alpha)$. □

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is by induction. The induction procedure is achieved through the following filtration of a module, see, e.g., [7, §1.4, Théorèmes 1 and 2 and §2.5, Remarque 1] or [22, Tag 00KY], that is sometimes referred to as a *prime filtration*.

Proposition 3.1. *Let M be a finitely generated $\mathcal{O}$-module. Then there exists a sequence of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M \tag{3.1}$$

such that

$$M_i/M_{i-1} \cong \mathcal{O}/P_i, \tag{3.2}$$

where $P_i \subseteq \mathcal{O}$ is a prime ideal contained in $\operatorname{supp} M$ for $i = 1, \dots, m$. The minimal prime ideals P_i (with respect to inclusion) appearing in (3.2) are exactly the minimal associated primes of M, and each such minimal prime P occurs exactly $\operatorname{length}_{\mathcal{O}_P} M_P$ times.

In general, also primes P_i appear in (3.2) that are not minimal primes of M, as in the following example. If only the minimal primes of M appear, then the filtration is said to be a *clean*, cf. [10].

Example 3.2. Let $\mathcal{J} = \mathcal{J}(xz, xw, yz, yw) \subseteq \mathcal{O}_{\mathbb{C}^4}$, which is the ideal generating the variety $\{x = y = 0\} \cup \{z = w = 0\}$. If we let

$$\mathcal{I}_0 = \mathcal{O},\ \mathcal{I}_1 = \mathcal{J}(x, y, w),\ \mathcal{I}_2 = \mathcal{J}(xz, y, w),\ \mathcal{I}_3 = \mathcal{J}(xz, xw, y),\ \mathcal{I}_4 = \mathcal{J},$$

then $M_j := \mathcal{O}/\mathcal{I}_j$ for $j = 1, \dots, 4$, is a prime filtration of $M := \mathcal{O}/\mathcal{J}$. Indeed, $M_j/M_{j-1} \cong \mathcal{I}_{j-1}/\mathcal{I}_j \cong \mathcal{O}/P_j$, where

$$P_1 = \mathcal{J}(x, y, w),\ P_2 = \mathcal{J}(y, z, w),\ P_3 \cong \mathcal{J}(x, y),\ P_4 \cong \mathcal{J}(z, w).$$

Note that P_3 and P_4 are the two (minimal) associated primes of M, which have codimension 2, while P_1 and P_2 are of codimension 3 and contained in the support of M but not associated primes of M. □

Corollary 3.3. *Let $\mathcal{F}$ be a coherent sheaf of codimension p and let $z_0 \in \operatorname{supp} \mathcal{F}$. For z in a neighborhood of z_0, outside a subvariety of positive codimension in* $\operatorname{supp} \mathcal{F}$*, $\mathcal{F}_z$ has a clean filtration where all the modules in the filtration have pure codimension p.*

Proof. Take a filtration of $\mathcal{F}_{z_0}$ as in Proposition 3.1 and choose a neighborhood $z_0 \in \mathcal{U} \subset X$ such that all M_i are defined in $\mathcal{U}$. Moreover let W be the union of the varieties of the P_i that have codimension $\geq p+1$. Take $z \in \mathcal{U} \setminus W$. For each i, at z, either $M_{i+1} = M_i$ or $M_{i+1}/M_i \cong \mathcal{O}/P_j$ for some j, where P_j is an associated prime of $\mathcal{F}_z$ of codimension p. Thus if we remove the M_i such that $M_{i+1} = M_i$ we are left with a clean filtration of $\mathcal{F}_z$. Since the sequence $0 \subset M_0 \subset \cdots \subset M_k$ gives a filtration of M_k, by Proposition 3.1, the only minimal primes of M_k are P_i for $i = 1, \dots, k$, and thus M_k has pure codimension p for $k = 1, \dots, m$. □

Lemma 3.4. *Let $P \subset \mathcal{O}$ be a prime ideal of codimension p and let (E, φ) be a hermitian free resolution of $\mathcal{O}/P$. Then*

$$\frac{1}{(2\pi i)^p p!} \operatorname{tr} D\varphi_1 \cdots D\varphi_p R^E_p = [\mathcal{O}/P]. \tag{3.3}$$

Proof. Since both sides of (3.3) are pseudomeromorphic (p, p)-currents with support on the variety Z of P, it is by the dimension principle enough to prove that (3.3) holds locally on Z_{reg}. We may thus assume that we have local coordinates $(z_1, \dots, z_n)$ such that $Z = \{z_1 = \cdots = z_p = 0\}$. Since the left-hand side of (3.3) is independent of the choice of locally free resolution (E, φ) by Lemma 2.4, we can assume that (E, φ) is the Koszul complex of $z_1, \dots, z_p$. In this case, it follows from (2.16) that the left-hand side of (3.3) equals $[z_1 = \cdots = z_p = 0] = [Z] = [\mathcal{O}/P]$. □

Proposition 3.5. *Let*

$$0 \to A' \to A \to A'' \to 0$$

be an exact sequence of $\mathcal{O}$-modules of codimension p, and let (E, φ), (F, ψ), and (G, η) be hermitian free resolutions of A, A', and A'', respectively. Then

$$\operatorname{tr} D\varphi_1 \cdots D\varphi_p R^E_p = \operatorname{tr} D\psi_1 \cdots D\psi_p R^F_p + \operatorname{tr} D\eta_1 \cdots D\eta_p R^G_p. \tag{3.4}$$

Proof. By the dimension principle it is enough to prove (3.4) outside a subvariety of codimension $p+1$, since all currents in the equation are pseudomeromorphic (p, p)-currents. Since a module of codimension p is Cohen-Macaulay outside a subvariety of codimension $\geq p+1$, we may thus assume that A, A', A'' are all Cohen-Macaulay. By Lemma 2.4 we may assume that (E, φ), (F, ψ), and (G, η) are any free resolutions of A, A', and A'', respectively. In particular, we may assume that (E, φ) and (F, ψ) have length p. Moreover, by Propositions 2.3 and 2.5 we may assume that (G, η) is the mapping cone of a morphism $a : (F, \psi) \to (E, \varphi)$ that extends the inclusion $A' \to A$. Note that (G, η) then has length $p+1$. Let $b : (E, \varphi) \to (G, \eta)$ be the morphism θ as defined in (2.17). Since b_0 is invertible, one gets by the comparison formula, (2.13), that

(3.5)
$$D\eta_1 \cdots D\eta_p R_p^G = D\eta_1 \cdots D\eta_p b_p R_p^E b_0^{-1} + D\eta_1 \cdots D\eta_p \eta_{p+1} M_{p+1}^0 b_0^{-1} =: W_1 + W_2.$$

By Lemma 2.1, $W_1 = (\alpha_0 + \beta_0 + \gamma_0) R_p^E b_0^{-1}$, where $\alpha_0, \beta_0, \gamma_0$ are as in the lemma. Since, by Lemma 2.4, $\operatorname{tr} D\eta_1 \cdots D\eta_p R_p^G$ is independent of the choice of connections on $G_0, \dots, G_p$, we may assume that the connection on G_j is such that it respects the direct sum $G_j = E_j \oplus \widetilde{F}_{j-1}$ for $j \geq 1$ and that it coincides with D_{E_j} on $E_j \subseteq G_j$ for $j \geq 0$. With this connection $Db_j = 0$, so $\alpha_0 = \beta_0 = 0$, and, using (2.2), we conclude that

$$\operatorname{tr} W_1 = \operatorname{tr} \gamma_0 R_p^E b_0^{-1} = \operatorname{tr} b_0 D\varphi_1 \cdots D\varphi_p R_p^E b_0^{-1} = \operatorname{tr} D\varphi_1 \cdots D\varphi_p R_p^E. \tag{3.6}$$

We now consider the term $\operatorname{tr} W_2$. Since A'' is Cohen-Macaulay of codimension p, A has codimension p, and (G, η) is a free resolution of A'' of length $p+1$, by [16, Lemma 3.3 and equation (3.11)], $M_{p+1}^0 = -\sigma_{p+1}^G b_p R_p^E$, where σ_{p+1}^G is a smooth $\operatorname{Hom}(G_p, G_{p+1})$-valued morphism, such that

$$\sigma_{p+1}^G \eta_{p+1} = \operatorname{Id}_{G_{p+1}}. \tag{3.7}$$

Therefore, in view of (2.5) and (2.2),

$$\operatorname{tr} W_2 = -\operatorname{tr} D\eta_2 \cdots D\eta_{p+1} \sigma_{p+1}^G b_p R_p^E b_0^{-1} \eta_1.$$

Using that $b_0 = \operatorname{Id}_{E_0}$, that $\eta_1 = \begin{bmatrix} -\varphi_1 & a_0 \varepsilon^{-1} \end{bmatrix}$, (2.11), and the comparison formula, (2.14), for $a : (F, \psi) \to (E, \varphi)$, we get that

$$R_p^E b_0^{-1} \eta_1 = R_p^E \begin{bmatrix} 0 & a_0 \varepsilon^{-1} \end{bmatrix} = \begin{bmatrix} 0 & a_p R_p^F \varepsilon^{-1} \end{bmatrix}.$$

Note that

$$\eta_{p+1} = \begin{bmatrix} a_p \varepsilon^{-1} \\ \tilde{\psi}_p \end{bmatrix} \text{ and thus } D\eta_2 \cdots D\eta_{p+1} = \begin{bmatrix} * \\ D\tilde{\psi}_1 \cdots D\tilde{\psi}_p \end{bmatrix}.$$

It follows that

$$\begin{aligned} \operatorname{tr} W_2 = -\operatorname{tr} \begin{bmatrix} * \\ D\tilde{\psi}_1 \cdots D\tilde{\psi}_p \end{bmatrix} \sigma_{p+1}^G b_p \begin{bmatrix} 0 & a_p R_p^F \varepsilon^{-1} \end{bmatrix} \\ = -\operatorname{tr} D\tilde{\psi}_1 \cdots D\tilde{\psi}_p \sigma_{p+1}^G b_p a_p R_p^F \varepsilon^{-1}. \end{aligned} \tag{3.8}$$

Moreover,

$$b_p a_p = \begin{bmatrix} (-1)^p a_p \\ 0 \end{bmatrix} = (-1)^p \begin{bmatrix} a_p \varepsilon^{-1} \\ 0 \end{bmatrix} \varepsilon = (-1)^p \left(\eta_{p+1} - \begin{bmatrix} 0 \\ \tilde{\psi}_p \end{bmatrix} \right) \varepsilon.$$

In view of (2.6) and (2.10), note that $\tilde{\psi}_p \varepsilon R_p^F = \varepsilon \psi_p R_p^F = 0$. Therefore

$$D\tilde{\psi}_1 \cdots D\tilde{\psi}_p \sigma_{p+1}^G b_p a_p R_p^F = (-1)^p D\tilde{\psi}_1 \cdots D\tilde{\psi}_p \varepsilon R_p^F = \varepsilon D\psi_1 \cdots D\psi_p R_p^F,$$

cf. (3.7) and (2.7). Plugging this into (3.8) and using (2.2), we get

$$\operatorname{tr} W_2 = -\operatorname{tr} \varepsilon D\psi_1 \cdots D\psi_p R_p^F \varepsilon^{-1} = -\operatorname{tr} D\psi_1 \cdots D\psi_p R_p^F. \tag{3.9}$$

We thus conclude that (3.4) holds by combining (3.5), (3.6), and (3.9). □

Proof of Theorem 1.2. It is enough to prove (1.6) locally and by the dimension principle, since both sides of (1.6) are pseudomeromorphic currents of bidegree (p,p), it is enough to prove (1.6) outside a subvariety of $Z := \operatorname{supp} \mathcal{F}$ of positive codimension. By Corollary 3.3 we may thus assume that we are at a point $z \in Z$ such that $\mathcal{F}_z$ has a clean filtration (3.1), where each M_i is of pure codimension p; in particular each P_i is of codimension p.

Since $\mathcal{F}_z = M_m$, we may prove the theorem by proving it for M_i by induction over i. The basic case $i = 1$ follows by Lemma 3.4. Next assume that Theorem 1.2 holds for M_i. Consider the short exact sequence

$$0 \to M_i \to M_{i+1} \to \mathcal{O}/P_{i+1} \to 0$$

and assume that (F,ψ), (E,φ), and (G,η) are hermitian free resolutions of M_i, M_{i+1}, and $\mathcal{O}/P_{i+1}$, respectively. Then

$$\frac{1}{(2\pi i)^p p!} \operatorname{tr} D\varphi_1 \cdots D\varphi_p R^E = \frac{1}{(2\pi i)^p p!} \operatorname{tr} D\psi_1 \cdots D\psi_p R^F_p + \frac{1}{(2\pi i)^p p!} \operatorname{tr} D\eta_1 \cdots D\eta_p R^G = [M_i] + [\mathcal{O}/P_{i+1}] = [M_{i+1}],$$

where we have used Proposition 3.5 for the first equality, and the induction hypothesis and Lemma 3.4 for the second. For the last equality, we use the fact that if we have a short exact sequence of $\mathcal{O}$-modules, $0 \to A' \to A \to A'' \to 0$, then

$$[A] = [A'] + [A''].$$

□

4. Proof of Theorem 1.1

The proof of Theorem 1.1 is by induction over the number of nonvanishing homology groups. In order to achieve a complex with one less nonvanishing homology group, we will use the following lemma.

Lemma 4.1. *Let (E,φ) be a complex of free $\mathcal{O}_{X,x}$-modules of length $k+p$, where $k \geq 1$, such that $H_\ell(E) = 0$ for $\ell > k$. Then there exists a neighborhood $x \in \mathcal{U} \subset X$ and a subvariety $W \subset \mathcal{U}$ of codimension $\geq p+1$ such that for $y \in \mathcal{U} \setminus W$ one can find free $\mathcal{O}_{X,y}$-modules G_ℓ and morphisms η_ℓ and b_ℓ for $\ell = k+1, \dots, k+p-1$ such that the diagram*

(4.1)
$$\begin{tikzcd}
 & 0 \arrow[r] & G_{k+p-1} \arrow[r] & G_{k+p-2} \arrow[r] & \cdots \arrow[r] & G_{k+1} \arrow[r] & E_k \arrow[r] & E_{k-1} \arrow[r] & \cdots \arrow[r] & E_0 \arrow[r] & 0 \\
0 \arrow[r] & E_{k+p} \arrow[r] \arrow[u] & E_{k+p-1} \arrow[r] \arrow[u, "b_{k+p-1}"] & E_{k+p-2} \arrow[r] \arrow[u, "b_{k+p-2}"] & \cdots \arrow[r] & E_{k+1} \arrow[r] \arrow[u, "b_{k+1}"] & E_k \arrow[r] \arrow[u, "\cong"] & E_{k-1} \arrow[r] \arrow[u, "\cong"] & \cdots \arrow[r] & E_0 \arrow[r] \arrow[u, "\cong"] & 0 \\
0 \arrow[r] & E_{k+p} \arrow[r] \arrow[u, "\cong"] & E_{k+p-1} \arrow[r] \arrow[u, "\cong"] & F_{k+p-2} \arrow[r] \arrow[u, "a_{k+p-2}"] & \cdots \arrow[r] & F_{k+1} \arrow[r] \arrow[u, "a_{k+1}"] & \widetilde{G}_{k+1} \arrow[r] \arrow[u, "a_k"] & 0 \arrow[u] & & &
\end{tikzcd}$$

is commutative and the rows (G,η) and (F,ψ) are complexes, and $a : (F,\psi) \to (E,\varphi)$ and $b : (E,\varphi) \to (G,\eta)$ are morphisms of complexes. Here $F_\ell = \widetilde{G}_{\ell+1} \oplus E_\ell$ for

$\ell = k+1,\dots,k+p-2$ and

$$a_k = \left[\begin{array}{c} b_k^{-1}\eta_{k+1}\varepsilon^{-1} \end{array}\right],\ a_\ell = \left[\begin{array}{cc} 0 & \mathrm{Id}_{E_\ell} \end{array}\right] \text{ for } \ell = k+1,\dots,k+p-2,$$

$$\psi_{k+p-1} = \begin{bmatrix} \varepsilon b_{k+p-1} \\ \varphi_{k+p-1} \end{bmatrix},\ \psi_{k+\ell} = \begin{bmatrix} -\tilde\eta_{k+\ell+1} & \varepsilon b_{k+\ell} \\ 0 & \varphi_{k+\ell} \end{bmatrix} \text{ for } \ell = 2,\dots,p-2, \text{ and}$$

$$\psi_{k+1} = \left[\begin{array}{cc} -\tilde\eta_{k+2} & \varepsilon b_{k+1} \end{array}\right].$$

Moreover, the complexes in the rows have the following homology groups:

$$\begin{array}{ccccccc}
\cdots & 0 & 0 & H_{k-1}(E) & H_{k-2}(E) & \cdots & H_0(E) \\
\\
\cdots & 0 & H_k(E) & H_{k-1}(E) & H_{k-2}(E) & \cdots & H_0(E) \\
\\
\cdots & 0 & H_k(E) & 0 & 0 & \cdots & 0
\end{array} \tag{4.2}$$

The left-most part of the complex (G,η) is a free resolution of $\operatorname{coker}\varphi_k$ and the complex (F,ψ) is essentially the mapping cone of $b:(E,\varphi)\to(G,\eta)$.

Proof. Let $\mathcal{F} = \operatorname{coker}\varphi_k$. Given any locally free resolution (K,κ) of $\mathcal{F}$, there are associated (germs of) subvarieties Z_k^K where κ_k does not have maximal rank. By uniqueness of minimal free resolutions, these sets are independent of the choice of resolution (K,κ) and thus associated with $\mathcal{F}$, and we may denote them by $Z_k^{\mathcal{F}}$ instead. Take $\mathcal{U}$ to be any neighborhood of x where (E,φ) is defined and W to be $Z_{p+1}^{\mathcal{F}}$. By the Buchsbaum-Eisenbud criterion $\operatorname{codim} W \geq p+1$, see [11, Theorem 20.9].

Assume that we are outside W, and take locally a free resolution

$$0 \to K_{N_0} \xrightarrow{\kappa_{N_0}} \cdots \to K_2 \xrightarrow{\kappa_2} E_k \xrightarrow{\varphi_k} E_{k-1}$$

of $\mathcal{F}$. Since we are outside W, $\operatorname{im}\kappa_{p+1}$ is free so if we replace K_p by $K_p/\operatorname{im}\kappa_{p+1}$, we can assume that the free resolution is of the form

$$0 \to K_p \xrightarrow{\kappa_p} \cdots \to K_2 \xrightarrow{\kappa_2} E_k \xrightarrow{\varphi_k} E_{k-1}.$$

We let $G_{k+\ell-1} = K_\ell$ and $\eta_{k+\ell-1} = \kappa_\ell$ for $\ell = 2,\dots,p$, which then gives the complex in the top row of (4.1). This complex has the stated homology groups in the first row of (4.2), since (G,η) by construction is exact at levels $\geq k$.

Since the top row of (4.1) is exact at levels $\geq k$ and the modules in the middle row are free, one can, locally, by a diagram chase inductively construct $b_{k+1},\dots,b_{k+p-1}$ so that the diagram (4.1) commutes in the top two rows, cf., e.g., the proof of [11, Proposition A3.13].

We now turn to the bottom row of (4.1). Let (C,μ) be the mapping cone of the morphism $b:(E,\varphi)\to(G,\eta)$ and let $\vartheta:(C,\mu)\to(E,\varphi)$ be the induced morphism of complexes of degree -1 as defined by (2.18). Recall that

$$\mu_\ell = \begin{bmatrix} -\eta_\ell & b_{\ell-1}\varepsilon^{-1} \\ 0 & \tilde\varphi_{\ell-1} \end{bmatrix}.$$

On C_ℓ for $\ell = 0,\dots,k+1$, we do the change of basis given by the isomorphism

$$\alpha_\ell = \begin{bmatrix} \mathrm{Id}_{G_\ell} & 0 \\ \varepsilon b_{\ell-1}^{-1}\eta_\ell & \mathrm{Id}_{\tilde E_{\ell-1}} \end{bmatrix},$$

i.e., we replace μ_ℓ by $\alpha_{\ell-1}^{-1}\mu_\ell\alpha_\ell$ for $\ell=1,\dots,k+1$, μ_{k+2} by $\alpha_{k+1}^{-1}\mu_{k+2}$, and ϑ_ℓ by $\vartheta_\ell\alpha_{\ell+1}$ for $\ell=0,\dots,k$. Note that for these ℓ, $b_{\ell-1}$ is the identity and thus invertible. In this new basis, using that $\varepsilon b_{\ell-1}^{-1}\eta_\ell b_\ell\varepsilon^{-1}=\tilde{\varphi}_\ell$ for $\ell=1,\dots,k+1$, we get that

$$\mu_{k+2}=\begin{bmatrix} -\eta_{k+2} & b_{k+1}\varepsilon^{-1} \\ 0 & 0 \end{bmatrix}, \mu_\ell=\begin{bmatrix} 0 & b_{\ell-1}\varepsilon^{-1} \\ 0 & 0 \end{bmatrix} \text{ for } \ell=1,\dots,k+1, \text{ and}$$

$$\vartheta_\ell=\begin{bmatrix} b_\ell^{-1}\eta_{\ell+1} & \varepsilon^{-1} \end{bmatrix} \text{ for } \ell=0,\dots,k.$$

Hence (C,μ) contains as summands the trivial complexes

$$0\to \widetilde{E}_{\ell-1}\stackrel{b_{\ell-1}\varepsilon^{-1}}{\longrightarrow} G_{\ell-1}\to 0 \text{ for } 1\le \ell\le k+1. \tag{4.3}$$

We let (F,ψ) be the complex (C,μ) where we use the new basis as described above and remove the trivial summands (4.3), and where we moreover shift the degree by 1 and change the signs so that $F_\ell=\widetilde{C}_{\ell+1}$ and the morphisms are adjusted accordingly. The morphisms μ and ϑ given by the mapping cone, adjusted accordingly, are then indeed the morphisms ψ and a as in the statement of the lemma. Since $(b_\ell)_*: H_\ell(E)\to H_\ell(G)$ is an isomorphism for $\ell=0,\dots,k-1$, $H_\ell(G)=0$ for $\ell\ge k$, and $H_\ell(E)=0$ for $\ell\ge k+1$, it follows from the long exact sequence (2.19) that $H_\ell(F)=H_{\ell+1}(C)=0$ for $\ell\ne k$ and that $H_k(F)=H_{k+1}(C)\stackrel{\cong}{\to} H_k(E)$. □

For a complex (E,φ) of length $N=k+p$, we will introduce the shorthand notation

$$(\operatorname{tr} D\varphi R^E)_p := \sum_{\ell=0}^{k}(-1)^\ell \operatorname{tr} D\varphi_{\ell+1}\cdots D\varphi_{\ell+p}(R^E)^\ell_{\ell+p}.$$

Proposition 4.2. *Let (E,φ), (F,ψ), and (G,η) be as in Lemma 4.1 and assume they are generically exact hermitian complexes. Then*

$$(\operatorname{tr} D\varphi R^E)_p = (\operatorname{tr} D\psi R^F)_p + (\operatorname{tr} D\eta R^G)_p. \tag{4.4}$$

Proof of Theorem 1.1. We prove by induction over k that for each generically exact hermitian complex (E,φ) of length $\le k+p$ such that $H_\ell(E)=0$ for $\ell>k$, the associated residue current satisfies (1.5). Since (E,φ) in Theorem 1.1 has length N and $H_\ell(E)$ has pure codimension p, (E,φ) has this property for $k=N$, and thus the theorem follows.

First note that the case $k=0$ is Theorem 1.2. Next assume that (1.5) holds for residue currents associated with complexes (E,φ) of length $k-1+p$ such that that $H_\ell(E)=0$ for $\ell>k-1$. It is enough to prove (1.5) locally and since both sides in (1.5) are pseudomeromorphic currents of bidegree (p,p), by the dimension principle, it is enough to prove it outside a subvariety of codimension $p+1$. Therefore we can assume that we have generically exact hermitian complexes (F,ψ) and (G,η) as in Lemma 4.1. It follows from Theorem 1.2 and (4.2) that

$$\frac{1}{(2\pi i)^p p!}(\operatorname{tr} D\psi R^F)_p = (-1)^k[H_k(E)].$$

Moreover, by the induction hypothesis and (4.2)

$$\frac{1}{(2\pi i)^p p!}(\operatorname{tr} D\eta R^G)_p = \sum_{\ell=0}^{k-1}(-1)^\ell[H_\ell(E)].$$

Now (1.5) follows from Proposition 4.2. □

4.1. **Proof of Proposition 4.2.** We will compute the two terms on the right-hand side of (4.4) separately.

4.1.1. *Computing* $(\operatorname{tr} D\eta R^G)_p$. Let us consider the currents

$$D\eta_{\ell+1}D\eta_{\ell+2}\cdots D\eta_{\ell+p}(R^G)^\ell_{\ell+p} \tag{4.5}$$

that one takes the trace of in $(\operatorname{tr} D\eta R^G)_p$. Note that (4.5) vanishes for $\ell \geq k$ since then $G_{\ell+p}=0$. It remains to consider the cases $\ell = 0,\dots,k-1$. For these ℓ, b_ℓ is an isomorphism and thus it is invertible. By the comparison formula, (2.12), we have

$$(R^G)^\ell_{\ell+p}b_\ell = b_{\ell+p}(R^E)^\ell_{\ell+p} + M^{\ell-1}_{\ell+p}\varphi_\ell + \eta_{\ell+p+1}M^\ell_{\ell+p+1}.$$

It follows that (4.5) equals

$$D\eta_{\ell+1}\cdots D\eta_{\ell+p}b_{\ell+p}(R^E)^\ell_{\ell+p}b_\ell^{-1}+ \\ D\eta_{\ell+1}\cdots D\eta_{\ell+p}M^{\ell-1}_{\ell+p}\varphi_\ell b_\ell^{-1} + D\eta_{\ell+1}\cdots D\eta_{\ell+p}\eta_{\ell+p+1}M^\ell_{\ell+p+1}b_\ell^{-1}.$$

We rewrite the trace of the last term as

$$\operatorname{tr} D\eta_{\ell+1}\cdots D\eta_{\ell+p}\eta_{\ell+p+1}M^\ell_{\ell+p+1}b_\ell^{-1} = \operatorname{tr}\eta_{\ell+1}D\eta_{\ell+2}\cdots D\eta_{\ell+p+1}M^\ell_{\ell+p+1}b_\ell^{-1} = \\ \operatorname{tr} D\eta_{\ell+2}\cdots D\eta_{\ell+p+1}M^\ell_{\ell+p+1}b_\ell^{-1}\eta_{\ell+1} = \operatorname{tr} D\eta_{\ell+2}\cdots D\eta_{\ell+p+1}M^\ell_{\ell+p+1}\varphi_{\ell+1}b_{\ell+1}^{-1}.$$

Here we have used (2.5) for the first equality and

$$b_\ell^{-1}\eta_{\ell+1} = \varphi_{\ell+1}b_{\ell+1}^{-1} \tag{4.6}$$

for the last equality; indeed, since $\ell < k$, $b_{\ell+1}$ is invertible. For the middle equality we have used (2.2); note that the sign is 1 since both $D\eta_{\ell+2}\cdots D\eta_{\ell+p+1}M^\ell_{\ell+p+1}b_\ell^{-1}$ and $\eta_{\ell+1}$ have odd total and endomorphism degrees. It follows that

$$\sum_{\ell=0}^{k-1}(-1)^\ell\Big(\operatorname{tr} D\eta_{\ell+1}\cdots D\eta_{\ell+p}M^{\ell-1}_{\ell+p}\varphi_\ell b_\ell^{-1} + \operatorname{tr} D\eta_{\ell+1}\cdots D\eta_{\ell+p}\eta_{\ell+p+1}M^\ell_{\ell+p+1}b_\ell^{-1}\Big) = \\ \operatorname{tr} D\eta_1\cdots D\eta_p M^{-1}_p\varphi_0 b_0^{-1} + (-1)^{k-1}\operatorname{tr} D\eta_k\cdots D\eta_{k+p-1}\eta_{k+p}M^{k-1}_{k+p}b_{k-1}^{-1} = 0,$$

since $\varphi_0 = 0$ and $\eta_{k+p}=0$. Thus

$$(\operatorname{tr} D\eta R^G)_p = \sum_{\ell=0}^{k-1}(-1)^\ell \operatorname{tr} D\eta_{\ell+1}\cdots D\eta_{\ell+p}(R^G)^\ell_{\ell+p} = \tag{4.7}$$

$$\sum_{\ell=0}^{k-1}(-1)^\ell\operatorname{tr} D\eta_{\ell+1}\cdots D\eta_{\ell+p}b_{\ell+p}(R^E)^\ell_{\ell+p}b_\ell^{-1} = \sum_{\ell=0}^{k-1}(-1)^\ell\operatorname{tr}(\alpha_\ell+\beta_\ell+\gamma_\ell)(R^E)^\ell_{\ell+p}b_\ell^{-1},$$

where we have used Lemma 2.1 for the last equality and $\alpha_\ell,\beta_\ell,\gamma_\ell$ are as in the lemma.

From the definitions of α_ℓ and β_ℓ it follows that

$$\operatorname{tr}\alpha_\ell(R^E)^\ell_{\ell+p}b_\ell^{-1} = \operatorname{tr}\eta_{\ell+1}\delta_{\ell+1}(R^E)^\ell_{\ell+p}b_\ell^{-1} = \operatorname{tr}\delta_{\ell+1}(R^E)^\ell_{\ell+p}b_\ell^{-1}\eta_{\ell+1} = \\ \operatorname{tr}\delta_{\ell+1}(R^E)^\ell_{\ell+p}\varphi_{\ell+1}b_{\ell+1}^{-1} = \operatorname{tr}\delta_{\ell+1}\varphi_{\ell+p+1}(R^E)^{\ell+1}_{\ell+p+1}b_{\ell+1}^{-1} = \operatorname{tr}\beta_{\ell+1}(R^E)^{\ell+1}_{\ell+p+1}b_{\ell+1}^{-1} \tag{4.8}$$

for $\ell = 0,\dots,k-1$. Here we have used (2.2) for the second equality; indeed, note that the sign is 1 since $\delta_{\ell+1}(R^E)^\ell_{\ell+p}b_\ell^{-1}$ and $\eta_{\ell+1}$ have odd total and endomorphism degrees. Moreover, we have used (4.6) for the third equality and (2.9) for the fourth equality. By (2.10), we then get that

$$\operatorname{tr}\beta_0(R^E)^0_p b_0^{-1} = \operatorname{tr}\delta_0\varphi_p(R^E)^0_p b_0^{-1} = 0. \tag{4.9}$$

Moreover note that

$$\operatorname{tr}\gamma_\ell(R^E)^\ell_{\ell+p}b_\ell^{-1} = \operatorname{tr} b_\ell D\varphi_{\ell+1}\cdots D\varphi_{\ell+p}(R^E)^\ell_{\ell+p}b_\ell^{-1} = \operatorname{tr} b_\ell^{-1}b_\ell D\varphi_{\ell+1}\cdots D\varphi_{\ell+p}(R^E)^\ell_{\ell+p} = \operatorname{tr} D\varphi_{\ell+1}\cdots D\varphi_{\ell+p}(R^E)^\ell_{\ell+p}, \tag{4.10}$$

where we have used (2.2) for the second equality; indeed, the sign is 1 since b_ℓ is of even total and endomorphism degree. From (4.7), (4.8), (4.9), and (4.10) we conclude

$$(\operatorname{tr} D\eta R^G)_p = (-1)^{k-1}\operatorname{tr}\beta_k(R^E)^k_{k+p}b_k^{-1} + \sum_{\ell=0}^{k-1}(-1)^\ell \operatorname{tr} D\varphi_{\ell+1}\cdots D\varphi_{\ell+p}(R^E)^\ell_{\ell+p}. \tag{4.11}$$

4.1.2. *Computing* $(\operatorname{tr} D\psi R^F)_p$. Since $F_\ell = 0$ for $\ell < k$, the only nonvanishing current that one takes the trace of in $(\operatorname{tr} D\psi R^F)_p$ is $(-1)^k D\psi_{k+1}\cdots D\psi_{k+p}(R^F)^k_{k+p}$. A computation yields[4]

$$D\psi_{k+1}\cdots D\psi_{k+p-1} = D\begin{bmatrix} -\tilde\eta_{k+2} & \varepsilon b_{k+1}\end{bmatrix} D\begin{bmatrix} -\tilde\eta_{k+3} & \varepsilon b_{k+2} \\ 0 & \varphi_{k+2}\end{bmatrix}\cdots D\begin{bmatrix} -\tilde\eta_{k+p-1} & \varepsilon b_{k+p-2} \\ 0 & \varphi_{k+p-2}\end{bmatrix} D\begin{bmatrix} \varepsilon b_{k+p-1} \\ \varphi_{k+p-1}\end{bmatrix} = \sum_{m=k+1}^{k+p-1}(-1)^{m-k-1}D\tilde\eta_{k+2}\cdots D\tilde\eta_m D(\varepsilon b_m)D\varphi_{m+1}\cdots D\varphi_{k+p-1}. \tag{4.12}$$

Recall that by (2.4)

$$D(\varepsilon b_m) = D\varepsilon b_m + (-1)^{\deg\varepsilon}\varepsilon Db_m = -\varepsilon Db_m. \tag{4.13}$$

Moreover by (2.7) we have

$$D\tilde\eta_\kappa\varepsilon = (-1)^{\deg_f(D\eta_\kappa)}\varepsilon D\eta_\kappa = -\varepsilon D\eta_\kappa. \tag{4.14}$$

Using (4.13) and then (4.14) repeatedly for $\kappa = m, m-1, \dots, k+2$ we get

$$D\tilde\eta_{k+2}\cdots D\tilde\eta_m D(\varepsilon b_m)D\varphi_{m+1}\cdots D\varphi_{k+p-1} = (-1)^{m-k}\varepsilon D\eta_{k+2}\cdots D\eta_m Db_m D\varphi_{m+1}\cdots D\varphi_{k+p-1}. \tag{4.15}$$

Now

$$D\psi_{k+1}\cdots D\psi_{k+p} = -\Big(\sum_{m=k+1}^{k+p-1}\varepsilon D\eta_{k+2}\cdots D\eta_m Db_m D\varphi_{m+1}\cdots D\varphi_{k+p-1}\Big)D\varphi_{k+p} = -\sum_{m=k+1}^{k+p}\varepsilon D\eta_{k+2}\cdots D\eta_m Db_m D\varphi_{m+1}\cdots D\varphi_{k+p} = -\varepsilon\delta_{k+1}, \tag{4.16}$$

[4] In the sum $D\tilde\eta_{k+2}\cdots D\tilde\eta_m$ and $D\varphi_{m+1}\cdots D\varphi_{k+p-1}$ are to be interpreted as 1 if $m = k+1$ and $m = k+p-1$, respectively.

where δ_{k+1} is as in Lemma 2.1; here we have used $\psi_{k+p} = \varphi_{k+p}$, cf. (4.1), (4.12), and (4.15) for the first equality, and $b_{k+p} = 0$ for the second. It follows that

$$\operatorname{tr} D\psi_{k+1} \cdots D\psi_{k+p}(R^F)^k_{k+p} = \operatorname{tr} D\psi_{k+1} \cdots D\psi_{k+p}(R^E)^k_{k+p} a_k =$$
$$\operatorname{tr} a_k D\psi_{k+1} \cdots D\psi_{k+p}(R^E)^k_{k+p} = \operatorname{tr} b_k^{-1}\eta_{k+1}\varepsilon^{-1} D\psi_{k+1} \cdots D\psi_{k+p}(R^E)^k_{k+p} =$$
$$-\operatorname{tr} b_k^{-1}\eta_{k+1}\varepsilon^{-1}\varepsilon\delta_{k+1}(R^E)^k_{k+p} = -\operatorname{tr} b_k^{-1}\alpha_k(R^E)^k_{k+p},$$

where α_k is in Lemma 2.1. Here we have used the comparison formula (2.14) and that $a_{k+p} = \mathrm{Id}_{E_{k+p}}$, (2.2), the definition of a_k, and (4.16) for the first, second, third, and fourth equality, respectively. Since $\eta_{k+p} = 0$, cf. (4.1), by Lemma 2.1 we get that

$$-b_k^{-1}\alpha_k = b_k^{-1}(\beta_k + \gamma_k) = b_k^{-1}\beta_k + D\varphi_{k+1} \cdots D\varphi_{k+p}.$$

Thus

$$(\operatorname{tr} D\psi R^F)_p = (-1)^k \operatorname{tr} D\psi_{k+1} \cdots D\psi_{k+p}(R^F)^k_{k+p} =$$
$$(-1)^k \operatorname{tr} b_k^{-1}\beta_k(R^E)^k_{k+p} + (-1)^k \operatorname{tr} D\varphi_{k+1} \cdots D\varphi_{k+p}(R^E)^k_{k+p}. \tag{4.17}$$

Finally using (2.2) we conclude from (4.11) and (4.17) that

$$(\operatorname{tr} D\psi R^F)_p + (\operatorname{tr} D\eta R^G)_p = (-1)^k \operatorname{tr} b_k^{-1}\beta_k(R^E)^k_{k+p} + (-1)^{k-1} \operatorname{tr} \beta_k(R^E)^k_{k+p} b_k^{-1}$$
$$+ \sum_{\ell=0}^{k} (-1)^\ell \operatorname{tr} D\varphi_{\ell+1} \cdots D\varphi_{\ell+p}(R^E)^\ell_{\ell+p} = (\operatorname{tr} D\varphi R^E)_p.$$

5. The Koszul complex

Let (E, φ) be the Koszul complex of a tuple f of holomorphic functions as in Section 2.3, and assume that it is equipped with the trivial metric. Recall that if $m = p = \operatorname{codim} Z(f)$, then (1.5) just equals (1.7). In this section we describe the currents in (1.5) when $m > p$. Then

$$[E] = \sum (-1)^\ell [\mathcal{H}_\ell(E)] = 0,$$

see, e.g., [20, Corollary 5.2.9 (ii)]; in particular the Koszul complex cannot be exact at all levels $\ell > 0$.

To describe the left-hand side of (1.5), let us recall the construction of R. Let $\sigma = \sum \bar{f}_i e_i/|f|^2$. Then R^ℓ_k is defined as multiplication with the analytic continuation to $\lambda = 0$ of the form $\bar\partial |f|^{2\lambda} \wedge \sigma \wedge (\bar\partial\sigma)^{k-\ell-1}$, see [1]. Since $D\varphi_j$ is just contraction with $\sum df_j \wedge e_j^*$, a computation yields that

$$\operatorname{tr} D\varphi_{\ell+1} \cdots D\varphi_k R^\ell_k = \binom{m-(k-\ell)}{\ell} \operatorname{tr} D\varphi_1 \cdots D\varphi_{k-\ell} R^0_{k-\ell},$$

cf. (2.15). Since (E, φ) ends at level m, $(2\pi i)^p p!$ times the left-hand side of (1.5) equals

$$\sum_{\ell=0}^{m-p} (-1)^\ell \operatorname{tr} D\varphi_{\ell+1} \cdots D\varphi_{\ell+p} R^\ell_{\ell+p} = \sum_{\ell=0}^{m-p} (-1)^\ell \binom{m-p}{\ell} \operatorname{tr} D\varphi_1 \cdots D\varphi_p R^0_p = 0.$$

To conclude, (1.5) holds since both sides vanish, so we get an explicit proof of Theorem 1.1 in this case.

Next, let us consider the individual terms in the left-hand side of (1.5) and in $[E]$. First note that since the image of φ_1 equals $\mathcal{J}(f)$, $[\mathcal{H}_0(E)]$ is just the cycle of $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}(f)$. In [2], Andersson proved that

$$\frac{1}{(2\pi i)^p p!}\operatorname{tr} D\varphi_1 \cdots D\varphi_p R_p^0 = \sum \alpha_j [Z_j^p], \tag{5.1}$$

where Z_j^p are the irreducible components of Z of codimension p and α_j is the *geometric* or *Hilbert-Samuel* multiplicity of $\mathcal{J}(f)$ along Z_j^p. For a complete intersection ideal, the geometric multiplicities coincide with the algebraic multiplicities and so (5.1) generalizes (1.7). In general, however, the multiplicities are different, cf. Example 5.1 below, and thus it is not true in general that the individual terms $\operatorname{tr} D\varphi_{\ell+1} \cdots D\varphi_{\ell+p} R_{\ell+p}^{\ell}$ and $[\mathcal{H}_\ell(E)]$ at level ℓ coincide.

Example 5.1. If $\mathcal{J}(f)$ is generated by monomials and $Z(f) = \{0\}$, then the algebraic multiplicity equals $n!$ times the volume of $\mathbb{R}_+^n \setminus \Gamma$, where the Γ is the convex hull in $\mathbb{R}^n$ of the exponents of the monomials in $\mathcal{J}(f)$, see, e.g., [20, exercise 2.8]. If $\mathcal{J}(f)$ is not a complete intersection ideal, this does not coincide with the geometric multiplicity, which is just the number of monomials that are not in $\mathcal{J}(f)$.

For example, if $f = (z_1^2, z_1 z_2, z_2^2)$ in $\mathbb{C}^2$, then the algebraic multiplicity of $\mathcal{J}(f)$ is 4, while the geometric multiplicity is 3. Thus in this case the first term in (1.5) equals

$$\frac{1}{(2\pi i)^2 2!}\operatorname{tr} D\varphi_1 D\varphi_2 (R^E)_2^0 = 4[0]$$

whereas $[\mathcal{H}_0(E)] = 3[0]$. □

6. Non-pure dimensional homology

In [17] we get a version [17, Theorem 1.5] of Theorem 1.2 when $\mathcal{F} = \mathcal{O}_Z$ for a general, not necessarily pure dimensional, analytic space Z. By the same arguments we get a version for general coherent sheaves $\mathcal{F}$.

Corollary 6.1. *Let $\mathcal{F}$ be a coherent sheaf, let (E, φ) be a hermitian locally free resolution of $\mathcal{F}$, and let D be the connection on* $\operatorname{End} E$ *induced by arbitrary* $(1,0)$*-connections on $E_0, \dots, E_N$. Moreover, let W_k be the union of all irreducible components of* $\operatorname{supp}\mathcal{F}$ *of codimension k. Then*

$$\sum_k \frac{1}{(2\pi i)^k k!}\operatorname{tr} D\varphi_1 \cdots D\varphi_k \mathbf{1}_{W_k} R_k^0 = [\mathcal{F}]. \tag{6.1}$$

Pseudomeromorphic currents allow for multiplication by characteristic functions of varieties or, more generally, constructible sets, see [5, Theorem 3.1], and thus $\mathbf{1}_{W_k} R_k^0$ is a well-defined pseudomeromorphic current.

It is natural to ask whether we also obtain a version of Theorem 1.1 when the homology groups do not have pure dimension or are not of the same dimension. However, this does not seem to follow as easily. Since $[E]$ is an alternating sum of cycles of sheaves, there are in general components $m_i[Z_i]$ and $m_j[Z_j]$ of $[E]$ such that Z_i is a proper subvariety of Z_j. If we remove these "embedded components" of $[E]$ we can get a formula like (1.5): Let $W = \cup \operatorname{supp} \mathcal{H}_\ell(E)$ and let

$$[E]_W = \sum (-1)^\ell [\mathcal{H}_\ell(E)]_W,$$

where $[\mathcal{H}_\ell(E)]_W$ is the cycle of $\mathcal{H}_\ell(E)$ but where we only include the irreducible components Z_i that are minimal primes of W. Moreover, let W_k be the union of the irreducible components of W of codimension k. Then by the same arguments as in the proof of [17, Theorem 1.5] we get

$$\sum_{k,\ell} \frac{1}{(2\pi i)^k k!} \operatorname{tr} D\varphi_{\ell+1} \cdots D\varphi_{\ell+k} \mathbf{1}_{W_k} R^\ell_{\ell+k} = [E]_W. \tag{6.2}$$

Maybe one could get a similar formula for $[E]$ by considering characteristic functions of different sets at different levels. For example if W_k^ℓ is the union of the irreducible components of $\operatorname{supp} \mathcal{H}_\ell(E)$ of codimension k, one could hope that

$$\sum_{k,\ell} \frac{1}{(2\pi i)^k k!} \operatorname{tr} D\varphi_{\ell+1} \cdots D\varphi_{\ell+k} \mathbf{1}_{W_k^\ell} R^\ell_{\ell+k} = [E].$$

However, this does not seem to follow as immediately from the dimension principle as (6.2).

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