RESIDUE CURRENTS AND FUNDAMENTAL CYCLES

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ABSTRACT. We give a factorization of the fundamental cycle of an analytic space in terms of certain differential forms and residue currents associated with a locally free resolution of its structure sheaf. Our result can be seen as a generalization of the classical Poincaré-Lelong formula. It is also a current version of a result by Lejeune-Jalabert, who similarly expressed the fundamental class of a Cohen-Macaulay analytic space in terms of differential forms and cohomological residues.

1. INTRODUCTION

Given a holomorphic function f on a complex manifold X, recall that the classical *Poincaré-Lelong formula* asserts that $\bar{\partial}\partial \log |f|^2 = 2\pi i[Z]$, where $[Z]$ is the current of integration (or Lelong current) of the divisor Z of f counted with multiplicities, or, more precisely, (the current of integration of) the fundamental cycle of Z. Formally we can rewrite the Poincaré-Lelong formula as

$$(1.1) \qquad \frac{1}{2\pi i}\bar{\partial}\frac{1}{f} \wedge df = [Z].$$

This factorization of $[Z]$ can be made rigorous if we construe $\bar{\partial}(1/f)$ as the *residue current* of $1/f$, introduced by Dolbeault, [D], and Herrera and Lieberman, [HL], and defined, e.g., as

$$(1.2) \qquad \lim_{\epsilon \to 0} \bar{\partial}\chi(|f|^2/\epsilon)\frac{1}{f},$$

where $\chi(t)$ is (a smooth approximand of) the characteristic function of the interval $[1, \infty)$. The current $\bar{\partial}(1/f)$ satisfies that a holomorphic function g on X is in the ideal (sheaf) $\mathcal{J}(f)$ generated by f if and only if $g\bar{\partial}(1/f) = 0$. This is referred to as the *duality principle* and it is central to many applications of residue currents; in a way $\bar{\partial}(1/f)$ can be thought of as a current representation of the ideal $\mathcal{J}(f)$. In this paper we prove that (the current of integration along) the fundamental cycle of any analytic space admits a natural factorization as a smooth "Jacobian" factor times a residue current, analogous to (1.1).

Let $Z \subset X$ be a (not necessarily reduced) analytic space. The *fundamental cycle* of Z, seen as a current on X, is the current

$$(1.3) \qquad [Z] = \sum m_i[Z_i],$$

where Z_i are the irreducible components of Z_{red}, $[Z_i]$ are the currents of integration of the (reduced) subspaces Z_i, and m_i are the geometric multiplicities of Z_i in Z.

For a generic $z \in Z_i$, $\mathcal{O}_{Z,z}$ is a free $\mathcal{O}_{Z_i,z}$-module of constant rank. One way of defining the *geometric multiplicity* m_i of Z_i in Z is as this rank. Equivalently m_i can be defined as the length of the Artinian ring $\mathcal{O}_{Z,Z_i}$, see, e.g., [F, Chapter 1.5]. The

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equivalence of the two definitions can be proved with the help of [F, Lemma 1.7.2]. If $Z_{\mathrm{red}} = \{z\}$ is a point, and Z is defined by an ideal sheaf $\mathcal{J}$, i.e., $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}$, then the geometric multiplicity of $(Z_{\mathrm{red}}$ in) Z is $\dim_{\mathbb{C}} \mathcal{O}_{X,z}/\mathcal{J}_z$. If $\dim Z_i > 0$, then for generic $z \in Z_i$ and $H \subset X$ a complex manifold transversal to (Z_i, z), $m_i = \dim_{\mathbb{C}} \mathcal{O}_{X,z}/(\mathcal{J}+\mathcal{J}_H)_z$, where $\mathcal{J}_H$ is the ideal of holomorphic functions vanishing on H.

We will consider Z such that $\mathcal{O}_Z$ has a global locally free resolution over $\mathcal{O}_X$. Such a resolution exists for any Z for example when X is projective. If X is Stein, then any Z has a semi-global resolution, i.e., it has a free resolution on every compact in X. Assume that

$$(1.4) \qquad 0 \to E_\nu \xrightarrow{\varphi_\nu} E_{\nu-1} \xrightarrow{\varphi_{\nu-1}} \cdots \xrightarrow{\varphi_2} E_1 \xrightarrow{\varphi_1} E_0,$$

is this locally free resolution, i.e., (1.4) is an exact complex of locally free $\mathcal{O}_X$-modules such that $\mathrm{coker}\, \varphi_1 \cong \mathcal{O}_Z$. If the corresponding vector bundles are equipped with Hermitian metrics we say that (E, φ) is a *Hermitian locally free resolution* of $\mathcal{O}_Z$ over $\mathcal{O}_X$. Given such an (E, φ), in [AW1] Andersson and the second author constructed an $\mathrm{End}E$-valued residue current $R^E = \sum R_k^E$, where $E = \bigoplus E_k$, and R_k^E takes values in $\mathrm{Hom}\,(E_0, E_k)$. This current satisfies a duality principle and it has found many applications; e.g., it has been used to obtain new results on the $\bar{\partial}$-equation on singular varieties, [AS], and a global effective Briançon-Skoda-Huneke theorem, [AW3].

If f is a holomorphic function on X and $E_0 \cong \mathcal{O}_X$ and $E_1 \cong \mathcal{O}_X$ are trivial line bundles, then

$$0 \to \mathcal{O}_X \xrightarrow{\varphi_1} \mathcal{O}_X,$$

where φ_1 is the 1×1-matrix $[f]$, gives a locally free resolution of $\mathcal{O}_Z := \mathcal{O}/\mathcal{J}(f)$. In this case (the coefficient of) $R^E = R_1^E$ is just $\bar{\partial}(1/f)$, and the Poincaré-Lelong formula (1.1) can be written as[1]

$$(1.5) \qquad \frac{1}{2\pi i} d\varphi_1 R_1^E = [Z].$$

Our main result is the following generalization of (1.5).

Theorem 1.1. *Let $Z \subset X$ be an analytic space of pure codimension p, let (E, φ) be a Hermitian locally free resolution of $\mathcal{O}_Z$ over $\mathcal{O}_X$, where $\mathrm{rank}\, E_0 = 1$, and let D be the connection[2] on $\mathrm{End}E$ induced by arbitrary connections on $E_0, \ldots, E_p$. Then*

$$(1.6) \qquad \frac{1}{(2\pi i)^p p!} D\varphi_1 \cdots D\varphi_p R_p^E = [Z].$$

Note that the endomorphism $D\varphi_1 \cdots D\varphi_p$ depends on the choice of connections on $E_0, \ldots, E_p$ and the current R_p^E in general depends on the choice of Hermitian metrics on $E_0, \ldots, E_p$. There is no assumption of any relation between the connections and the Hermitian metrics.

Various special cases of Theorem 1.1 and related results have been proved earlier: Assume that Z is a complete intersection of codimension p, i.e., $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}$, where $\mathcal{J}$ is a complete intersection ideal, generated by, say, $f = (f_1, \ldots, f_p)$. Then

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[1]The relation between the signs in (1.1) and (1.5) is explained in Section 2.5.

[2]The connection D is defined by (2.3).

Coleff and Herrera proved in [CH] the following generalization of the Poincaré-Lelong formula (1.1):

$$(1.7) \qquad \frac{1}{(2\pi i)^p} \bar{\partial}\frac{1}{f_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1} \wedge df_1 \wedge \cdots \wedge df_p = [Z],$$

where $\bar{\partial}(1/f_p) \wedge \cdots \wedge \bar{\partial}(1/f_1)$ is the so-called *Coleff-Herrera product* of f. In this situation, one may choose the resolution (E, φ) such that (1.6) becomes precisely (1.7), see (2.13).

In [DP] Demailly and Passare extended (1.7) to the case when Z is a locally complete intersection, cf. Remark 4.3. The result of Demailly-Passare was further extended by Andersson in [A1], where he proved that if one considers so-called Bochner-Martinelli residue currents associated to generators of the defining ideal of an analytic space Z of pure dimension, and form a current similar to the left-hand side of (1.6), then a similar formula holds. This is a variant of the so-called King's formula, where the right-hand side of (1.6) is a current of integration like (1.3), but where the multiplicities m_i are the corresponding *algebraic* (or *Hilbert-Samuel*) multiplicities, see, e.g., [F, Chapter 4.3].

If Z and (E, φ) are as in Theorem 1.1, then by [A3, Example 1], there exists *some* holomorphic $\mathrm{Hom}\,(E_p, E_0)$-valued form ξ such that $\xi R_p^E = [Z]$. Our Theorem 1.1 thus states that $(1/(2\pi i)^p p!)D\varphi_1 \cdots D\varphi_p$ is an explicit such ξ.

In previous works, [LW] and [W], we proved Theorem 1.1 for certain resolutions of monomial ideals by explicitly computing the residue currents R^E and the Jacobian factors $D\varphi_1 \cdots D\varphi_p$ respectively.

Another result that is closely related to ours, although not formulated in terms of residue currents, is a cohomological version of Theorem 1.1 in the Cohen-Macaulay case due to Lejeune-Jalabert, [LJ1]. Given a free resolution (E, φ) of $\mathcal{O}_{Z,z}$ of minimal length, where Z is a Cohen-Macaulay analytic space, she constructed a generalization of the Grothendieck residue pairing, which in a sense is a cohomological version of the current in [AW1], and proved that the fundamental class of Z at z is represented by $D\varphi_1 \cdots D\varphi_p$. In Section 6 we describe this in more details and also discuss the relation to our results. The relationship between Lejeune-Jalabert's residue pairing and the residue currents in [AW1] is elaborated in [Lä3], see also [Lu1, Lu2].

To be precise, the current in the left-hand side of (1.6) takes values in $\mathrm{End}\,E_0$. However, since E_0 has rank 1, it is naturally identified with a scalar-valued current. In fact, it is possible to drop the assumption that rank $E_0 = 1$, but to make sense of (1.6) we then need to turn the $\mathrm{End}\,E_0$-valued current

$$\Theta := \frac{1}{(2\pi i)^p p!} D\varphi_1 \cdots D\varphi_p R_p^E$$

into a scalar-valued current. We will describe two natural ways of doing this. The first one is to take the trace $\mathrm{tr}\,\Theta$ of Θ. Secondly, let τ be the natural surjection $\tau : E_0 \to \mathrm{coker}\,\varphi_1 \cong \mathcal{O}_Z$. Since $R_p^E \varphi_1 = 0$, see (2.6) below, one gets a well-defined $\mathrm{Hom}\,(\mathcal{O}_Z, E_p)$-valued current $R_p^E \tau^{-1}$ by (locally) letting $R_p^E \tau^{-1} f := R_p^E f_0$ for any section f_0 of E_0 such that $\tau f_0 = f$. It follows that $\tau\Theta\tau^{-1}$ is a well-defined $\mathrm{End}(\mathcal{O}_Z)$-valued current, which can be identified with a scalar-valued current (annihilated by $\mathcal{J}$, where $\mathcal{J} \subset \mathcal{O}_X$ is the ideal defining Z). Note that if rank $E_0 = 1$, then $\mathrm{tr}\,\Theta$ and $\tau\Theta\tau^{-1}$ coincide with Θ (regarded as scalar currents).

Theorem 1.2. *Let $Z \subset X$ be an analytic space of pure codimension p, let (E, φ) be a Hermitian locally free resolution of $\mathcal{O}_Z$ over $\mathcal{O}_X$, and let D be the connection on*

End$E$ *induced by arbitrary connections on* $E_0, \ldots, E_p$. *Then*

(1.8) $$\frac{1}{(2\pi i)^p p!} \operatorname{tr}\left(D\varphi_1 \cdots D\varphi_p R_p^E\right) = [Z]$$

and

(1.9) $$\frac{1}{(2\pi i)^p p!} \tau D\varphi_1 \cdots D\varphi_p R_p^E \tau^{-1} = [Z],$$

where τ *is the natural surjection* $\tau : E_0 \to \operatorname{coker} \varphi_1 \cong \mathcal{O}_Z$.

In view of the discussion above, note that Theorem 1.1 is just a special case of Theorem 1.2.

The proof of Theorem 1.2 is given in Section 4. The first key ingredient is two lemmas, Lemmas 4.1 and 4.2, which assert that the left-hand sides of (1.8) and (1.9), respectively, only depend on Z and not on the choice of (E, φ) or D. In particular, it follows that the left-hand side of (1.8) coincides with the left-hand side of (1.9), cf. (4.15). Thus, to prove Theorem 1.2 it is enough to prove (1.8) for a specific choice of resolution and connection. The proofs of Lemmas 4.1 and 4.2 rely on a comparison formula for residue currents due to the first author, [Lä2], see Section 2.4.

By the *dimension principle*, Proposition 2.1, for so-called *pseudomeromorphic currents*, see Section 2.1, it suffices to prove (1.8) generically on Z_{red} (i.e., outside a hypersurface of Z_{red}). For z generically on Z_{red} we can use a certain *universal free resolution* of $\mathcal{O}_{Z,z}$, based on a construction by Scheja and Storch, [SS], and Eisenbud, Riemenschneider and Schreyer, [ERS]; this is described in Section 3. The inspiration to use this universal free resolution comes from [LJ1]. The resolution is in general far from being minimal, in particular, rank $E_0 > 1$ in general, but it is explicit enough so that we can explicitly compute (1.8), see Lemma 4.5.

In Theorems 1.1 and 1.2 we assume that Z has pure codimension, or, equivalently, pure dimension. In fact, for the proofs we only need that Z has *pure dimension* in the weak sense that all irreducible components of Z_{red} have the same dimension, in other words, all minimal primes of $\mathcal{J}$ have the same dimension. In particular, we allow $\mathcal{J}$ to have embedded primes.

Example 1.3. Let $Z \subset \mathbb{C}^2$ be defined by $\mathcal{J} = \mathcal{J}(y^k, x^\ell y^m) \subset \mathcal{O}_{\mathbb{C}^2}$, where $m < k$. Then Z has pure dimension, since Z_{red} equals $\{y = 0\}$, which is irreducible. However, note that $\mathcal{J}$ has an embedded prime $\mathcal{J}(x, y)$ of dimension 0.

Example 1.4. Let $Z \subset \mathbb{C}^3$ be defined by $\mathcal{J} = \mathcal{J}(xz, yz) \subset \mathcal{O}_{\mathbb{C}^3}$. Then Z does not have pure dimension, since its irreducible components $\{z = 0\}$ and $\{x = y = 0\}$ have dimension 2 and 1, respectively.

We get a version of Theorem 1.1 also when Z does not have pure dimension, without much extra work. However, the formulation becomes slightly more involved. Since the residue currents R_k^E are pseudomeromorphic, see Section 2.1, it follows that one can give a natural meaning to the restrictions $\mathbf{1}_W R_k^E$ if W is a subvariety of X.

Theorem 1.5. *Let* $Z \subset X$ *be an analytic space. Assume that* $\dim X = N$ *and* $\operatorname{codim} Z = p$. *Let* (E, φ) *be a Hermitian locally free resolution of* $\mathcal{O}_Z$ *over* $\mathcal{O}_X$, *where* rank $E_0 = 1$, *and let* D *be the connection on* End$E$ *induced by arbitrary connections on* $E_0, \ldots, E_N$. *Let* W_k *be the union of the components of* Z_{red} *of codimension* k,

and define $R_{[k]} := \mathbf{1}_{W_k} R_k^E$. Then

$$(1.10) \qquad \sum_{k=p}^{N} \frac{1}{(2\pi i)^k k!} D\varphi_1 \cdots D\varphi_k R_{[k]} = [Z].$$

Remark 1.6. As in Theorem 1.2 we could drop the assumption that rank $E_0 = 1$. Using the notation from above, we get

$$(1.11) \quad \sum_{k=p}^{N} \frac{1}{(2\pi i)^k k!} \operatorname{tr}(D\varphi_1 \cdots D\varphi_k R_{[k]}) = \sum_{k=p}^{N} \frac{1}{(2\pi i)^k k!} \tau D\varphi_1 \cdots D\varphi_k R_{[k]} \tau^{-1} = [Z],$$

see Remark 4.7.

It is natural to also consider the "full" currents $D\varphi_1 \cdots D\varphi_k R_k$ and it would be interesting to investigate whether they may capture geometric or algebraic information (in addition to the fundamental cycle). In Section 5 we compute the current $D\varphi_1 D\varphi_2 R_2^E$ for a Hermitian resolution of Z from Example 1.3. We also illustrate Theorem 1.5 by explicitly computing the currents in (1.10) in the situation of Example 1.4.

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2. Preliminaries

Throughout this paper X will be a complex manifold of dimension N, and $\chi(t)$ will be (a smooth approximand of) the characteristic function of the interval $[1, \infty)$. Let f be a holomorphic function on X or, more generally, a holomorphic section of a line bundle over X. Then there is an associated *principal value current* $1/f$, [D, HL], defined, e.g., as the limit

$$\lim_{\epsilon \to 0} \chi(|f|^2/\epsilon) \frac{1}{f}.$$

The associated *residue current* is defined as $\bar{\partial}(1/f)$, cf. (1.2).

2.1. **Pseudomeromorphic currents.** Following [AW2] we say that a current of the form

$$\frac{1}{z_1^{a_1}} \cdots \frac{1}{z_k^{a_k}} \bar{\partial} \frac{1}{z_{k+1}^{a_{k+1}}} \wedge \cdots \wedge \bar{\partial} \frac{1}{z_m^{a_m}} \wedge \xi,$$

where $z_1, \ldots, z_N$ is a local coordinate system and ξ is a smooth form with compact support, is an *elementary current*. Moreover a current on X is said to be *pseudomeromorphic* if it can be written as a locally finite sum of push-forwards of elementary currents under compositions of modifications, open inclusions, or projections.[3] Note that if T is pseudomeromorphic, then so is $\bar{\partial} T$.

The sheaf of pseudomeromorphic currents, denoted $\mathcal{PM}$, was introduced to obtain a coherent approach to questions concerning principal value and residue currents; in fact, all principal value and residue currents in this paper are pseudomeromorphic. It follows from, e.g., [A1] that currents of integration along analytic subvarieties $W \subset X$ are pseudomeromorphic.

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[3] In [AW2] only modifications were allowed. This more general class of pseudomeromorphic currents appeared in [AS].

In many ways pseudomeromorphic currents behave like normal currents, i.e., currents T such that T and dT are of order 0. In particular, they satisfy the following *dimension principle*, [AW2, Corollary 2.4].

Proposition 2.1. *If $T \in \mathcal{PM}(X)$ is a (p,q)-current with support on a subvariety $W \subset X$, and $\operatorname{codim} W > q$, then $T = 0$.*

Moreover, pseudomeromorphic currents admit natural restrictions to analytic subvarieties, see [AW2, Section 3] and also [AS, Proposition 2.3]. If $T \in \mathcal{PM}(X)$, $W \subset X$ is a subvariety of X, and h is a tuple of holomorphic functions such that $W = \{h = 0\}$, the restriction $\mathbf{1}_W T$ can be defined, e.g., as

$$\mathbf{1}_W T := \lim_{\epsilon \to 0} \left(1 - \chi(|h|^2/\epsilon) \right) T.$$

This definition is independent of the choice of χ and the tuple h, and $\mathbf{1}_W T$ is a pseudomeromorphic current with support on W. If $\mathbf{1}_W T = 0$ for all subvarieties $W \subset X$ of positive codimension, then T is said to have the *standard extension property, SEP*.

2.2. **Superstructure.** Let

$$0 \to E_\nu \xrightarrow{\varphi_\nu} E_{\nu-1} \xrightarrow{\varphi_{\nu-1}} \cdots \xrightarrow{\varphi_2} E_1 \xrightarrow{\varphi_1} E_0$$

be a complex of locally free $\mathcal{O}_X$-modules. Then $E := \bigoplus E_k$ has a natural superstructure, i.e., a $\mathbb{Z}_2$-grading, which splits E into odd and even elements E^+ and E^-, where $E^+ = \bigoplus E_{2k}$ and $E^- = \bigoplus E_{2k+1}$. Also $\operatorname{End} E$ gets a superstructure by letting the even elements be the endomorphisms preserving the degree, and the odd elements the endomorphisms switching degrees.

We let $\mathcal{E}$ and $\mathcal{E}^\bullet$ denote the sheaves of smooth functions and forms, respectively, on X and we let $\mathcal{E}^\bullet(E) = \mathcal{E}^\bullet \otimes_{\mathcal{E}} \mathcal{E}(E)$ and $\mathcal{E}^\bullet(\operatorname{End} E) = \mathcal{E}^\bullet \otimes_{\mathcal{E}} \mathcal{E}(\operatorname{End} E)$ be the sheaves of form-valued sections of E and $\operatorname{End} E$, respectively. Given a section $\gamma = \omega \otimes \eta$, where ω is a smooth form and η is a smooth section of E or $\operatorname{End} E$, we let $\deg_f \gamma := \deg \omega$ and $\deg_e \gamma := \deg \eta$. Then $\mathcal{E}^\bullet(E)$ and $\mathcal{E}^\bullet(\operatorname{End} E)$ inherit superstructures by letting $\deg \gamma := \deg_f \gamma + \deg_e \gamma$. Both $\mathcal{E}^\bullet(E)$ and $\mathcal{E}^\bullet(\operatorname{End} E)$ are naturally left $\mathcal{E}^\bullet$-modules. We make them into right $\mathcal{E}^\bullet$-modules by letting

$$(2.1) \qquad\qquad \gamma\omega = (-1)^{(\deg \gamma)(\deg \omega)} \omega\gamma,$$

where ω is a smooth form, and γ is a section of $\mathcal{E}^\bullet(E)$ or $\mathcal{E}^\bullet(\operatorname{End} E)$. Moreover, if $\beta = \alpha \otimes \xi$ and $\gamma = \omega \otimes \eta, \gamma' = \omega' \otimes \eta'$ are sections of $\mathcal{E}^\bullet(E)$ and $\mathcal{E}^\bullet(\operatorname{End} E)$, respectively, we let

$$\gamma(\beta) = (-1)^{(\deg_e \gamma)(\deg_f \beta)} \omega \wedge \alpha \otimes \eta(\xi),$$
$$(2.2) \qquad\qquad \gamma\gamma' = (-1)^{(\deg_e \gamma)(\deg_f \gamma')} \omega \wedge \omega' \otimes \eta\eta'.$$

Note that if $\gamma = \alpha \otimes \operatorname{Id}$ then $\gamma\beta = \alpha\beta$, $\gamma\gamma' = \alpha\gamma'$, and $\gamma'\gamma = \gamma'\alpha$, cf. (2.1). Thus we can regard a form α as a (form-valued) endomorphism. Moreover, we have the following associativity: $(\gamma\gamma')\beta = \gamma(\gamma'\beta)$ and $(\gamma\gamma')\gamma'' = \gamma(\gamma'\gamma'')$ if γ'' is a section of $\mathcal{E}^\bullet(\operatorname{End} E)$. Analogously the sheaves $\mathcal{C}^\bullet(E) = \mathcal{C}^\bullet \otimes_{\mathcal{E}} \mathcal{E}(E)$ and $\mathcal{C}^\bullet(\operatorname{End} E) = \mathcal{C}^\bullet \otimes_{\mathcal{E}} \mathcal{E}(\operatorname{End} E)$ of current-valued sections of E and $\operatorname{End} E$, respectively, inherit superstructures.

If $E_0, \ldots, E_\nu$ (considered as vector bundles) are equipped with connections $D_{E_0}, \ldots, D_{E_\nu}$, and D_E is the connection $\bigoplus D_{E_i}$ on E, we equip $\operatorname{End} E$ with the induced connection D_{End} defined by

$$(2.3) \qquad\qquad D_E(\gamma(\xi)) = D_{\operatorname{End}}(\gamma)\xi + (-1)^{\deg \gamma} \gamma(D_E \xi),$$

where ξ is a section of $\mathcal{E}^\bullet(E)$ and γ is a section of $\mathcal{E}^\bullet(\mathrm{End}\,E)$. It is then straightforward to verify that for arbitrary sections γ, γ' of $\mathcal{E}^\bullet(\mathrm{End}\,E)$,

$$(2.4) \qquad D_{\mathrm{End}}(\gamma\gamma') = D_{\mathrm{End}}\gamma\gamma' + (-1)^{\deg\gamma}\gamma D_{\mathrm{End}}\gamma'.$$

Moreover, note that if $\gamma = \alpha \otimes \mathrm{Id}$, then $D_{\mathrm{End}}\gamma = d\alpha$, so, again, we can regard a form α as a (form-valued) endomorphism.

Throughout this paper we will use the sign conventions associated with this superstructure, cf. Section 2.5.

Example 2.2. We consider the situation when (E, φ) is the Koszul complex $(K, \phi) = (\bigwedge \mathcal{O}_X^{\oplus p}, \delta_f)$ associated to a tuple $(f_1, \ldots, f_p)$ of holomorphic functions, and $e_1, \ldots, e_p$ is the standard basis of $\mathcal{O}_X^{\oplus p}$ so that δ_f is contraction with $f = f_1 e_1^* + \cdots + f_p e_p^*$, see Section 3.2. If we assume that D is trivial with respect to the induced bases e_I of (K, ϕ), then $D\delta_f$ is contraction with $df_1 \wedge e_1^* + \cdots + df_p \wedge e_p^*$. As $df_i \wedge e_i^*$ is even, we thus get that $D\phi_1 \cdots D\phi_p$ is contraction with

$$(df_1 \wedge e_1^* + \cdots + df_p \wedge e_p^*)^p = p!\; df_1 \wedge e_1^* \wedge \cdots \wedge df_p \wedge e_p^* = p!\; df_1 \wedge \cdots \wedge df_p \wedge e_{\{1,\ldots,p\}}^*,$$

where $e_{\{1,\ldots,p\}}$ and $e_\emptyset$ are frames of E_p and E_0, see Section 3.2 for notation. Thus

$$(2.5) \qquad D\phi_1 \cdots D\phi_p = p!\; df_1 \wedge \cdots \wedge df_p \wedge e_\emptyset \wedge e_{\{1,\ldots,p\}}^*.$$

2.3. Residue currents associated with Hermitian locally free resolutions.
Let $\mathcal{G}$ be a coherent sheaf on X of codim $p > 0$ with a Hermitian locally free resolution (E, φ), cf. the introduction. In [AW1] Andersson and the second author defined a $(\mathrm{Hom}\,(E_0, E)$-valued) current R^E associated with (E, φ). We will write $R^E = \sum R_k^E$, where R_k^E is the part of R^E which takes values in $\mathrm{Hom}\,(E_0, E_k)$. The current R_k^E is a $(0, k)$-current with support on $\mathrm{supp}\,\mathcal{G}$ and thus $R_k^E = 0$ if $k < p$ by the dimension principle, Proposition 2.1. The current R^E satisfies that if α is a holomorphic section of E_0, then $R^E\alpha = 0$ if and only if α belongs to $\mathrm{im}\,\varphi_1$, [AW1, Theorem 1.1]; this can be seen as a duality principle. In particular,

$$(2.6) \qquad R_p^E\varphi_1 = 0.$$

The current R^E is ∇-closed, where $\nabla = \varphi - \bar\partial$, i.e., $\varphi_k R_k^E - \bar\partial R_{k-1}^E = 0$ for all k. In particular,

$$(2.7) \qquad \varphi_p R_p^E = 0.$$

For details about the construction of these residue currents, we refer to [AW1]. For further reference, we mention that the construction is related to certain *singularity subvarieties* associated to a coherent analytic sheaf, see [ST]. The singularity subvariety Z_k^E is defined as the set where φ_k does not have optimal rank[4]. By uniqueness of minimal free resolutions, these sets are in fact independent of the choice of (E, φ), and indeed only depend on $\mathcal{G}$.

Example 2.3. Assume that Z is a complete intersection of codimension p, i.e., $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}$, where $\mathcal{J}$ is a complete intersection ideal, generated by, say, $f = (f_1, \ldots, f_p)$. Let (E, φ) be the Koszul complex of f. Then the corresponding sheaf complex is a free resolution of $\mathcal{O}_Z$ and

$$(2.8) \qquad R_p^E = \bar\partial\frac{1}{f_p} \wedge \cdots \wedge \bar\partial\frac{1}{f_1} \wedge e_{\{1,\ldots,p\}} \wedge e_\emptyset^*,$$

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[4]In [ST], the singularity subvariety $S_\ell(\mathcal{G})$ is defined as the set of x such that $\mathrm{depth}_x\,\mathcal{G}_x \leq \ell$, and by the Auslander-Buchsbaum formula, $Z_k^E = S_{N-k}(\mathcal{G})$.

where $e_{\{1,\dots,p\}}$ and $e_\emptyset$ are frames of E_p and E_0, see Section 3.2 for notation. This was proven in [PTY, Theorem 4.1][5] and [A2, Corollary 3.5].

2.4. **A comparison formula for residue currents.** Let $\alpha : H \to G$ be a homomorphism of finitely generated $\mathcal{O}_{X,\zeta}$-modules, and let (F, ψ) and (E, φ) be free resolutions of H and G respectively. We say that a morphism of complexes $a : (F, \psi) \to (E, \varphi)$ *extends* α if the map coker $\psi_1 \cong H \to G \cong$ coker φ_1 induced by a_0 equals α.

Proposition 2.4. *Let $\alpha : H \to G$ be a homomorphism of finitely generated $\mathcal{O}_{X,\zeta}$-modules, and let (F, ψ) and (E, φ) be free resolutions of H and G respectively. Then, there exists a morphism $a : (F, \psi) \to (E, \varphi)$ of complexes which extends α.*

If $\tilde{a} : (F, \psi) \to (E, \varphi)$ is any other such morphism, then there exists a morphism $s_0 : F_0 \to E_1$ such that $a_0 - \tilde{a}_0 = \varphi_1 s_0$.

The existence of a follows from defining it inductively by a relatively straightforward diagram chase, see [E, Proposition A3.13], and the existence of s_0 follows by a similar argument.

The residue currents associated with (E, φ) and (F, ψ) are related by the following comparison formula, see [Lä2, Theorem 3.2].

Theorem 2.5. *Assume that H and G are two finitely generated $\mathcal{O}_{X,\zeta}$-modules with Hermitian free resolutions (F, ψ) and (E, φ), respectively. If $a : (F, \psi) \to (E, \varphi)$ is a morphism of complexes, then*

$$(2.9) \qquad\qquad R^E a_0 - a R^F = \nabla M$$

where M is a pseudomeromorphic $\mathrm{Hom}\,(F_0, E)$-valued current with support on $\mathrm{supp}\,H \cup \mathrm{supp}\,G$.

If we write $M = \sum M_\ell$, where M_ℓ is the part of M with values in $\mathrm{Hom}\,(F_0, E_\ell)$, and if H and G have codimension $\geq k$, then $M_k = 0$ by [Lä2, Corollary 3.6]. In particular, if H and G have codimension p, then (2.9) implies that (by taking the $\mathrm{Hom}\,(F_0, E_p)$-valued part)

$$(2.10) \qquad\qquad R^E_p a_0 = a_p R^F_p + \varphi_{p+1} M_{p+1}.$$

If, in addition, G is Cohen-Macaulay, i.e., it has a free resolution of length p, and (E, φ) is such a resolution, then

$$(2.11) \qquad\qquad R^E_p a_0 = a_p R^F_p.$$

Finally, we will also need to consider the situation when G is Cohen-Macaulay, but when the free resolution does not have minimal length p. The following Lemma follows from [Lä2, Lemma 3.3 and Corollary 3.6].

Lemma 2.6. *We use the notation from Theorem 2.5. Assume that H and G have codimension p and moreover that G is Cohen-Macaulay. Then*

$$M_{p+1} = -\sigma^E_{p+1} a_p R^F_p,$$

where σ^E_{p+1} is smooth.

---

[5]In fact, in [PTY] it was proved that $\bar{\partial}(1/f_p) \wedge \cdots \wedge \bar{\partial}(1/f_1)$ equals the so-called Bochner-Martinelli residue current of f, which by [A2, Corollary 3.5] is the coefficient of R^E (i.e., the current in front of $e_{\{1,\dots,p\}} \wedge e^*_\emptyset$).

2.5. **Matrix notation.** For a section γ of $\mathcal{E}^\bullet(\mathrm{End}\,E)$ (or $\mathcal{C}^\bullet(\mathrm{End}\,E)$), let $\{\gamma\}$ denote the matrix representing γ in a local frame of E.

From (2.2) it follows that if β and γ are sections of $\mathcal{E}^\bullet(\mathrm{End}\,E)$, then

$$(2.12) \qquad \{\beta\gamma\} = (-1)^{(\deg_e\beta)(\deg_f\gamma)}\{\beta\}\{\gamma\}.$$

If we consider the main formula (1.6) as a product of matrices in a local frame, then by repeatedly using (2.12), the formula becomes

$$[Z] = \frac{1}{(2\pi i)^p p!}(-1)^{p(p-1)/2+p^2}\{D\varphi_1\}\cdots\{D\varphi_p\}\{R_p^E\}.$$

In [LW], we explicitly computed the current $D\varphi_1\cdots D\varphi_p R_p^E$, when (E,φ) is a certain free resolution of a 2-dimensional Artinian monomial ideal, by multiplying matrices, and this is the reason for why the constant $C_p = (-1)^{p(p-1)/2+p^2} = (-1)^{\lceil p/2\rceil}$ appeared in [LW, (7.4)].

When (E,φ) is the Koszul complex of a tuple $(f_1,\ldots,f_p)$ of holomorphic functions defining a complete intersection ideal $\mathcal{J}(f)$ of codimension p, then

$$
\begin{aligned}
\frac{1}{(2\pi i)^p p!}\{D\varphi_1\cdots D\varphi_p R_p^E\} &= \frac{1}{(2\pi i)^p p!}(-1)^{p^2}\{D\varphi_1\cdots D\varphi_p\}\{R_p^E\} = \\
&= \frac{1}{(2\pi i)^p}(-1)^{p^2} df_1\wedge\cdots\wedge df_p\wedge\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1} = \\
&= \frac{1}{(2\pi i)^p}\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge df_1\wedge\cdots\wedge df_p = [Z],
\end{aligned}
$$

(2.13)

where $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}(f)$ and where we have used (2.5) and (2.8) in the second equality and the Poincaré-Lelong formula, (1.7), in the last equality.

Assume that β and γ are $\mathrm{Hom}\,(E_k,E_\ell)$- and $\mathrm{Hom}\,(E_\ell,E_k)$-valued forms, respectively. Using that for scalar-valued $(i\times j)$- and $(j\times i)$-matrices B and C, $\mathrm{tr}(BC) = \mathrm{tr}(CB)$, together with (2.12), one gets that

$$
\begin{aligned}
\mathrm{tr}\{\beta\gamma\} &= (-1)^{(\deg_e\beta)(\deg_f\gamma)}\,\mathrm{tr}\left(\{\beta\}\{\gamma\}\right) = \\
&= (-1)^{(\deg_e\beta)(\deg_f\gamma)+(\deg_f\beta)(\deg_f\gamma)}\,\mathrm{tr}\left(\{\gamma\}\{\beta\}\right) = \\
&= (-1)^{(\deg_e\beta)(\deg_f\gamma)+(\deg_f\beta)(\deg_f\gamma)+(\deg_e\gamma)(\deg_f\beta)}\,\mathrm{tr}\{\gamma\beta\}.
\end{aligned}
$$

Hence,

$$(2.14) \qquad \mathrm{tr}\left(\beta\gamma\right) = (-1)^{(\deg\beta)(\deg\gamma)-(\deg_e\beta)(\deg_e\gamma)}\,\mathrm{tr}\left(\gamma\beta\right).$$

Note that both (2.12) and (2.14) hold also when either β or γ is a section of $\mathcal{C}^\bullet(\mathrm{End}\,E)$.

3. Universal free resolutions

A key ingredient in the proof of Theorem 1.2 is a specific universal free resolution of $\mathcal{O}_{Z,\zeta}$ for ζ where Z is Cohen-Macaulay. It is in general far from minimal, but on the other hand the construction is explicit. The universal free resolution, which is a Koszul complex over a certain ring A that we describe below, is a special case of a universal free resolution of Cohen-Macaulay ideals due to Scheja and Storch, [SS, p. 87–88], and Eisenbud, Riemenschneider and Schreyer, [ERS, Theorem 1.1 and Example 1.1], who however do this in an algebraic setting.

In order to prove Theorem 1.2, it will be enough to have a free resolution generically on Z. Generically on Z, a Noether normalization $\pi: Z \to W$ is given by a projection to $W := Z_{\mathrm{red}}$, and one can there describe $\mathcal{O}_{Z,\zeta}$ as a free $\mathcal{O}_{W,\zeta}$-module in an explicit way, see Lemma 3.1. In Lemma 4.5, which we use to prove Theorem 1.2, we will

use this description of $\mathcal{O}_{Z,\zeta}$ as a free $\mathcal{O}_{W,\zeta}$-module. In this case, we can give a direct proof that the construction of [ERS] and [SS] indeed gives a free resolution of $\mathcal{O}_Z$; this is Theorem 3.4.

3.1. **The ring** A. For a tuple $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{N}^p$, where $\mathbb{N} = \{0, 1, 2 \ldots\}$, we use the multi-index notation $z^\alpha := z_1^{\alpha_1} \cdots z_p^{\alpha_p}$, and, in addition, we let $|\alpha| := \alpha_1 + \cdots + \alpha_p$.

Lemma 3.1. *Let X be a complex manifold of dimension N, and assume that $\mathcal{J} \subset \mathcal{O}_X$ is the defining ideal of an analytic subspace Z of X, of pure codimension p, and let $n = N - p$. Let $W = Z_{\mathrm{red}}$ and assume that $\xi \in W_{\mathrm{reg}}$. Assume that we near ξ have coordinates $(z, w) \in \mathbb{C}^p \times \mathbb{C}^n$ on X, such that $(z, w)(\xi) = 0$ and that in these coordinates, $W = \{z_1 = \cdots = z_p = 0\}$. Let m denote the geometric multiplicity of W in Z near ξ.*

Then there exist a neighbourhood $U \subset W$ of ξ, a hypersurface $Y \subset U$, and tuples $\alpha^1, \ldots, \alpha^m \in \mathbb{N}^p$ such that for $\zeta \in U \setminus Y$, $\mathcal{O}_{Z,\zeta}$ is a free $\mathcal{O}_{W,\zeta}$-module with a basis $z^{\alpha^1}, \ldots, z^{\alpha^m}$. Moreover, the tuples α^i satisfy $|\alpha^1| \geq |\alpha^2| \geq \cdots \geq |\alpha^m|$ and if we express any monomial z^γ in terms of the z^{α^i}, $z^\gamma = \sum f_i(w) z^{\alpha^i} + \mathcal{J}$, then for all i such that $f_i \not\equiv 0$, we have that $|\alpha^i| \geq |\gamma|$.

Note that if one considers a tuple $\beta \in \mathbb{N}^p$, then, by the last statement of the lemma, we have for each j,

$$(3.1) \qquad z^\beta z^{\alpha^j} = \sum_{i \leq j} f_i(w) z^{\alpha^i} + \mathcal{J},$$

and if $\beta \neq 0$, then the sum can be taken just over $i < j$.

Proof. By the Nullstellensatz in $\mathcal{O}_{X,\xi}$, we can choose β_i such that $z_i^{\beta_i} \in \mathcal{J}$ for $i = 1, \ldots, p$. In particular, the finite set of monomials z^α such that $\alpha_i < \beta_i$ for $i = 1, \ldots, p$ must generate $\mathcal{O}_{Z,\xi}$ as an $\mathcal{O}_{W,\xi}$-module. By coherence, these monomials also generate $\mathcal{O}_{Z,\zeta}$ as an $\mathcal{O}_{W,\zeta}$-module for ζ in some neighbourhood $U \subset W$ of ξ.

We let a^i be an enumeration of the tuples α with $\alpha_k < \beta_k$ for $k = 1, \ldots, p$, ordered so that $|a^i| \geq |a^j|$ if $i \leq j$. We now choose $\alpha^1, \ldots, \alpha^M$ inductively among the a^i so that $z^{\alpha^1}, \ldots, z^{\alpha^M}$ are independent over $\mathcal{O}_{W,\xi}$ in the following way: First, we let $\alpha^1 = a^{i_1}$, where i_1 is the first index i such that $f_1(w) z^{a^i} \equiv 0$ in $\mathcal{O}_{Z,\xi}$ implies that $f_1 \equiv 0$. Then, if we have already chosen $\alpha^1, \ldots, \alpha^k$, $\alpha^j = a^{i_j}$, we define inductively $\alpha^{k+1} = a^{i_{k+1}}$ as the next a^i such that if $f_1(w) z^{\alpha^1} + \cdots + f_k(w) z^{\alpha^k} + f_{k+1}(w) z^{a^i} \equiv 0$, then $f_{k+1} \equiv 0$. Clearly, $|\alpha^1| \geq \cdots \geq |\alpha^M|$.

Note that if a^k is not among the α^i, then there exists a relation $f_k(w) z^{a^k} = \sum_{j : i_j < k} g_{k,j}(w) z^{\alpha^j}$ in $\mathcal{O}_{Z,\xi}$, where $f_k \not\equiv 0$. By possibly shrinking U, we can assume that all the f_k's are defined on U. Let $Y := \bigcup_{k \notin \{i_1, \ldots, i_M\}} \{f_k = 0\}$. Then, outside the hypersurface Y, any such z^{a^k} can be expressed uniquely in terms of z^{α^j} with $i_j < k$. Thus, for $\zeta \in U \setminus Y$, $\mathcal{O}_{Z,\zeta}$ is a free $\mathcal{O}_{W,\zeta}$-module with basis $z^{\alpha^1}, \ldots, z^{\alpha^M}$. Therefore $M = m$. In addition, since each z^{a^k} not among the z^{α^i} can be written in terms of z^{α^j}, with $i_j < k$, by the ordering of the a^i, those α^i will satisfy that $|\alpha^i| \geq |a^k|$. $\square$

Definition 3.2. We consider the situation in Lemma 3.1. Given $\zeta \in U \setminus Y$, we define the $\mathcal{O}_{W,\zeta}$-module

$$A = A_\zeta := \mathcal{O}_{X,\zeta} \otimes_{\mathcal{O}_{W,\zeta}} \mathcal{O}_{Z,\zeta}.$$

Note that by Lemma 3.1, $\mathcal{O}_{Z,\zeta}$ is a free $\mathcal{O}_{W,\zeta}$-module of rank m, so A is a free $\mathcal{O}_{X,\zeta}$-module of rank m, i.e., $A \cong \mathcal{O}_{X,\zeta}^{\oplus m}$. We will denote an element $f \otimes g \in A$ by $f[g]$. We will also sometimes use the short-hand notation $f := f[1]$ and $[g] := 1[g]$. Note that since $\mathcal{O}_{X,\zeta}$ and $\mathcal{O}_{Z,\zeta}$ are $\mathcal{O}_{W,\zeta}$-algebras, so is A, and the multiplication is defined by $(f_1[g_1])(f_2[g_2]) = f_1 f_2[g_1 g_2]$.

Remark 3.3. Using the notation from above, for $\zeta \in U \backslash Y$, we have a basis $z^{\alpha^1}, \ldots, z^{\alpha^m}$ of $\mathcal{O}_{Z,\zeta}$ as a free $\mathcal{O}_{W,\zeta}$-module. This gives a basis $\left[z^{\alpha^1}\right], \ldots, \left[z^{\alpha^m}\right]$ of A as a free $\mathcal{O}_{X,\zeta}$-module. If z^γ is a monomial, then we can consider (multiplication with) $[z^\gamma]$ as an element in $\mathrm{End}_{\mathcal{O}_{X,\zeta}}(A)$, and the matrix of $[z^\gamma]$ with respect to the basis $\left[z^{\alpha^1}\right], \ldots, \left[z^{\alpha^m}\right]$ from Lemma 3.1 is upper triangular by (3.1), and it has zeros along the diagonal unless $\gamma = 0$, in which case $[z^\gamma]$ is the identity matrix.

3.2. **Universal free resolutions.** Let R be a commutative ring, and let $x_1, \ldots, x_p$ be elements of R. To fix notation, we remind that the *Koszul complex* of $x = (x_1, \ldots, x_p)$ is the complex $(\bigwedge^\bullet R^{\oplus p}, \delta_x)$, where the differential δ_x is defined by inner multiplication with x, i.e., if we choose as a standard basis $e_1, \ldots, e_p$ of $R^{\oplus p}$, then

$$\delta_x : e_I \mapsto \sum_{i=1}^k (-1)^{i-1} x_{I_i} e_{I \backslash I_i},$$

where $I = (I_1, \ldots, I_k)$, and we use the short-hand notation $e_I = e_{I_1} \wedge \cdots \wedge e_{I_k}$. In particular, we use the notation $e_\emptyset$ for the basis of $\bigwedge^0 R^{\oplus p} \cong R$. If the sequence x is a regular sequence, then it is well-known that $(\bigwedge^\bullet R^{\oplus p}, \delta_x)$ is a free resolution of $R/(x_1, \ldots, x_p)$, see for example [E, Corollary 17.5]. When $R = \mathcal{O}_{X,\zeta}$, then $f = (f_1, \ldots, f_p)$ is a regular sequence if and only if codim $\{f_1 = \cdots = f_p = 0\} = p$. Hence, for complete intersection ideals, we have an explicitly defined free resolution. The universal free resolution gives an explicit free resolution for more general ideals in $\mathcal{O}_{X,\zeta}$, but then one considers a Koszul complex over the ring A instead of over $\mathcal{O}_{X,\zeta}$.

Theorem 3.4. *Assume that we are in the situation of Lemma 3.1, and that we fix some $\zeta \in U \setminus Y$. Let A be as in Definition 3.2, and let $\mathbf{z}_i := z_i - [z_i] \in A$ for $i = 1, \ldots, p$. Then, the Koszul complex $(K, \phi) := (\bigwedge^\bullet A^{\oplus p}, \delta_\mathbf{z})$ of $\mathbf{z} := (\mathbf{z}_1, \ldots, \mathbf{z}_p)$ is a free resolution of $\mathcal{O}_{Z,\zeta}$ over A and $\mathcal{O}_{X,\zeta}$.*

For tuples $\gamma, \eta \in \mathbb{N}^p$ we use the partial ordering that $\gamma \leq \eta$ if and only if $\gamma_i \leq \eta_i$ for $i = 1, \ldots, p$. We also use the short-hand notation $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^p$.

For the convenience of the reader, we provide a proof of Theorem 3.4 in our situation. In [SS] and [ERS], the corresponding theorem is in an algebraic setting, and does not immediately apply to our setting, although it should be possible to adapt the proof to our setting using analytic tensor products (cf., Section 2 in [ABM]).

Proof. By construction, K consists of free A-modules, and since, as explained above, A is a free $\mathcal{O}_{X,\zeta}$-module, K is also a complex of free $\mathcal{O}_{X,\zeta}$-modules. Exactness is independent of whether we consider the complex as $\mathcal{O}_{X,\zeta}$-modules or A-modules, so it is sufficient to prove that (K, ϕ) is a free resolution as $\mathcal{O}_{X,\zeta}$-modules.

We first prove that coker $\phi_1 \cong \mathcal{O}_{Z,\zeta}$. We get a surjective mapping $\pi : K_0 \to \mathcal{O}_{Z,\zeta}$ by letting $\pi(f[g]) := fg$. Note that $\pi(\mathbf{z}_i) = 0$ for $i = 1, \ldots, p$, so we get a well-defined induced mapping $\tilde{\pi} : K_0/(\mathrm{im}\,\phi_1) \to \mathcal{O}_{Z,\zeta}$. Clearly, $\tilde{\pi}$ is surjective since π is surjective. Next, we claim that

$$(3.2) \qquad\qquad f[g] = [fg] + \sum \mathbf{z}_i \eta_i,$$

for some $\eta_i \in A$, $i = 1, \ldots, p$. To prove the claim, we first choose β_i such that $z_i^{\beta_i} = 0$ in $\mathcal{O}_{Z,\zeta}$ for $i = 1, \ldots, p$, which is possible by the Nullstellensatz. We then make a finite Taylor expansion of f,

$$f = \sum_{\alpha \leq \beta - \mathbf{1}} f_\alpha(w) z^\alpha + \sum_{i=1}^{p} z_i^{\beta_i} f_i(z, w).$$

Using this Taylor expansion, in combination with the formula

$$z_i^k = \left[z_i^k \right] + (z_i^{k-1} + z_i^{k-2}\left[z_i \right] + \cdots + \left[z_i^{k-1} \right])(z_i - [z_i]),$$

and the fact that $\left[z_i^{\beta_i} \right] = 0$ and $f_\alpha(w)\left[g \right] = [f_\alpha(w)g]$, we get that $f[g]$ is of the form (3.2). If $\tilde{\pi}(\sum f_i \left[g_i \right] e_\emptyset) = 0$, then $\sum f_i g_i = 0$ in $\mathcal{O}_{Z,\zeta}$, and by (3.2),

$$\sum f_i \left[g_i \right] e_\emptyset = \sum \mathbf{z}_j \eta_j e_\emptyset = \phi_1 \eta,$$

for some $\eta = (\eta_1, \ldots, \eta_p) \in K_1$, i.e., $\sum f_i \left[g_i \right] = 0$ in $\mathrm{coker}\, \phi_1$, so $\tilde{\pi}$ is injective. We thus get that $\mathrm{coker}\, \phi_1 \cong \mathcal{O}_{Z,\zeta}$.

It remains to see that (K, ϕ) is exact at levels $k \geq 1$. In order to prove this, we first prove that ϕ_1 is pointwise surjective outside of $W = \{z_1 = \cdots = z_p = 0\}$. If $(z, w) \notin W$, we can assume that, say, $z_i \neq 0$. Then $\mathbf{z}_i$ is invertible, with inverse

$$\gamma_i := \sum_{k=0}^{\infty} \frac{1}{z_i^{k+1}} \left[z_i^k \right],$$

where the series is in fact a finite sum, since $z_i^k = 0$ in $\mathcal{O}_{Z,\zeta}$ for $k \gg 1$ by the Nullstellensatz. Then, $\phi_1(f[g]\gamma_i e_i) = f[g]e_\emptyset$, so ϕ_1 is surjective as a morphism of sheaves. Since the image is K_0, which is a vector bundle, it is also pointwise surjective. To conclude, ϕ_1 is pointwise surjective outside of W, i.e., $Z_1^K \subset W$, where Z_1^K is the first singularity subvariety associated to (K, ϕ).

We next prove that the complex is exact as a complex of sheaves at level $k \geq 1$ outside of W. As above, if (z, w) is outside of W, and, say, $z_i \neq 0$, and $\alpha \in K_k$ is such that $\phi_k \alpha = 0$, then $\phi_{k+1}(\gamma_i e_i \wedge \alpha) = (\delta_{\mathbf{z}} \gamma_i e_i) \wedge \alpha = \alpha$, so the complex is exact as a complex of sheaves outside of W. For a free resolution (E, φ), $Z_{k+1}^E \subset Z_k^E$, for $k \geq 1$, see [E, Corollary 20.12]. Hence, $Z_k^K \setminus W \subset Z_1^K \setminus W = \emptyset$, i.e., $Z_k^K \subset W$ for $k \geq 1$.

To conclude, the complex (K, ϕ) of length p is pointwise exact outside of W, which has codimension p. Thus, it is exact as a complex of sheaves by the Buchsbaum-Eisenbud criterion, [E, Theorem 20.9], because $\mathrm{codim}\, Z_k^K \geq p \geq k$ and the pointwise exactness of (K, ϕ) outside of W implies that $\mathrm{rank}\, K_k = \mathrm{rank}\, \phi_k + \mathrm{rank}\, \phi_{k+1}$. $\square$

In general, for $\zeta \in U \setminus Y$, the universal free resolution of $\mathcal{O}_{Z,\zeta}$ is not minimal as a free resolution of $\mathcal{O}_{X,\zeta}$-modules. To see this, note that $K_0 \cong A$, so if Z has geometric multiplicity $m > 1$ near ζ, then $\mathrm{rank}_{\mathcal{O}_{X,\zeta}} K_0 = m > 1$, while a minimal free resolution (E, φ) of $\mathcal{O}_{Z,\zeta}$ would have $\mathrm{rank}_{\mathcal{O}_{X,\zeta}} E_0 = 1$.

4. Proofs of Theorem 1.2 and Theorem 1.5

A key part in the proof of Theorem 1.2 is to prove that the currents on the left-hand sides of (1.8) and (1.9) are independent of the choice of locally free resolution (E, φ) of $\mathcal{O}_Z$ and the choice of connections on $E_0, \ldots, E_p$.

Lemma 4.1. *Let G be a finitely generated $\mathcal{O}_{X,\zeta}$-module of codimension p, and let (E, φ) and (F, ψ) be Hermitian free resolutions of G. Then,*

$$\text{(4.1)} \qquad \text{tr}(D\varphi_1 \cdots D\varphi_p R_p^E) = \text{tr}(D\psi_1 \cdots D\psi_p R_p^F),$$

where D is the connection on $\text{End}(E \oplus F)$ induced by arbitrary connections on $E_0, \ldots, E_p$ and $F_0, \ldots, F_p$.

Lemma 4.2. *Let G, (E, φ), (F, ψ) and D be as in Lemma 4.1, and let η and τ be the natural surjections $\eta : F_0 \to \text{coker}\,\psi_1 \cong G$ and $\tau : E_0 \to \text{coker}\,\varphi_1 \cong G$. Then*

$$\text{(4.2)} \qquad \eta D\psi_1 \cdots D\psi_p R_p^F \eta^{-1} = \tau D\varphi_1 \cdots D\varphi_p R_p^E \tau^{-1}.$$

Here $R_p^F \eta^{-1}$ and $R_p^E \tau^{-1}$ are defined as in the text preceding Theorem 1.2.

Remark 4.3. In case $\text{rank}\,E_0 = \text{rank}\,F_0 = 1$ these lemmas coincide. When $G = \mathcal{O}_{X,\zeta}/\mathcal{I}$, where $\mathcal{I}$ is a complete intersection ideal of codimension p, and (E, φ) and (F, ψ) are Koszul complexes of two minimal sets of generators of $\mathcal{I}$, then (4.1) and (4.2) follows rather easily from the transformation law and duality principle for Coleff-Herrera products. This was a key observation which allowed for global versions of the Poincaré-Lelong formula (1.7) for locally complete intersections in [DP]. In order to prove Lemma 4.1 and Lemma 4.2, we use the comparison formula, Theorem 2.5, which is a generalization of the transformation law.

The proofs of both of these lemmas use the following lemma.

Lemma 4.4. *Let (E, φ) and (F, ψ) be complexes of free $\mathcal{O}_{X,\zeta}$-modules, and let $b : (E, \varphi) \to (F, \psi)$ be a morphism of complexes. Let D be the connection on $\text{End}(E \oplus F)$ induced by connections on $E_0, \ldots, E_p$ and $F_0, \ldots, F_p$. Then,*

$$\text{(4.3)} \qquad D\psi_1 \cdots D\psi_p b_p = b_0 D\varphi_1 \ldots D\varphi_p + \psi_1 \alpha + \beta \varphi_p$$

for a smooth $\text{Hom}\,(E_p, F_1)$-valued $(p, 0)$-form α and a smooth $\text{Hom}\,(E_{p-1}, F_0)$-valued $(p, 0)$-form β.

Proof. We claim that for any $1 \le k \le p$,

$$\text{(4.4)} \qquad \begin{aligned} &D\psi_1 \cdots D\psi_k b_k D\varphi_{k+1} \cdots D\varphi_p = \\ &D\psi_1 \cdots D\psi_{k-1} b_{k-1} D\varphi_k \cdots D\varphi_p + \psi_1 \alpha_k + \beta_k \varphi_p \end{aligned}$$

for a smooth $\text{Hom}\,(E_p, F_1)$-valued $(p, 0)$-form α_k and a smooth $\text{Hom}\,(E_{p-1}, F_0)$-valued $(p, 0)$-form β_k. By using this repeatedly for $k = p, \ldots, 1$, we get (4.3).

To prove the claim, we note first that since b is a morphism of complexes, $\psi_k b_k = b_{k-1} \varphi_k$, and thus,

$$\text{(4.5)} \qquad \begin{aligned} D\psi_k b_k &= D(\psi_k b_k) + \psi_k D b_k = D(b_{k-1} \varphi_k) + \psi_k D b_k = \\ &D b_{k-1} \varphi_k + b_{k-1} D\varphi_k + \psi_k D b_k, \end{aligned}$$

where the signs depend on that b is an even mapping, while ψ is odd.

We now replace $D\psi_k b_k$ in the first line of (4.4) by the expression in the second line of (4.5). Note first that the term coming from the term $b_{k-1} D\varphi_k$ in the second line of (4.5) equals the first term of the second line of (4.4).

We consider next the term

$$\text{(4.6)} \qquad D\psi_1 \cdots D\psi_{k-1} D b_{k-1} \varphi_k D\varphi_{k+1} \cdots D\varphi_p,$$

coming from the term $Db_{k-1}\varphi_k$ in the second line of (4.5). Since $\varphi_\ell\varphi_{\ell+1} = 0$, we get by the Leibniz rule (2.4) and the fact that φ_ℓ has odd degree that $\varphi_\ell D\varphi_{\ell+1} = D\varphi_\ell\varphi_{\ell+1}$. Using this repeatedly for $\ell = k, \ldots, p - 1$, we get that (4.6) equals

$$D\psi_1 \cdots D\psi_{k-1}Db_{k-1}D\varphi_k D\varphi_{k+1} \cdots D\varphi_{p-1}\varphi_p =: \beta_k\varphi_p.$$

Finally, we consider the term coming from the term $\psi_k Db_k$ in the second line of (4.5). By using that $\psi_\ell\psi_{\ell+1} = 0$ and the Leibniz rule, we get that $D\psi_\ell\psi_{\ell+1} = \psi_\ell D\psi_{\ell+1}$, and using this repeatedly we get that this term equals

$$\psi_1 D\psi_2 \cdots D\psi_k Db_k D\varphi_{k+1} \cdots D\varphi_p =: \psi_1\alpha_k.$$

To conclude, when replacing $D\psi_k b_k$ in the first line of (4.4) by the last line of (4.5), we obtain three terms of the form as in the second line of (4.4), and we have thus proved (4.4). $\qquad\square$

Proof of Lemma 4.1. Since G has codimension p, it is Cohen-Macaulay outside of a subvariety of codimension $p + 1$. Since both sides of (4.1) are pseudomeromorphic (p, p)-currents, it is by the dimension principle, Proposition 2.1, enough to prove (4.1) where G is Cohen-Macaulay. We will thus assume for the remainder of the proof that G is Cohen-Macaulay.

Let (H, η) by any free resolution of G. Using (2.14) and (2.6), we get that if $\xi : H_p \to H_1$ is any smooth morphism, then

$$(4.7) \qquad \operatorname{tr}(\eta_1\xi R_p^H) = \pm\operatorname{tr}(\xi R_p^H \eta_1) = 0.$$

We let $a : (F, \psi) \to (E, \varphi)$ and $b : (E, \varphi) \to (F, \psi)$ be morphisms of complexes extending the identity morphism on G, see Section 2.4. Then, $b \circ a : (F, \psi) \to (F, \psi)$ extends the identity morphism on G. Since the identity morphism on (F, ψ) trivially also extends the identity morphism on G, we get by Proposition 2.4 that there exists $s_0 : F_0 \to F_1$ such that

$$(4.8) \qquad \operatorname{Id}_{F_0} = b_0 a_0 + \psi_1 s_0.$$

We let $W = \operatorname{tr}(D\psi_1 \cdots D\psi_p R_p^F)$. We then get by (2.6) and (4.8) that

$$(4.9) \qquad W = \operatorname{tr}(D\psi_1 \cdots D\psi_p R_p^F) = \operatorname{tr}(D\psi_1 \cdots D\psi_p R_p^F b_0 a_0),$$

and by (2.14),

$$W = \operatorname{tr}(a_0 D\psi_1 \cdots D\psi_p R_p^F b_0).$$

By the comparison formula (2.10), applied to $b : (E, \varphi) \to (F, \psi)$, and Lemma 2.6,

$$R_p^F b_0 = b_p R_p^E - \psi_{p+1}\sigma_{p+1}^F b_p R_p^E,$$

where σ_{p+1}^F is smooth. Since $D\psi_1 \cdots D\psi_p \psi_{p+1} = \psi_1 D\psi_2 \cdots D\psi_{p+1}$, see the previous proof, we get that

$$W = \operatorname{tr}(a_0 D\psi_1 \cdots D\psi_p b_p R_p^E) - \operatorname{tr}(a_0\psi_1\alpha' R_p^E),$$

where α' is smooth. Thus, by Lemma 4.4,

$$(4.10) \qquad W = \operatorname{tr}(a_0 b_0 D\varphi_1 \cdots D\varphi_p R_p^E) + \operatorname{tr}(a_0\psi_1(\alpha - \alpha')R_p^E) + \operatorname{tr}(a_0\beta\varphi_p R_p^E).$$

The last term in the right-hand side of (4.10) vanishes by (2.7). In addition, since a is a morphism of complexes, $a_0\psi_1 = \varphi_1 a_1$, so the middle term in the right-hand side of (4.10) vanishes by (4.7). Thus, only the first term in the right-hand side of (4.10) remains, i.e.,

$$W = \operatorname{tr}(a_0 b_0 D\varphi_1 \cdots D\varphi_p R_p^E).$$

From (2.14) and (4.9) (with the roles of (E, φ) and (F, ψ) reversed), we finally conclude that

$$W = \operatorname{tr}(D\varphi_1 \cdots D\varphi_p R_p^E).$$

$\square$

Proof of Lemma 4.2. Since the currents in (4.2) are pseudomeromorphic (p, p)-currents, we may as in the previous proof assume that G is Cohen-Macaulay. In addition, it is enough to prove (4.2) under the assumption that one of the free resolutions, say, (F, ψ), has minimal length, p. We let $a : (F, \psi) \to (E, \varphi)$ and $b : (E, \varphi) \to (F, \psi)$ be morphisms of complexes extending the identity morphism on G.

We claim that

$$(4.11) \qquad\qquad R_p^F \eta^{-1} = b_p R_p^E \tau^{-1}.$$

To see this, let $g \in \mathcal{O}_{Z,\zeta}$, and let g_0 be such that $\tau g_0 = g$. Then, by definition,

$$(4.12) \qquad\qquad b_p R_p^E \tau^{-1} g = b_p R_p^E g_0,$$

cf. the text right before Theorem 1.2. By (2.11), the right-hand side of (4.12) equals $R_p^F b_0 g_0$. Since b extends the identity morphism, $\eta b_0 g_0 = \tau g_0 = g$. Thus, $R_p^F b_0 g_0$ equals by definition $R_p^F \eta^{-1} g$, which proves the claim.

By (4.11),

$$(4.13) \qquad\qquad \eta D\psi_1 \cdots D\psi_p R_p^F \eta^{-1} = \eta D\psi_1 \cdots D\psi_p b_p R_p^E \tau^{-1}.$$

By Lemma 4.4, the right-hand side of (4.13) equals

$$(4.14) \qquad\qquad \eta b_0 D\varphi_1 \cdots D\varphi_p R_p^E \tau^{-1} + \eta \psi_1 \alpha R_p^E \tau^{-1} + \eta \beta \varphi_p R_p^E \tau^{-1}.$$

Since $\eta\psi_1 = 0$, the second term in the right-hand side of (4.14) vanishes, and the last term also vanishes by (2.7). To conclude, using that $\eta b_0 = \tau$, we thus get (4.2). $\quad\square$

By Lemma 4.1 and Lemma 4.2,

$$\operatorname{tr}(D\varphi_1 \cdots D\varphi_p R_p^E) \text{ and } \tau D\varphi_1 \cdots D\varphi_p R_p^E \tau^{-1}$$

only depend on G and not on the choice of free resolution (E, φ) of G and connection D. When rank $E_0 = 1$, these currents coincide. If $G = \mathcal{O}_Z$, then there always exists a free resolution (F, ψ) of $\mathcal{O}_Z$ with rank $F_0 = 1$, and thus, we get that for any free resolution (E, φ) of $\mathcal{O}_Z$,

$$(4.15) \qquad\qquad \operatorname{tr}(D\varphi_1 \cdots D\varphi_p R_p^E) = \tau D\varphi_1 \cdots D\varphi_p R_p^E \tau^{-1}.$$

Proof of Theorem 1.2. Note that by (4.15), it is enough to prove (1.8).

Let $W = Z_{\mathrm{red}}$. We first consider a point $\xi \in W_{\mathrm{reg}}$, and apply Lemma 3.1. We fix a neighbourhood $V \subset X$ of ξ contained in the coordinate chart from Lemma 3.1 such that $W = \{z_1 = \cdots = z_p = 0\}$ on V, and $V \cap W = U$. We first prove that (1.8) holds on V. Note that on V, $[Z] = m[z_1 = \cdots = z_p = 0]$, so we thus want to prove that

$$(4.16) \qquad\qquad \frac{1}{(2\pi i)^p p!} \operatorname{tr}(D\varphi_1 \cdots D\varphi_p R_p^E) = m[z_1 = \cdots = z_p = 0].$$

Lemma 4.5. *Let $\zeta \in U \setminus Y$, and let (K, ϕ) be the universal free resolution of $\mathcal{O}_{Z,\zeta}$ from Theorem 3.4. Then*

$$(4.17) \qquad\qquad \frac{1}{(2\pi i)^p p!} \operatorname{tr}(D\phi_1 \cdots D\phi_p R_p^K) = m[z_1 = \cdots = z_p = 0]$$

in a neighbourhood of ζ.

Taking this lemma for granted, using Lemma 4.1 and Theorem 3.4, we get first that (4.16) holds in a neighbourhood of each $\zeta \in U \setminus Y$. Thus, (4.16) holds in a neighbourhood of $U \setminus Y$, and since both sides of (4.16) have their support on $V \cap W = U$, (4.16) holds in fact on $V \setminus Y$. Since Y is a hypersurface of W, and W has codimension p in V, Y has codimension $p+1$ in V. As both sides of (4.16) are pseudomeromorphic (p,p)-currents on V which coincide outside of Y, (4.16) holds on all of V by the dimension principle, Proposition 2.1.

We have thus proven that any point $\xi \in W_{\mathrm{reg}}$ has a neighbourhood such that (1.8) holds, and since both sides of (1.8) have support on W, (1.8) holds on $X \setminus W_{\mathrm{sing}}$. Both sides of (1.8) are pseudomeromorphic (p,p)-currents on X, and W_{sing} has codimension $\geq p+1$ in X, so we get by the dimension principle that (1.8) holds on all of X. $\qquad\square$

Proof of Lemma 4.5. We here use the notation from Section 3, and we let $e_1, \ldots, e_p$ be the standard basis for $A^{\oplus p}$ over A. Note that over $\mathcal{O}_{X,\zeta}$, $\bigwedge^k A^{\oplus p}$ has the basis $\left[z^{\alpha^i} \right] e_I$, where $i = 1, \ldots, m$ and $I \subset \{1, \ldots, p\}$, $|I| = k$. Since by Lemma 4.1, the left-hand side of (4.17) is independent of the choice of connection, we may assume that D is trivial with respect to these bases.

In order to prove (4.17), we first write out the left-hand side as

$$(4.18) \qquad \mathrm{tr}(D\phi_1 \cdots D\phi_p R_p^K) = \sum_{i=1}^{m} \left(\left[z^{\alpha^i} \right] e_\emptyset \right)^* D\phi_1 \cdots D\phi_p R_p^K \left[z^{\alpha^i} \right] e_\emptyset,$$

where $\left(\left[z^{\alpha^1} \right] e_\emptyset \right)^*, \ldots, \left(\left[z^{\alpha^m} \right] e_\emptyset \right)^*$ is the dual basis of the basis $\left[z^{\alpha^1} \right] e_\emptyset, \ldots, \left[z^{\alpha^m} \right] e_\emptyset$ of K_0.

We will use the comparison formula, Theorem 2.5, to compute the currents $R_p^K \left[z^{\alpha^i} \right] e_\emptyset$ appearing in the sum in the right-hand side of (4.18). First of all, by the Nullstellensatz, there exist β_i such that $z_i^{\beta_i} \in \mathcal{J}$ for $i = 1, \ldots, p$. Throughout this proof, we will let $\beta_1, \ldots, \beta_p$ denote such a choice. We let $\epsilon_1, \ldots, \epsilon_p$ be the standard basis of $\mathcal{O}_{X,\zeta}^{\oplus p}$ over $\mathcal{O}_{X,\zeta}$. We let (L, ψ) be the Koszul complex over $\mathcal{O}_{X,\zeta}$ of the tuple $(z_1^{\beta_1}, \ldots, z_p^{\beta_p})$, and we let $\mathcal{I}$ be the ideal generated by this tuple.

Since $\mathcal{I}$ is contained in $\mathcal{J}$, there exists a morphism $c : (L, \psi) \to (K, \phi)$ extending the natural surjection $\mathcal{O}_{X,\zeta}/\mathcal{I} \to \mathcal{O}_{Z,\zeta}$, see Proposition 2.4. We construct explicitly such a morphism c. We let c_k be the map $L_k = \bigwedge^k \mathcal{O}_{X,\zeta}^{\oplus p} \to \bigwedge^k A^{\oplus p} = K_k$ induced by the map $c_1 : \mathcal{O}_{X,\zeta}^{\oplus p} \to A^{\oplus p}$,

$$c_1 : \epsilon_i \mapsto \sum_{\gamma_i=0}^{\beta_i-1} z_i^{\beta_i-\gamma_i-1} \left[z_i^{\gamma_i} \right] e_i,$$

i.e., c_k is defined by

$$c_k : \epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_k} \mapsto c_1(\epsilon_{i_1}) \wedge \cdots \wedge c_1(\epsilon_{i_k}).$$

Here, $c_0 : L_0 \to K_0$ is to be interpreted as $\epsilon_\emptyset \mapsto [1] e_\emptyset$. It is straightforward to check that c is a morphism of complexes extending the natural surjection $\mathcal{O}_{X,\zeta}/\mathcal{I} \to \mathcal{O}_{Z,\zeta}$ by using the formula

$$(z_j - [z_j]) \left(\sum_{\gamma_j=0}^{\beta_j-1} z_j^{\beta_j-\gamma_j-1} \left[z_j^{\gamma_j} \right] \right) = z_j^{\beta_j} [1] - \left[z_j^{\beta_j} \right] = z_j^{\beta_j} [1],$$

where the last equality comes from that $z_j^{\beta_j} = 0$ in $\mathcal{O}_{Z,\zeta}$.

We now fix some $i \in \{1, \ldots, m\}$, and let $\tilde{c} := (\left[z^{\alpha^i}\right] c) : (L, \psi) \to (K, \phi)$ (i.e., $\tilde{c}$ equals c composed with multiplication with $\left[z^{\alpha^i}\right]$). This is clearly a morphism of complexes, with $\tilde{c}_0(\epsilon_\emptyset) = \left[z^{\alpha^i}\right] e_\emptyset$. Thus, using the comparison formula, (2.11), for $\tilde{c}$,

$$R_p^K \left[z^{\alpha^i}\right] e_\emptyset \epsilon_\emptyset^* = \left[z^{\alpha^i}\right] c_p R_p^L.$$

Applying this to each term in the sum in (4.18), we get that

$$\operatorname{tr}(D\phi_1 \cdots D\phi_p R_p^K) = \sum e_\emptyset^* \left[z^{\alpha^i}\right]^* D\phi_1 \cdots D\phi_p R_p^K \left[z^{\alpha^i}\right] e_\emptyset \epsilon_\emptyset^* \epsilon_\emptyset =$$
$$= \sum e_\emptyset^* \left[z^{\alpha^i}\right]^* D\phi_1 \cdots D\phi_p \left[z^{\alpha^i}\right] c_p R_p^L \epsilon_\emptyset.$$

We write the map c_p as

$$c_p : \epsilon_{\{1,\ldots,p\}} \mapsto \tilde{B} \wedge e_{\{1,\ldots,p\}},$$

where

$$\tilde{B} = \sum_{\gamma \leq \beta - 1} z^{\beta - \gamma - 1} \left[z^\gamma\right].$$

Since $\left[z^{\alpha^i}\right]$ and $\tilde{B}$ commute, being elements of A, we get that

$$\operatorname{tr}(D\phi_1 \cdots D\phi_p R_p^K) = \sum e_\emptyset^* \left[z^{\alpha^i}\right]^* D\phi_1 \cdots D\phi_p \tilde{B} \left[z^{\alpha^i}\right] e_{\{1,\ldots,p\}} \epsilon_{\{1,\ldots,p\}}^* R_p^L \epsilon_\emptyset.$$

We let B be the form-valued $\mathcal{O}_{X,\zeta}$-linear map $A \to A$ given by

$$B := e_\emptyset^* D\phi_1 \cdots D\phi_p \tilde{B} e_{\{1,\ldots,p\}}.$$

Using that $e_\emptyset^*$ and $\left[z^{\alpha^i}\right]^*$ commute, and that $e_{\{1,\ldots,p\}}$ and $\left[z^{\alpha^i}\right]$ commute, we then get that

$$\operatorname{tr}(D\phi_1 \cdots D\phi_p R_p^K) = \sum \left[z^{\alpha^i}\right]^* B \left[z^{\alpha^i}\right] \epsilon_{\{1,\ldots,p\}}^* R_p^L \epsilon_\emptyset = (\operatorname{tr} B) \epsilon_{\{1,\ldots,p\}}^* R_p^L \epsilon_\emptyset.$$

Note that by (2.8) and (2.2),

$$\epsilon_{\{1,\ldots,p\}}^* R_p^L \epsilon_\emptyset = (-1)^{p^2} \bar{\partial} \frac{1}{z_p^{\beta_p}} \wedge \cdots \wedge \bar{\partial} \frac{1}{z_1^{\beta_1}}.$$

Moreover, in view of the Poincaré-Lelong formula (1.7), note that

$$(-1)^{p^2} \frac{1}{(2\pi i)^p} z^{\beta - 1} dz_1 \wedge \cdots \wedge dz_p \wedge \bar{\partial} \frac{1}{z_p^{\beta_p}} \wedge \cdots \wedge \bar{\partial} \frac{1}{z_1^{\beta_1}} = [z_1 = \cdots = z_p = 0].$$

Thus, from Lemma 4.6 below, we conclude that (4.17) holds. $\qquad \square$

Lemma 4.6. *Let B be as in the proof of Lemma 4.5. Then*

$$(4.19) \qquad \operatorname{tr} B = p! m z^{\beta - 1} dz_1 \wedge \cdots \wedge dz_p.$$

Proof. As ϕ_k is contraction with $\mathbf{z}_1 e_1 + \cdots + \mathbf{z}_p e_p$, and D is assumed to be trivial with respect to the bases $\left[z^{\alpha^i}\right] e_I$, we get in the same way as in Example 2.2 that

$$e_\emptyset^* D\phi_1 \cdots D\phi_p e_{\{1,\ldots,p\}} = p! D\mathbf{z}_1 \cdots D\mathbf{z}_p.$$

Since $\mathbf{z}_i = z_i - [z_i]$, we thus get that B is a sum of terms of the form

$$(4.20) \qquad \pm p! dz_I \wedge (D\left[z_{J_1}\right]) \cdots (D\left[z_{J_\ell}\right]) z^{\beta - \gamma - 1} \left[z^\gamma\right],$$

where $|I| + |J| = p$, and $I \cup J = \{1, \ldots, p\}$.

We claim that the traces of all such terms are zero, unless $|J| = 0$ and $\gamma = 0$. Recall from Remark 3.3 that, in the basis of A given by $\left[z^{\alpha^1}\right], \ldots, \left[z^{\alpha^m}\right]$, the matrix for multiplication with any monomial $\left[z^\delta\right]$ is upper triangular, and in addition, it will have zeros on the diagonal if and only if $\delta \neq 0$. Thus, the matrix of each $D\left[z_{J_i}\right]$ is a (form-valued) upper triangular matrix with zeros on the diagonal, since D is assumed to be trivial with respect to the bases $\left[z^{\alpha^i}\right] e_I$. Since $[z^\gamma]$ is also upper-triangular, the full product (4.20) is upper-triangular, and with zeros on the diagonal if $|J| > 0$ or $\gamma \neq 0$. Thus, the trace is zero in case $|J| > 0$ or $\gamma \neq 0$, which proves the claim.

To conclude,
$$\operatorname{tr} B = p! \, dz_1 \wedge \cdots \wedge dz_p z^{\beta-1} \operatorname{tr}[1],$$

and since $\operatorname{tr}[1] = \operatorname{rank}_{\mathcal{O}_{X,\zeta}} A = m$, we obtain (4.19). $\qquad \square$

Proof of Theorem 1.5. We let $[Z]_{[k]}$ be the part of the fundamental cycle $[Z]$ of codimension k, i.e., $[Z]_{[k]} = \sum m_i[Z_i]$, where the sum is over the irreducible components Z_i of Z_{red} of codimension k, and m_i is the geometric multiplicity of Z_i in Z. Thus,
$$[Z] = \sum_k [Z]_{[k]},$$

and it is enough to prove that

(4.21)
$$\frac{1}{(2\pi i)^k k!} D\varphi_1 \cdots D\varphi_k R_{[k]} = [Z]_{[k]},$$

for $k = \operatorname{codim} Z, \ldots, N$. Let $V_k = W_k \cap (\cup_{q \neq k} W_q)$; then V_k has codimension $\geq k+1$. Note that both sides of (4.21) have support on W_k, and that Z has pure codimension k on $W_k \backslash V_k$. Thus, (4.21) holds on $X \backslash V_k$ by Theorem 1.1. Since codim $V_k \geq k+1$ and both sides of (4.21) are pseudomeromorphic (k,k)-currents, (4.21) holds everywhere by the dimension principle, Proposition 2.1. $\qquad \square$

Remark 4.7. By analogous arguments we can prove (1.11). First
$$\frac{1}{(2\pi i)^k k!} \operatorname{tr}(D\varphi_1 \cdots D\varphi_k R_{[k]}) = \frac{1}{(2\pi i)^k k!} \tau D\varphi_1 \cdots D\varphi_k R_{[k]} \tau^{-1} = [Z]_{[k]},$$

holds on $X \setminus V_k$ by Theorem 1.2 and thus it holds everywhere by the dimension principle.

5. EXAMPLES OF HIGHER DEGREE CURRENTS

We will start by illustrating Theorem 1.5 by explicitly computing the left-hand side of (1.10) in the situation of Example 1.4.

Example 5.1. Let Z be as in Example 1.4. Then $\mathcal{O}_Z$ has a (minimal) free resolution
$$0 \to \mathcal{O}_{\mathbb{C}^3} \overset{\varphi_2}{\to} \mathcal{O}_{\mathbb{C}^3}^{\oplus 2} \overset{\varphi_1}{\to} \mathcal{O}_{\mathbb{C}^3},$$

where
$$\{\varphi_2\} = \begin{bmatrix} -y \\ x \end{bmatrix} \text{ and } \{\varphi_1\} = \begin{bmatrix} xz & yz \end{bmatrix}.$$

Let D be (induced by) the trivial connections on $E_0 = \mathcal{O}_{\mathbb{C}^3}$, $E_1 = \mathcal{O}_{\mathbb{C}^3}^{\oplus 2}$, and $E_2 = \mathcal{O}_{\mathbb{C}^3}$. In [Lä1, Example 5], the current $R^E = R_1^E + R_2^E$ was computed explicitly:

$$\{R_1^E\} = \frac{1}{|x|^2 + |y|^2} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \bar{\partial}\frac{1}{z}$$

$$\{R_2^E\} = \frac{1}{z}\bar{\partial}\frac{1}{y} \wedge \bar{\partial}\frac{1}{x} + \bar{\partial}\left(\frac{\begin{bmatrix} -\bar{y} & \bar{x} \end{bmatrix}}{|x|^2 + |y|^2}\right)\frac{1}{|x|^2 + |y|^2}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \wedge \bar{\partial}\frac{1}{z} =: \mu_1 + \mu_2.$$

Note that the irreducible components $Z_1 := \{z = 0\}$ and $Z_2 := \{x = y = 0\}$ of Z are of codimension 1 and 2, respectively; thus $R_{[k]}^E = \mathbf{1}_{Z_k} R_k^E$ for $k = 1, 2$. Since R_1^E has support on Z_1 it follows that $R_{[1]}^E = R_1^E$. Since $\operatorname{supp}\mu_2 \subseteq \{z = 0\}$, $\mathbf{1}_{Z_2}\mu_2$ has support on $Z_2 \cap \{z = 0\} = \{x = y = z = 0\}$, which has codimension 3, and thus it vanishes by the dimension principle. Since $\operatorname{supp}\mu_1 \subseteq \{x = y = 0\} = Z_2$, we get that $\mathbf{1}_{Z_2}\mu_1 = \mu_1$. Thus, to conclude,

$$\{R_{[1]}^E\} = \frac{1}{|x|^2 + |y|^2}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}\bar{\partial}\frac{1}{z} \text{ and } \{R_{[2]}^E\} = \frac{1}{z}\bar{\partial}\frac{1}{y} \wedge \bar{\partial}\frac{1}{x}.$$

By a straightforward calculation, one can then verify (1.10) in this case.

It would be interesting to consider the full currents

(5.1) $$D\varphi_1 \cdots D\varphi_k R_k^E$$

(and not only $D\varphi_1 \cdots D\varphi_k R_{[k]}^E$) and investigate whether these capture algebraic or geometric information (in addition to the fundamental cycle). If (E, φ) is the Koszul complex of a holomorphic tuple f it was shown in [ASWY] that the currents (5.1) satisfy a generalized King's formula, generalizing [A1]; in particular, the Lelong numbers are the so-called *Segre numbers* of the ideal generated by f.

We remark that in the above example we do not know how to interpret the current $D\varphi_1 D\varphi_2 R_2^E$ or rather the part $D\varphi_1 D\varphi_2 \mu_2$. Below, however, we will consider an example where $D\varphi_1 D\varphi_2 R_2^E$ is a current of integration along the (only) associated prime of codimension 2. For an ideal $\mathcal{J}$ over a local ring R, there is a notion of the *length along an associated prime* $\mathfrak{p}$, defined as the length of the largest ideal in $R_\mathfrak{p}/\mathcal{J}R_\mathfrak{p}$ of finite length, see for example [EH, Sect. II.3, p. 68]. The length of $\mathcal{J}$ along $\mathfrak{p}$ coincides with the geometric multiplicity of $\mathcal{J}(\mathfrak{p})$ in $\mathcal{J}$ if $\mathfrak{p}$ is a minimal associated prime of $\mathcal{J}$. It would be interesting to see whether these numbers could be recovered from the currents (5.1). However, in view of the example below this is not clear how to do.

Example 5.2. Let Z be as in Example 1.3. Then

$$0 \to \mathcal{O}_{\mathbb{C}^2} \xrightarrow{\varphi_2} \mathcal{O}_{\mathbb{C}^2}^{\oplus 2} \xrightarrow{\varphi_1} \mathcal{O}_{\mathbb{C}^2} \to \mathcal{O}_Z,$$

where

$$\{\varphi_2\} = \begin{bmatrix} -x^\ell \\ y^{k-m} \end{bmatrix} \text{ and } \{\varphi_1\} = \begin{bmatrix} y^k & x^\ell y^m \end{bmatrix},$$

is a free resolution of $\mathcal{O}_Z$. Note that, since Z_{red} only has one irreducible component $\{y = 0\}$ of codimension 1, $R_{[2]}^E = 0$.

Let D be (induced by) the trivial connections on $E_0 = \mathcal{O}_{\mathbb{C}^2}$, $E_1 = \mathcal{O}_{\mathbb{C}^2}^{\oplus 2}$, and $E_2 = \mathcal{O}_{\mathbb{C}^2}$. Then a direct computation yields

$$\{D\varphi_1 D\varphi_2\} = -\ell(2k - m)x^{\ell-1}y^{k-1}dx \wedge dy =: -Cx^{\ell-1}y^{k-1}dx \wedge dy,$$

where, as above, we have used the notation from Section 2.5. Next, let (F, ψ) be the Koszul complex of (y, x) and let $a_0 : F_0 \to E_0$ be given by $\{a_0\} = \left[\begin{array}{c} x^{\ell-1}y^{k-1} \end{array} \right]$. Then $\{R_2^F\} = \bar{\partial}(1/x) \wedge \bar{\partial}(1/y)$ and a_0 can be extended to a morphism of complexes $a : (F, \psi) \to (E, \varphi)$, where

$$\{a_2\} = \left[\begin{array}{c} 1 \end{array} \right] \text{ and } \{a_1\} = \left[\begin{array}{cc} x^{\ell-1} & 0 \\ 0 & y^{k-m-1} \end{array} \right].$$

If we apply the comparison formula, (2.9), and identify the components that takes values in $\mathrm{Hom}\,(F_0, E_2)$ we get that

$$R_2^E a_0 - a_2 R_2^F = \varphi_3 M_3 - \bar{\partial}M_2.$$

Note that $M_3 = 0$ since (E, φ) has length 2. Moreover, since $Z_2^E = Z_1^F = \{x = y = 0\}$ has codimension ≥ 2, $M_2 = 0$ by [Lä2, Proposition 3.5]. Hence $R_2^E a_0 = a_2 R_2^F$. Thus, we get that

$$\{D\varphi_1 D\varphi_2 R_2^E\} = -Cx^{\ell-1}y^{k-1}dx \wedge dy\{R_2^E\} = -Cdx \wedge dy\{R_2^E a_0\} =$$
$$- Cdx \wedge dy\{a_2 R_2^F\} = -Cdx \wedge dy \wedge \bar{\partial}\frac{1}{x} \wedge \bar{\partial}\frac{1}{y} = (2\pi i)^2 C[0],$$

cf. (2.12).

We conclude that

$$(5.2) \qquad\qquad D\varphi_1 D\varphi_2 R_2^E = (2\pi i)^2 \ell(2k - m)[0],$$

i.e., $D\varphi_1 D\varphi_2 R_2^E$ is the current of integration along the (only) associated prime $\mathfrak{m}_{\mathbb{C}^2,0} = \mathcal{J}(x, y)$ of $\mathcal{J}$ with mass $(2\pi i)^2 \ell(2k - m)$. However, a computation yields that the length of $\mathcal{J}_0$ along $\mathfrak{m}_{\mathbb{C}^2,0}$ equals $\ell(k - m)$; it it not clear to us how to relate these numbers.

6. Relation to the results of Lejeune-Jalabert

Our results are closely related to results by Lejeune-Jalabert, [LJ1, LJ2], and we will in this section compare our results with hers.

Throughout this section, we let Z be a (not necessarily reduced) analytic space of pure dimension n. Assume that Z is a subspace of codimension p of the complex manifold X of dimension $N = n+p$, and let Z be defined by the ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$.

6.1. The Grothendieck dualizing sheaf and residue currents. If Z is Cohen-Macaulay, then the *Grothendieck dualizing sheaf* ω_Z is

$$\omega_Z := \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega_X^N),$$

where Ω_X^N is the sheaf of holomorphic N-forms on X. If Z is smooth, then ω_Z coincides with Ω_Z.

One way of realizing ω_Z is as $H^p\big(\mathcal{H}om\,(E_\bullet, \Omega_X^N)\big)$, where (E, φ) is a locally free resolution of $\mathcal{O}_Z$, and another is as $H^p\big(\mathcal{H}om\,(\mathcal{O}_Z, \mathcal{C}^{N,\bullet})\big)$, where $(\mathcal{C}^{N,\bullet}, \bar{\partial})$ is the Dolbeault complex of $(N, \bullet)$-currents on X. There is a canonical isomorphism between these representations of ω_Z,

$$(6.1) \qquad\qquad \mathrm{res} : H^p\big(\mathcal{H}om\,(E_\bullet, \Omega_X^N)\big) \overset{\cong}{\to} H^p\big(\mathcal{H}om\,(\mathcal{O}_Z, \mathcal{C}^{N,\bullet})\big),$$

and by [A3, Theorem 1.5 and Example 1], this isomorphism can be realized concretely by the residue current[6] R_p^E:

$$\text{(6.2)} \qquad \text{res} : [\xi] \mapsto \left[\frac{1}{(2\pi i)^p} \xi R_p^E \tau^{-1} \right],$$

where τ is the natural surjection $\tau : E_0 \to \operatorname{coker} \varphi_1 \cong \mathcal{O}_Z$ and we consider $\xi R_p^E \tau^{-1}$ as a scalar current in a similar way as in the introduction.

6.2. **Coleff-Herrera currents.** A (q, p)-current μ on X is a *Coleff-Herrera current* on Z_{red}, denoted $\mu \in \mathcal{CH}^q_{Z_{\text{red}}}$, if $\bar{\partial}\mu = 0$, $\overline{\psi}\mu = 0$ for all holomorphic functions ψ vanishing on Z_{red}, and μ has the *SEP with respect to Z_{red}*, i.e., for any hypersurface V of Z_{red}, the limit $\mathbf{1}_V \mu := \lim_{\epsilon \to 0}(1 - \chi(|f|/\epsilon))\mu$ exists and $\mathbf{1}_V \mu = 0$, where f is a tuple of holomorphic functions defining V. This description of Coleff-Herrera currents is due to Björk, see [B1, Chapter 3], and [B2, Section 6.2].

Let $\mathcal{G}$ be a coherent sheaf of codimension p, with a locally free resolution (E, φ) of length p (so that in particular, $\mathcal{G}$ is Cohen-Macaulay). Then R_p^E is a $\operatorname{Hom}(E_0, E_p)$-valued Coleff-Herrera current on $V := \operatorname{supp} \mathcal{G}$. To see this, note first that, by the ∇-closedness of R^E and the fact that E has length p, $\bar{\partial} R_p^E = \varphi_{p+1} R_{p+1}^E = 0$. The fact that R_p^E has the SEP follows from the dimension principle, Proposition 2.1. Moreover that $\overline{\psi} R_p^E = 0$ for any holomorphic function ψ vanishing on V follows from the fact that R_p^E is a pseudomeromorphic current with support on V, see [AW2, Proposition 2.3].

We let $(\mathcal{C}^{N,\bullet}_{[Z_{\text{red}}]}, \bar{\partial})$ denote the Dolbeault complex of $(N, \bullet)$-currents on X with support on Z_{red}. It was proven in [DS1] (for Z_{red} a complete intersection) and [DS2, Proposition 5.2] (for Z_{red} arbitrary of pure dimension) that Coleff-Herrera currents are canonical representatives in moderate cohomology in the sense that

$$\left(\ker \bar{\partial} : \mathcal{C}^{N,p}_{[Z_{\text{red}}]} \to \mathcal{C}^{N,p+1}_{[Z_{\text{red}}]} \right) \cong \mathcal{CH}^N_{Z_{\text{red}}} \oplus \bar{\partial}\mathcal{C}^{N,p-1}_{[Z_{\text{red}}]},$$

i.e., each cohomology class in $H^p\big(\mathcal{C}^{N,\bullet}_{[Z_{\text{red}}]}\big)$ has a unique representative which is a Coleff-Herrera current. In particular,

$$\text{(6.3)} \qquad \mathcal{CH}^N_{Z_{\text{red}}} \cap \left(\operatorname{im} \bar{\partial} : \mathcal{C}^{N,p-1}_{[Z_{\text{red}}]} \to \mathcal{C}^{N,p}_{[Z_{\text{red}}]} \right) = \{0\}.$$

6.3. **Relation to the results in** [LJ1]**.** In this section, we discuss how the results of Lejeune-Jalabert give our results and vice versa. The main point is to describe how the result of [LJ1] give the following special case of Theorem 1.2.

Theorem 6.1. *Let $Z \subset X$ be an analytic space of pure codimension p which is Cohen-Macaulay. Assume that $\mathcal{O}_Z$ has a locally free resolution (E, φ) over $\mathcal{O}_X$ of length p, and let D be the connection on $\operatorname{End}E$ induced by connections on $E_0, \dots, E_p$. Then,*

$$\text{(6.4)} \qquad \frac{1}{(2\pi i)^p p!} \operatorname{tr}(D\varphi_1 \cdots D\varphi_p R_p^E) = [Z],$$

and

$$\text{(6.5)} \qquad \frac{1}{(2\pi i)^p p!} \tau D\varphi_1 \cdots D\varphi_p R_p^E \tau^{-1} = [Z],$$

where τ is the natural surjection $\tau : E_0 \to \operatorname{coker} \varphi_1 \cong \mathcal{O}_Z$.

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[6]We have introduced the factor $1/(2\pi i)^p$ for normalization reasons.

In order to prove Theorem 1.2 in full generality, without assuming that Z is Cohen-Macaulay or that (E, φ) has length p, one can then argue in the same way as in our proof of Theorem 1.2, but using Theorem 6.1 instead of Lemma 4.5. Indeed, first of all, by (4.15), it is sufficient to prove just (1.8). By combining Lemma 4.1 and Theorem 6.1, we first obtain (1.8) in a neighbourhood of each Cohen-Macaulay point. By the dimension principle, (1.8) then holds on all of X.

In [LJ1], the fundamental class of Z is considered as a map $c_Z : \Omega_Z^n \to \omega_Z$, where Ω_Z^n is the sheaf of holomorphic n-forms on Z. If α is a section of Ω_Z^n and $\tilde{\alpha}$ is a section of Ω_X^n, which is a representative of α, then $\gamma := \tilde{\alpha} \wedge \tau D\varphi_1 \cdots D\varphi_p$ is a section of $\mathcal{H}om\,(E_p, \Omega_X^N \otimes \mathcal{O}_Z)$. Since (E, φ) has length p, γ induces a section $[\gamma]$ of $\mathcal{E}xt^p(\mathcal{O}_Z, \Omega_X^N \otimes \mathcal{O}_Z)$. We now consider the isomorphism

$$(6.6) \qquad \omega_Z = \mathcal{E}xt^p(\mathcal{O}_Z, \Omega_X^N) \cong \mathcal{E}xt^p(\mathcal{O}_Z, \Omega_X^N \otimes \mathcal{O}_Z)$$

induced by the surjection $\Omega_X^N \to \Omega_X^N \otimes \mathcal{O}_Z$, see [ALJ, Proposition 4.6]. Since E_p is locally free, γ can locally be lifted to sections γ_i of $\mathcal{H}om\,(E_p, \Omega_X^N)$. Since (E, φ) has length p, these local liftings of γ define sections $[\gamma_i]$ of ω_Z locally. On overlaps, the γ_i's differ by sections of $\mathcal{H}om\,(E_p, \Omega_X^N) \otimes \mathcal{J}$, and since $\mathcal{J}\omega_Z = 0$, the sections $[\gamma_i]$ patch together to a global section of ω_Z, which we denote by $[\tilde{\alpha} \wedge \tau D\varphi_1 \cdots D\varphi_p]$. By construction, $[\tilde{\alpha} \wedge \tau D\varphi_1 \cdots D\varphi_p]$ maps to $[\gamma]$ using the isomorphism (6.6). The main theorem in [LJ1] asserts that this gives the fundamental class of α (times $p!$), i.e.,

$$(6.7) \qquad c_Z(\alpha) = \frac{1}{p!}[\tilde{\alpha} \wedge \tau D\varphi_1 \cdots D\varphi_p].$$

Note that where the local lifting γ_i of $[\tilde{\alpha} \wedge \tau D\varphi_1 \cdots D\varphi_p]$ is defined, $\gamma_i R_p^E$ coincides with $\gamma R_p^E = \tilde{\alpha} \wedge \tau D\varphi_1 \cdots D\varphi_p R_p^E$ (if we consider the currents as scalar currents). Thus combining (6.7) with the realization (6.2) of the isomorphism (6.1), we get that

$$(6.8) \qquad \operatorname{res} c_Z(\alpha) = \frac{1}{(2\pi i)^p p!} \tilde{\alpha} \wedge \tau D\varphi_1 \cdots D\varphi_p R_p^E \tau^{-1} + \bar{\partial}\mathcal{H}om\,(\mathcal{O}_Z, \mathcal{C}^{N,p-1}).$$

It is not entirely clear to us how the fundamental class is defined in [LJ1], but it is reasonable to assume that if one uses the isomorphism (6.1) to represent $c_Z(\alpha)$ as a current, then one should have

$$(6.9) \qquad \operatorname{res} c_Z(\alpha) = \tilde{\alpha} \wedge [Z],$$

where we by $[Z]$ mean the fundamental cycle (seen as a current on X) as defined in (1.3). Since we have an independent proof of Theorem 6.1 this assumption must indeed be correct, cf. the last paragraph below. Note that since the right-hand side of (6.9) is a pseudomeromorphic (p, p)-current, by the dimension principle, it is uniquely determined by its restriction to Z_{reg}, and hence, it is independent of the precise definition of Ω_Z^n as long as the forms in Ω_Z^n coincide with regular holomorphic n-forms on Z_{reg} and can be lifted to holomorphic n-forms on X.

If we assume (6.9), then (6.8) implies that

$$\mu := \tilde{\alpha} \wedge [Z] - \frac{1}{(2\pi i)^p p!} \tilde{\alpha} \wedge \tau D\varphi_1 \cdots D\varphi_p R_p^E \tau^{-1} \in \left(\operatorname{im} \bar{\partial} : \mathcal{C}_{[Z_{\text{red}}]}^{N,p-1} \to \mathcal{C}_{[Z_{\text{red}}]}^{N,p} \right).$$

By Lemma 4.2, $\tau D\varphi_1 \cdots D\varphi_p R_p^E \tau^{-1}$ is independent of the connection D, and we can thus assume that D is the trivial connection d in a trivialization of E. Then $D\varphi_1 \cdots D\varphi_p$ is a holomorphic $\operatorname{Hom}\,(E_p, E_0)$-valued morphism, and thus, since R_p^E is a $\operatorname{Hom}\,(E_0, E_p)$-valued Coleff-Herrera current, $\tau D\varphi_1 \cdots D\varphi_p R_p^E \tau^{-1} \in \mathcal{CH}_{Z_{\text{red}}}^p$. Hence, $\mu \in \mathcal{CH}_{Z_{\text{red}}}^N$, so by (6.3), $\mu = 0$. Since $\mu = 0$ for any choice of the holomorphic p-form

$\tilde{\alpha}$ on X, we get that (6.5) holds. Finally, using (4.15), we get that (6.4) holds. To conclude, assuming (6.9), Theorem 6.1 follows from the theorem in [LJ1].

On the other hand, Theorem 6.1 together with (6.9) implies (6.8), which in turn implies (6.7) since (6.2) is an isomorphism. Thus, Lejeune-Jalabert's result follows from Theorem 6.1 and (6.9). Finally, taking Theorem 6.1 and Lejeune-Jalabert's result for granted, it follows that (6.9) must be a correct assumption.

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