

COMPUTING RESIDUE CURRENTS OF MONOMIAL IDEALS USING COMPARISON FORMULAS

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ABSTRACT. Given a free resolution of an ideal $\mathfrak{a}$ of holomorphic functions, one can construct a vector-valued residue current R , which coincides with the classical Coleff-Herrera product if $\mathfrak{a}$ is a complete intersection ideal and whose annihilator ideal is precisely $\mathfrak{a}$.

We give a complete description of R in the case when $\mathfrak{a}$ is an Artinian monomial ideal and the resolution is the hull resolution (or a more general cellular resolution). The main ingredient in the proof is a comparison formula for residue currents due to the first author.

By means of this description, we obtain in the monomial case a current version of a factorization of the fundamental cycle of $\mathfrak{a}$ due to Lejeune-Jalabert.

1. INTRODUCTION

With a regular sequence $f_1, \dots, f_p$ of holomorphic functions at the origin in $\mathbf{C}^n$, there is a canonical associated residue current, the *Coleff-Herrera product* $R_{CH}^f = \bar{\partial}[1/f_p] \wedge \dots \wedge \bar{\partial}[1/f_1]$, introduced in [10]. It has support on $\{f_1 = \dots = f_p = 0\}$ and satisfies the *duality principle* ([11, 20]): *A holomorphic function ξ is locally in the ideal (f) generated by $f_1, \dots, f_p$ if and only if ξ annihilates R_{CH}^f , i.e., $\xi R_{CH}^f = 0$.* Given a free resolution of an ideal (sheaf) $\mathfrak{a}$ of holomorphic functions, Andersson and the second author constructed in [5] a vector-valued residue current R that satisfies the duality principle and that coincides with R_{CH}^f if $\mathfrak{a}$ is a complete intersection ideal, generated by a regular sequence $f_1, \dots, f_p$, see Section 2. This construction has recently been used, e.g., to obtain new results for the $\bar{\partial}$ -equation and effective solutions to polynomial ideal membership problems on singular varieties, see, e.g., [2, 3, 4, 7, 24].

In this paper we compute the current R for the *hull resolution* (and more general cellular resolutions), introduced by Bayer-Sturmfels [8], of Artinian, i.e., 0-dimensional, monomial ideals, extending previous results by the second author. The hull resolution of a monomial ideal M is encoded in the *hull complex* $\text{hull}(M)$, which is a labeled polyhedral cell complex in $\mathbf{R}^n$ of dimension $n - 1$ with one vertex for each

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minimal generator of M . The face $\sigma \in \text{hull}(M)$ is labeled by the least common multiple of the monomials corresponding to the vertices of σ , see Section 4.

Theorem 1.1. *Let M be an Artinian monomial ideal in $\mathbf{C}^n$ and let R be the residue current constructed from the hull resolution of M . Then R has one entry R_σ for each $(n-1)$ -dimensional face σ of $\text{hull}(M)$, and*

$$R_\sigma = \bar{\partial} \left[\frac{1}{z_n^{\alpha_n}} \right] \wedge \cdots \wedge \bar{\partial} \left[\frac{1}{z_1^{\alpha_1}} \right],$$

where $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is the label of σ .

If M is a complete intersection ideal, $\text{hull}(M)$ is an $(n-1)$ -simplex and the hull resolution is the Koszul complex. In general, $\text{hull}(M)$ is a polyhedral subdivision of an $(n-1)$ -simplex. In fact, Theorem 1.1 holds for more general cellular resolutions, where the underlying polyhedral cell complex is a polyhedral subdivision of the $(n-1)$ -simplex, see Theorem 5.1.

It was proved in [10] that if $f_1, \dots, f_p$ is a regular sequence, then

$$R_{\text{CH}}^f \wedge \frac{df_1 \wedge \cdots \wedge df_p}{(2\pi i)^p} = [(f)], \quad (1.1)$$

where $[(f)]$ is the fundamental cycle of the ideal (f) . Our main motivation to compute R explicitly was to understand a similar factorization of the fundamental cycle of an arbitrary ideal. By computing $d\varphi := d\varphi_0 \circ \cdots \circ d\varphi_{n-1}$, where φ_k are the maps in the (hull) resolution of a (generic) Artinian monomial ideal $\mathfrak{a}$, and using Theorem 1.1, we get

$$\frac{d\varphi}{n!(2\pi i)^n} \circ R = [\mathfrak{a}], \quad (1.2)$$

see Section 7. Since $\mathfrak{a}$ is Artinian, $[\mathfrak{a}] = m[0]$, where m is the *geometric multiplicity* $\dim_{\mathbf{C}} \mathcal{O}_0^n / \mathfrak{a}$ of $\mathfrak{a}$, see [14, Section 1.5]. Moreover, since $\mathfrak{a}$ is monomial, m equals the volume of the *staircase* $\mathbf{R}_+^n \setminus \bigcup_{z^\alpha \in \mathfrak{a}} \{\alpha + \mathbf{R}_+^n\}$ of $\mathfrak{a}$. If $\mathfrak{a}$ is a complete intersection ideal generated by $f_1, \dots, f_n$, then $d\varphi = n!df_1 \wedge \cdots \wedge df_n$, and thus (1.2) can be seen as a generalization of (1.1). We recently managed to prove a generalized version of (1.2) for arbitrary ideals of pure dimension; this is a current version of (a generalization of) a result due to Lejeune-Jalabert [17] and will be the subject of the forthcoming paper [16].

In [27] the current R was computed as the push-forward of a certain current in a toric resolution of the ideal M . The main result in that paper asserts that each R_σ is of the form $R_\sigma = c_\sigma \bar{\partial}[1/z_n^{\alpha_n}] \wedge \cdots \wedge \bar{\partial}[1/z_1^{\alpha_1}]$ for some $c_\sigma \in \mathbf{C}$. The coefficients c_σ appear as integrals that seem to be hard to compute in general, see Section 6. The proof of Theorem 1.1 given here is different and more direct. A key tool is a comparison

formula for residue currents due to the first author. If

$$0 \rightarrow \mathcal{O}(E_{n-1}) \xrightarrow{\varphi_{n-1}} \cdots \xrightarrow{\varphi_1} \mathcal{O}(E_0) \xrightarrow{\varphi_0} \mathcal{O}(E_{-1})$$

is a resolution of an Artinian ideal $\mathfrak{a}$ and $\cdots \rightarrow \mathcal{O}(F_k) \rightarrow \mathcal{O}(F_{k-1}) \rightarrow \cdots$ is a resolution of $\mathfrak{b} \subset \mathfrak{a}$, then there are (locally) maps $a_k : \mathcal{O}(F_k) \rightarrow \mathcal{O}(E_k)$, so that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(E_{n-1}) & \xrightarrow{\varphi_{n-1}} & \cdots & \xrightarrow{\varphi_1} & \mathcal{O}(E_0) \xrightarrow{\varphi_0} \mathcal{O}(E_{-1}) \\ & & \uparrow a_{n-1} & & & & \uparrow a_0 \\ 0 & \longrightarrow & \mathcal{O}(F_{n-1}) & \xrightarrow{\psi_{n-1}} & \cdots & \xrightarrow{\psi_1} & \mathcal{O}(F_0) \xrightarrow{\psi_0} \mathcal{O}(F_{-1}) \\ & & & & & & \uparrow a_{-1} \end{array}$$

commutes. Theorem 1.3 in [15] asserts that $R^E a_{-1} = a_{n-1} R^F$ if R^E and R^F are the currents associated with $\mathcal{O}(E_\bullet)$ and $\mathcal{O}(F_\bullet)$, respectively, see Section 2.1.

The main ingredient in the proof of Theorem 1.1 is Proposition 5.2, which gives an explicit description of mappings a_k when $\mathcal{O}(E_\bullet)$ and $\mathcal{O}(F_\bullet)$ are cellular resolutions such that the underlying polyhedral cell complex of $\mathcal{O}(E_\bullet)$ refines the polyhedral cell complex of $\mathcal{O}(F_\bullet)$, and which we have not managed to find in the literature. Letting $\mathcal{O}(E_\bullet)$ be the hull resolution of M and $\mathcal{O}(F_\bullet)$ the Koszul complex of a sequence $z_1^{b_1}, \dots, z_n^{b_n}$ contained in M , so that R^F is the simple Coleff-Herrera product $\bar{\partial}[1/z_n^{b_n}] \wedge \cdots \wedge \bar{\partial}[1/z_1^{b_1}]$, we can then easily compute R^E .

The paper is organized as follows. In Sections 2 and 4 we provide some background on residue currents and cellular resolutions, respectively. In Section 3 we prove some basic results concerning oriented polyhedral complexes, which are needed for the proof of Theorem 1.1 (and the slightly more general Theorem 5.1). The proof occupies Section 5. In Section 6 we compare Theorems 1.1 and 5.1 to previous results and also illustrate them by some examples. In Section 6.1 we consider residue currents of non-Artinian monomial ideals, and, finally, in Section 7 we discuss the relation to fundamental cycles.

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2. RESIDUE CURRENTS

Given a holomorphic function f we will write $[1/f]$ (or sometimes just $1/f$) for the *principal value distribution* of $1/f$, which can be realized, e.g., as the limit of the smooth approximands $\frac{\bar{f}}{|f|^2 + \epsilon}$. If f is a regular sequence of (germs of) holomorphic functions $f_1, \dots, f_p$ one can give meaning to products of principal values $[1/f_j]$ and *residue currents* $\bar{\partial}[1/f_k]$, as was first done in [10], see also [21]. The products can be defined, e.g., by taking the limit of products of the corresponding forms

$\frac{\bar{f}_j}{|f_j|^{2+\epsilon}}$ and $\bar{\partial}\frac{\bar{f}_k}{|f_k|^{2+\epsilon}}$. They are (anti-)commutative in the factors and satisfy Leibniz' rule: If $f_k = g_1 \cdots g_s$, then

$$\bar{\partial} \left[\frac{1}{f_k} \right] \wedge \cdots \wedge \bar{\partial} \left[\frac{1}{f_1} \right] = \sum_j \left[\frac{1}{g_1 \cdots \hat{g}_j \cdots g_s} \right] \bar{\partial} \left[\frac{1}{g_j} \right] \wedge \bar{\partial} \left[\frac{1}{f_{k-1}} \right] \wedge \cdots \wedge \bar{\partial} \left[\frac{1}{f_1} \right]. \quad (2.1)$$

We will denote the *Coleff-Herrera product* $\bar{\partial}[1/f_p] \wedge \cdots \wedge \bar{\partial}[1/f_1]$ of f by R_{CH}^f . If $f_j = z_j^{b_j}$ for $j = 1, \dots, n$, then the action of R_{CH}^f on the test form $\xi(z) dz_1 \wedge \cdots \wedge dz_n$ equals

$$\frac{(2\pi i)^n}{(b_1 - 1)! \cdots (b_n - 1)!} \frac{\partial^{b_1 + \cdots + b_n - n}}{\partial z_1^{b_1 - 1} \cdots \partial z_n^{b_n - 1}} \xi(0).$$

Consider a complex of Hermitian holomorphic vector bundles over a complex manifold X of dimension n ,

$$0 \rightarrow E_N \xrightarrow{\varphi_N} \cdots \xrightarrow{\varphi_2} E_1 \xrightarrow{\varphi_1} E_0 \xrightarrow{\varphi_0} E_{-1}, \quad (2.2)$$

that is exact outside an analytic variety $Z \subset X$ of positive codimension p . Suppose that the rank of E_{-1} is 1. In [5] Andersson and the second author constructed an $\text{End}(\bigoplus E_k)$ -valued current $R = R^E$ that in a certain sense measures the lack of exactness of the associated sheaf complex of holomorphic sections

$$0 \rightarrow \mathcal{O}(E_N) \xrightarrow{\varphi_N} \cdots \xrightarrow{\varphi_1} \mathcal{O}(E_0) \xrightarrow{\varphi_0} \mathcal{O}(E_{-1}). \quad (2.3)$$

The current R has support on Z and if $\xi \in \mathcal{O}(E_{-1})$ satisfies $R\xi = 0$ then $\xi \in \text{Im } \varphi_0$. If (2.3) is exact, i.e., if it is a locally free resolution of the sheaf $\mathcal{O}(E_{-1})/\text{Im } \varphi_0$, then $R\xi = 0$ if and only if $\xi \in \text{Im } \varphi_0$. The grading in (2.2) is somewhat unorthodox; in [5] the complex ends at E_0 . In this paper the grading is shifted by one step, in order to make it fit the grading of the hull complex better.

Let R_k^ℓ denote the component of R that takes values in $\text{Hom}(E_{\ell-1}, E_{k-1})$ and let $R^\ell = \sum_k R_k^\ell$. The shifting of the indices here is motivated by the shifting of the grading of (2.2) compared to [5]. If (2.3) is exact, then $R^\ell = 0$ for $\ell \geq 1$. We then write $R_k = R_k^0$ without any risk of confusion. The current R_k has bidegree $(0, k)$, and thus, by the dimension principle for residue currents (see [6], Corollary 2.4), $R_k = 0$ for $k < p$, and for degree reasons, $R_k = 0$ for $k > n$. In particular, if (2.3) is a resolution of length p of a *Cohen-Macaulay* ideal sheaf, i.e., at each $x \in X$, there is a resolution of length p (so that (2.3) ends at level $p - 1$), then $R = R_p$. In this case, R is independent of the Hermitian metrics on the bundles E_k . By Hilbert's syzygy theorem, each 0-dimensional ideal sheaf is Cohen-Macaulay.

The degree of explicitness of the current R of course depends on the degree of explicitness of the complex (2.2). In general it is hard to find explicit free resolutions. In Section 4 we will describe a method for constructing free resolutions of monomial ideals due to Bayer-Sturmfels [8].

Example 2.1. Let f be a sequence of holomorphic functions $f_1, \dots, f_p$ in a domain Ω in $\mathbf{C}^n$, and let (2.2) be the Koszul complex of f : Identify f with a section $f = \sum f_j e_j$ of a trivial vector bundle $\tilde{E}$ of rank p over Ω with frame e_j . Let E_{k-1} be the k th exterior product $\Lambda^k \tilde{E}^*$ of the dual bundle $\tilde{E}^*$, equipped with the trivial metric, and let φ_{k-1} be contraction δ_f with f , i.e.,

$$\delta_f : e_{i_1}^* \wedge \dots \wedge e_{i_k}^* \mapsto \sum_j (-1)^{j-1} f_{i_j} e_{i_1}^* \wedge \dots \wedge e_{i_{j-1}}^* \wedge e_{i_{j+1}}^* \wedge \dots \wedge e_{i_k}^*,$$

where e_j^* is the dual frame to e_j . Then the entries of R^E are the *Bochner-Martinelli residue currents of f* in the sense of Passare-Tsikh-Yger [22], see [1]. If f defines a complete intersection ideal $\mathfrak{a}$, then the Koszul complex of f is a resolution of $\mathfrak{a}$ and the current $R^E = R_p^E$ then equals the Coleff-Herrera product R_{CH}^f (times $e_1^* \wedge \dots \wedge e_p^*$), see [22, Theorem 4.1] or [1, Theorem 1.7]. The currents R^E can thus be seen as generalizations of the Coleff-Herrera products and the fact that $R^E \xi = 0$ if and only if $\xi \in \text{Im } \varphi_0$ when (2.3) is exact can be seen as an extension of the duality principle for Coleff-Herrera products. $\square$

2.1. A comparison formula for residue currents. Assume that $E_\bullet, \varphi_\bullet$ and $F_\bullet, \psi_\bullet$ are Hermitian complexes of vector bundles and that there are holomorphic mappings $a_k : \mathcal{O}(F_k) \rightarrow \mathcal{O}(E_k)$ so that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(E_N) & \xrightarrow{\varphi_N} & \dots & \xrightarrow{\varphi_1} & \mathcal{O}(E_0) \xrightarrow{\varphi_0} \mathcal{O}(E_{-1}) \\ & & \uparrow a_N & & & & \uparrow a_0 & & \uparrow a_{-1} \\ 0 & \longrightarrow & \mathcal{O}(F_N) & \xrightarrow{\psi_N} & \dots & \xrightarrow{\psi_1} & \mathcal{O}(F_0) \xrightarrow{\psi_0} \mathcal{O}(F_{-1}) \end{array} \quad (2.4)$$

commutes. For example, if the sheaf complex (2.3) is exact and $\text{Im } \psi_0 \subset \text{Im } \varphi_0$ one can always find maps $a_k : \mathcal{O}_x(F_k) \rightarrow \mathcal{O}_x(E_k)$ for each $x \in X$, so that the corresponding diagram commutes, see [12, Proposition A3.13].

In [15] the residue currents associated with $E_\bullet, \varphi_\bullet$ and $F_\bullet, \psi_\bullet$ are related in terms of the morphisms a_k . Assume that $\mathcal{O}(E_\bullet), \varphi_\bullet$ and $\mathcal{O}(F_\bullet), \psi_\bullet$ are locally free resolutions of minimal length of $\mathcal{O}(E_{-1})/\mathfrak{a}$ and $\mathcal{O}(F_{-1})/\mathfrak{b}$, respectively, where $\mathfrak{a}$ and $\mathfrak{b}$ are Cohen-Macaulay ideals of codimension p . Then Theorem 1.3 in [15] asserts that

$$R^E a_{-1} = a_{p-1} R^F. \quad (2.5)$$

We will apply (2.5) to the situation where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathcal{O}(E_{-1}) = \mathcal{O}(F_{-1})$ such that $\mathfrak{b} \subset \mathfrak{a}$ (and a_{-1} is the isomorphism $\mathcal{O}(F_{-1}) \cong \mathcal{O}(E_{-1})$).

If $E_\bullet, \varphi_\bullet$ and $F_\bullet, \psi_\bullet$ are Koszul complexes of regular sequences $f_1, \dots, f_p$ and $g_1, \dots, g_p$, respectively, such that $[g_p \dots g_1]^T = A[f_p \dots f_1]^T$ for

some holomorphic matrix A , then (2.5) is just the *transformation law* for Coleff-Herrera products:

$$R_{CH}^f = \det(A) R_{CH}^g, \quad (2.6)$$

see [15, Remark 2].

3. ORIENTED POLYHEDRAL CELL COMPLEXES

Recall that a *face* of a polytope σ is the intersection of σ and a supporting hyperplane of σ . A *polyhedral cell complex* X is a finite collection of convex polytopes in $\mathbf{R}^n$ for some n , the *faces* of X , that satisfy that if $\sigma \in X$ and τ is a face of σ , then $\tau \in X$, and moreover if σ and σ' are in X , then $\sigma \cap \sigma'$ is a face of both σ and σ' . For a reference on polytopes and polyhedral cell complexes, see, e.g., [30]. The dimension of a face σ , $\dim \sigma$, is defined as the dimension of its affine hull (in $\mathbf{R}^n$) and the dimension of X , $\dim X$, is defined as $\max_{\sigma \in X} \dim \sigma$. Let X_k denote the set of faces of X of dimension k ; X_{-1} should be interpreted as $\{\emptyset\}$. If $\dim \sigma = k$, then a face of σ of dimension $k - 1$ is said to be a *facet* of σ . Faces of dimension 0 are called *vertices* and faces of dimension 1 are called *edges*. A face σ is a *simplex* if the number of vertices is equal to $\dim \sigma + 1$. A polyhedral cell complex $X' \subset X$ is said to be a *subcomplex* of X .

We will write $|X|$ for the union of all faces in X . A *polyhedral subdivision* of a polytope $\sigma \subset \mathbf{R}^n$ is a polyhedral cell complex X , such that $|X| = \sigma$. If Y is a polyhedral cell complex such that $|X| = |Y|$ and each face in Y is a union of faces in X ; we say that X *refines* Y .

The following lemma can be proved by standard arguments, cf., e.g., [30]. Note that the assumption that $|X|$ is convex is crucial. For example, the lemma fails to hold if X consists of three edges meeting at a single vertex.

Lemma 3.1. *Let X be a polyhedral cell complex of dimension $k \geq 1$, such that $|X|$ is a convex polytope. Consider $\tau \in X_{k-1}$. If τ is contained in the boundary of $|X|$, there is a unique $\sigma \in X_k$ such that τ is a facet of σ . Otherwise there are precisely two faces $\sigma_1, \sigma_2 \in X_k$ such that τ is a facet of σ_1 and σ_2 .*

3.1. Orientation. For a convex set $S \subset \mathbf{R}^n$ we let $\text{span } S$ be the underlying vector space of the affine hull of S . In other words, $\text{span } S$ is the subspace of $\mathbf{R}^n$ generated by vectors of the form $\rho_1 - \rho_2$, where $\rho_1, \rho_2 \in S$. By an *oriented polytope* in $\mathbf{R}^n$ we will mean a polytope $\sigma \subset \mathbf{R}^n$ together with an orientation of the subspace $\text{span } \sigma$. Within this section will write $\underline{\sigma}$ for the polytope and reserve σ for the oriented polytope. Recall that an orientation of $\text{span } \underline{\sigma}$ is determined by a linear form, which we denote by ω_σ , on $\Lambda^k(\text{span } \underline{\sigma})$ if $\dim \underline{\sigma} = k \geq 1$; a basis $w_1, \dots, w_k$ of $\text{span } \underline{\sigma}$ is positively oriented if and only if $\omega_\sigma(w_1 \wedge \dots \wedge$

$w_k) > 0$. There is only one way of orienting polytopes of dimension 0 as well as the empty set.

Remark 3.2. An oriented simplex can equivalently be seen as a simplex together with an equivalence class of the total ordering of the vertices, where two orderings are equivalent if and only if they differ by an even permutation. We write $[v_1, \dots, v_{k+1}]$ for the simplex with vertices $v_1, \dots, v_{k+1}$ together with the equivalence class of the ordering $v_1 < \dots < v_{k+1}$, and $-[v_1, \dots, v_{k+1}]$ for the simplex with the opposite orientation, cf. for instance, [23, Chap. 4]. If $\underline{\sigma}$ is a simplex with vertices $v_1, \dots, v_{k+1}$, we identify $\sigma = [v_1, \dots, v_{k+1}]$ with σ oriented so that the basis $v_1 - v_{k+1}, \dots, v_k - v_{k+1}$ of $\text{span } \underline{\sigma}$ is positively oriented. $\square$

An oriented polytope σ of dimension $k \geq 2$ induces orientations of the facets of σ in the following way: Let $\underline{\tau}$ be a facet of $\underline{\sigma}$, and let η be a normal vector to the affine hull of $\underline{\tau}$ in the affine hull of $\underline{\sigma}$ pointing in the direction of $\underline{\sigma}$. We will say that such a vector η is a *normal vector to $\underline{\tau}$ pointing inwards to $\underline{\sigma}$* . Then, the orientation of $\text{span } \underline{\tau}$ induced by σ is defined by that a basis $w_1, \dots, w_{k-1}$ of $\text{span } \underline{\tau}$ is positively oriented if and only if $\eta, w_1, \dots, w_{k-1}$ is a positively oriented basis of $\text{span } \underline{\sigma}$. If σ is a simplex $[v_1, \dots, v_{k+1}]$ and $\underline{\tau}$ is obtained from $\underline{\sigma}$ by removing the vertex v_j , then it is easily verified that σ induces the orientation $(-1)^{j-1}[v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_{k+1}]$ of τ .

We say that a polyhedral cell complex is *oriented* if each face is equipped with an orientation. More precisely, an oriented polyhedral cell complex is a finite collection of oriented polytopes σ , such that the underlying polytopes $\underline{\sigma}$ form a polyhedral cell complex; we say that τ is a face of σ if $\underline{\tau}$ is a face of $\underline{\sigma}$ etc.

If X is an oriented polyhedral cell complex, $\sigma \in X_k$, and $\tau \in X_{k-1}$ is a facet of σ , let $\text{sgn}(\tau, \sigma) = 1$ if the orientation of τ induced by the orientation of σ coincides with the orientation of τ , and let $\text{sgn}(\tau, \sigma) = -1$ otherwise. If $w_1, \dots, w_{k-1}$ is a basis of $\text{span } \underline{\tau}$, and η is a normal vector of $\underline{\tau}$ pointing inwards to $\underline{\sigma}$, then

$$\text{sgn}(\tau, \sigma) = \text{sgn}(\omega_\sigma(\eta \wedge w_1 \wedge \dots \wedge w_{k-1})) / \text{sgn}(\omega_\tau(w_1 \wedge \dots \wedge w_{k-1})). \quad (3.1)$$

If $k = 1$, we interpret $\text{sgn}(\tau, \sigma)$ as 1 if the normal η pointing inwards to $\underline{\sigma}$ is positively oriented, and -1 otherwise, and if $k = 0$ we interpret $\text{sgn}(\tau, \sigma)$ as 1. This is consistent with (3.1) if we interpret ω_σ as 1 if $\dim \sigma \leq 0$.

Similarly if $\sigma \in X_k$ and σ' is any oriented polytope of dimension k that is contained in σ (i.e., $\underline{\sigma'} \subset \underline{\sigma}$), let $\text{sgn}(\sigma', \sigma) = 1$ if the orientation of $\text{span } \underline{\sigma'} = \text{span } \underline{\sigma}$ given by σ' coincides with the orientation given by σ and let $\text{sgn}(\sigma', \sigma) = -1$ otherwise. If $w_1, \dots, w_k$ is a basis of $\text{span } \underline{\sigma}$, then

$$\text{sgn}(\sigma', \sigma) = \text{sgn}(\omega_\sigma(w_1 \wedge \dots \wedge w_k)) / \text{sgn}(\omega_{\sigma'}(w_1 \wedge \dots \wedge w_k)). \quad (3.2)$$

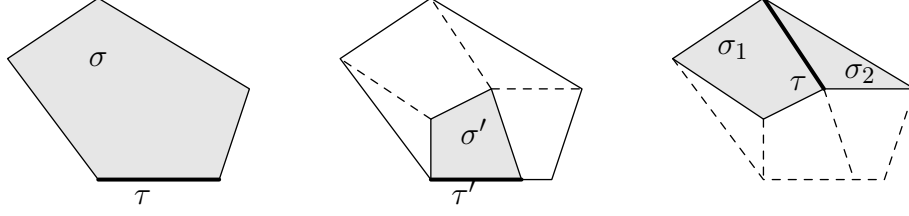


FIGURE 3.1. Examples of faces σ , τ , σ' , and τ' in Lemma 3.3 (in the left and middle figure) and faces σ_1 , σ_2 and τ in Lemma 3.4 (in the right figure).

If $k \leq 0$, $\text{sgn}(\sigma', \sigma)$ should be interpreted as 1.

Lemma 3.3. *Let X and X' be oriented polyhedral cell complexes such that X' refines X . Assume that $\sigma' \subset \sigma$, where $\sigma \in X_k$ and $\sigma' \in X'_k$. Moreover assume that $\tau \in X_{k-1}$ and $\tau' \in X'_{k-1}$ are facets of σ and σ' , respectively, and that $\tau' \subset \tau$. Then*

$$\text{sgn}(\sigma', \sigma) \text{sgn}(\tau', \sigma') = \text{sgn}(\tau, \sigma) \text{sgn}(\tau', \tau). \quad (3.3)$$

Proof. Let η be a normal vector of $\underline{\tau}'$ pointing inwards to $\underline{\sigma}'$. Then, η is also a normal vector of $\underline{\tau}$ pointing inwards to $\underline{\sigma}$. Let $w_1, \dots, w_{k-1}$ be a basis of $\text{span } \underline{\tau}' = \text{span } \underline{\tau}$. Then by (3.1) and (3.2), both sides of (3.3) are equal to

$$\text{sgn}(\omega_\sigma(\eta \wedge w_1 \wedge \dots \wedge w_{k-1})) / \text{sgn}(\omega_{\tau'}(w_1 \wedge \dots \wedge w_{k-1})).$$

□

Lemma 3.4. *Let σ be an oriented polytope of dimension $k \geq 1$, and let X be a polyhedral subdivision of σ . Assume that $\tau \in X_{k-1}$ is a facet of two faces $\sigma_1, \sigma_2 \in X_k$. Then*

$$\text{sgn}(\sigma_1, \sigma) \text{sgn}(\tau, \sigma_1) + \text{sgn}(\sigma_2, \sigma) \text{sgn}(\tau, \sigma_2) = 0. \quad (3.4)$$

Proof. Being in the same situation as in the second case in Lemma 3.1, it is easily verified that we may assume that $|X| = \underline{\sigma} \subset \mathbf{R}_{x_1, \dots, x_k}^k$, $\underline{\tau} \subset \{x_k = 0\}$, and $\underline{\sigma}_j \subset H_j, j = 1, 2$, where $H_1 = \{x_k \geq 0\}$ and $H_2 = \{x_k \leq 0\}$. Then the vector $\eta := (0, \dots, 0, 1)$ is a normal vector to $\underline{\tau}$ pointing inwards to $\underline{\sigma}_1$ and $-\eta$ is a normal vector to $\underline{\tau}$ pointing inwards to $\underline{\sigma}_2$. Letting $w_1, \dots, w_{k-1}$ be a basis of $\text{span } \underline{\tau}$, by (3.1) and (3.2) the first term in the left-hand side of (3.4) equals

$$\text{sgn}(\omega_\sigma(\eta \wedge w_1 \wedge \dots \wedge w_{k-1})) / \text{sgn}(\omega_\tau(w_1 \wedge \dots \wedge w_{k-1})) \quad (3.5)$$

and the second term equals (3.5) with the opposite sign.

□

4. CELLULAR RESOLUTIONS OF MONOMIAL IDEALS

Let us recall the construction of cellular resolutions due to Bayer-Sturmfels [8]. Let S be the polynomial ring $\mathbf{C}[z_1, \dots, z_n]$. We say that an (oriented) polyhedral cell complex X is *labeled* if there is a monomial m_i in S associated with each vertex v_i . An arbitrary face σ of X is then labeled by the least common multiple of the labels of the vertices of σ , i.e., by $m_\sigma = \text{lcm}\{m_i | i \in \sigma\}$; $m_\emptyset$ should be interpreted as 1. We will sometimes be sloppy and not differ between the faces of a labeled complex and their labels.

Definition 4.1. If X and Y are two labeled polyhedral cell complexes, we say that X *refines* Y if X refines Y as polyhedral cell complexes, i.e., $|X| = |Y|$, and each face of Y is a union of faces in X , and in addition, we require that if $\sigma' \in X$, $\sigma \in Y$, and $\sigma' \subset \sigma$, then $m_{\sigma'} | m_\sigma$. Note that this implies that the ideal generated by the labels of the vertices of Y must be contained in the ideal generated by the labels of the vertices of X .

Let M be a *monomial ideal* in S , i.e., M can be generated by monomials. We will use the shorthand notation z^α for the monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ in S . It is easy to check that a monomial ideal has a unique minimal set of generators that are monomials; assume that $\{m_1, \dots, m_r\}$ is a minimal set of monomial generators of M . Next, let X be an oriented polyhedral cell complex with vertices $\{1, \dots, r\}$ labeled by $\{m_1, \dots, m_r\}$. We will associate with X a graded complex of free S -modules: For $k = -1, \dots, \dim X$, let A_k be the free S -module with basis $\{e_\sigma\}_{\sigma \in X_k}$ and let the differential $\varphi_k : A_k \rightarrow A_{k-1}$ be defined by

$$\varphi_k : e_\sigma \mapsto \sum_{\text{facets } \tau \subset \sigma} \text{sgn}(\tau, \sigma) \frac{m_\sigma}{m_\tau} e_\tau. \quad (4.1)$$

Note that m_σ/m_τ is a monomial when τ is a face of σ . The complex

$$\mathbf{F}_X : 0 \rightarrow A_{\dim X} \xrightarrow{\varphi_{\dim X}} \cdots \xrightarrow{\varphi_1} A_0 \xrightarrow{\varphi_0} A_{-1}$$

is the *cellular complex supported on X* . Note that, with the identification $A_{-1} = S$, the cokernel of φ_0 equals S/M . The complex $\mathbf{F}_X$ is exact if the labeled complex X satisfies a certain acyclicity condition. More precisely, for $\beta \in \mathbf{N}^n$, where $\mathbf{N} = \{0, 1, \dots\}$, let $X_{\preceq \beta}$ denote the subcomplex of X consisting of all faces σ for which z^β is divisible by m_σ . Then $\mathbf{F}_X$ is exact if and only if $X_{\preceq \beta}$ is acyclic, which means that it is empty or has zero reduced homology, for all $\beta \in \mathbf{N}^n$, see [18, Proposition 4.5]. Note, in particular, that the acyclicity does not depend on the orientation of X . When $\mathbf{F}_X$ is exact we say that it is a *cellular resolution* of S/M .

To put the cellular resolutions into the context of [5], let us consider the vector bundle complex (2.2), where E_k for $k = -1, \dots, N = \dim X$ is a trivial bundle over $\mathbf{C}^n$ of rank equal to the number of faces in

X_k , with a global frame $\{e_\sigma\}_{\sigma \in X_k}$, endowed with the trivial metric, and where the differential φ_k is given by (4.1). We will say that the corresponding residue current R is associated with X and denote it by R^X , and we will use R_σ to denote the coefficient of $e_\sigma \otimes e_\emptyset^*$. The induced sheaf complex (2.3) is exact if and only if $\mathbf{F}_X$ is. This follows from the standard fact that the ring $\mathcal{O}_0$ of germs of holomorphic functions at $0 \in \mathbf{C}^n$ is flat over S , see for example [25, Theorem 13.3.5]. We will think of monomial ideals sometimes as ideals in the polynomial ring S , sometimes as ideals in the ring of entire functions in $\mathbf{C}^n$, and sometimes as ideals in the local ring $\mathcal{O}_0^n$.

4.1. The hull resolution. Given a monomial ideal M in S and $t \in \mathbf{R}$, let $\mathcal{P}_t = \mathcal{P}_t(M)$ be the convex hull in $\mathbf{R}^n$ of $\{(t^{\alpha_1}, \dots, t^{\alpha_n}) =: t^\alpha \mid z^\alpha \in M\}$. Then $\mathcal{P}_t$ is a unbounded polyhedron in $\mathbf{R}^n$ of dimension n and the face poset (i.e., the set of faces partially ordered by inclusion) of bounded faces of $\mathcal{P}_t$ is independent of t if $t \gg 0$. The *hull complex* $\text{hull}(M)$ of M , introduced in [8], is the polyhedral cell complex of all bounded faces of $\mathcal{P}_t$ for $t \gg 0$. The vertices of $\text{hull}(M)$ are precisely the points t^α , where z^α is a minimal generator of M , and thus $\text{hull}(M)$ admits a natural labeling. The corresponding complex $\mathbf{F}_{\text{hull}(M)}$ is a resolution of S/M ; it is called the *hull resolution*.

Example 4.2. Let N be the complete intersection ideal $(z_1^{b_1}, \dots, z_n^{b_n})$. Then, $\text{hull}(N)$ is the polyhedral cell complex consisting of the $(n-1)$ -simplex $\Delta = [v_1, \dots, v_n]$ in $\mathbf{R}^n$ and its faces, where $v_1 = (t^{b_1}, 1, \dots, 1)$, $v_2 = (1, t^{b_2}, 1, \dots, 1)$, $\dots$, $v_n = (1, \dots, 1, t^{b_n})$. The vertices $v_1, \dots, v_n$ of $\text{hull}(N)$ are labeled by $z_1^{b_1}, \dots, z_n^{b_n}$, respectively, and we assume the faces are oriented so that the simplex σ with vertices $v_{i_1}, \dots, v_{i_\ell}$ equals $[v_{i_1}, \dots, v_{i_\ell}]$ if $i_1 < \dots < i_\ell$. Then the corresponding cellular complex $\mathbf{F}_{\text{hull}(N)}$ is the Koszul complex of $(z_1^{b_1}, \dots, z_n^{b_n})$, and

$$R^{\text{hull}(N)} = \bar{\partial} \left[\frac{1}{z_n^{b_n}} \right] \wedge \dots \wedge \bar{\partial} \left[\frac{1}{z_1^{b_1}} \right] e_\Delta \otimes e_\emptyset^*, \quad (4.2)$$

cf. Section 3.1 and Example 2.1. Note that a different orientation of the top-dimensional simplex $\Delta = [v_1, \dots, v_n]$ would permute the residue factors in (4.2). $\square$

The example shows that the hull complex of the complete intersection ideal is the cellular complex consisting of an $(n-1)$ -simplex together with its faces. In general, if M is Artinian, $\text{hull}(M)$ is a polyhedral subdivision of such an $(n-1)$ -simplex or, rather, it can be embedded as one, see, e.g., (the proof of) Theorem 4.31 in [18]. We will need the following more precise description of this embedding. To begin with, we note that an Artinian monomial ideal has monomials of the form $z_1^{\beta_1}, \dots, z_n^{\beta_n}$ among its minimal monomial generators. Note also that every other minimal generator has degree smaller than β_i in z_i .

Proposition 4.3. *Let M be an Artinian monomial ideal with $(z_1^{b_1}, \dots, z_n^{b_n})$ among its minimal monomial generators. Let N be the complete intersection ideal $(z_1^{b_1}, \dots, z_n^{b_n})$. Then $\text{hull}(M)$ can be embedded as a refinement of $\text{hull}(N)$ as labeled polyhedral cell complexes.*

We will be sloppy and not always distinguish between the hull complex of M and its embedding.

Proof. That $\text{hull}(M)$ refines $\text{hull}(N)$ as polyhedral cell complexes is Theorem 4.31 in [18]. In fact, it follows from the proof in [18] of that theorem that it is a refinement also as *labeled* polyhedral cell complexes. To see this, we begin by recalling (slightly differently described) the construction of the embedding in that proof.

We know from Example 4.2 that $\text{hull}(N)$ consists of the faces of the simplex Δ with vertices $v_1 = (t^{b_1}, 1, \dots, 1), \dots, v_n = (1, \dots, 1, t^{b_n})$. For a point $p \neq \mathbf{1} := (1, \dots, 1)$, with $p_i \geq 1$, consider the line ℓ through $\mathbf{1}$ and p . Since $p_i \geq 1$, ℓ intersects Δ in a unique point, which we denote $\pi(p)$. Moreover, since $|\text{hull}(M)|$ is contained in the set where $p_i \geq 1$, we get a map $\pi : |\text{hull}(M)| \rightarrow \Delta$, which induces an embedding of $\text{hull}(M)$ into Δ by letting the faces of the embedded complex be the images $\pi(\sigma)$, where $\sigma \in \text{hull}(M)$ (with the same labeling).

Consider a face σ' of $\text{hull}(M)$ such that $\pi(\sigma') \subseteq \sigma = [v_{i_1}, \dots, v_{i_k}]$. Then the vertices of $\pi(\sigma')$ must be contained in the set $\{x \in \mathbf{R}^n \mid x_i = 1, i \neq i_1, \dots, i_k\}$, since the v_{i_j} are. A vertex v of $\text{hull}(M)$ with label $m_v = z^\alpha$ has coordinates $(t^{\alpha_1}, \dots, t^{\alpha_n})$, so if $\pi(v)$ is contained in $\{x_i = 1\}$, then we must have $\alpha_i = 0$ in m_v . It follows that $m_{\sigma'}$ is of the form $m_{\sigma'} = z_{i_1}^{\alpha_{i_1}} \dots z_{i_k}^{\alpha_{i_k}}$, and since each label of a minimal monomial generator is of degree at most b_i in z_i , the same must hold for $m_{\sigma'}$ since it is the common multiple of such labels. Hence, $m_{\sigma'} | m_\sigma = z_{i_1}^{b_{i_1}} \dots z_{i_k}^{b_{i_k}}$. $\square$

Recall that a graded free resolution $A_\bullet, \varphi_\bullet$ is *minimal* if and only if for each k , φ_k maps a basis of A_k to a minimal set of generators of $\text{Im } \varphi_k$, see, e.g., [13, Corollary 1.5]. The hull resolution is not minimal in general, cf. Example 6.1. However, if M is a *generic* monomial ideal in the sense of [9, 19], the hull complex is simplicial, i.e., all faces are simplices, and it coincides with the *Scarf complex* of M , which is a minimal resolution of S/M , see [9]. The ideal M is generic if whenever two distinct minimal generators m_i and m_j have the same positive degree in some variable, then there exists a third generator m_k that strictly divides the least common multiple z^α of m_i and m_j , meaning that m_k divides $z_1^{\alpha_1-1} \dots z_n^{\alpha_n-1}$. Note that when $n \leq 2$ all monomial ideals are generic. The Scarf complex of M is the collection of subsets $I \subset \{1, \dots, r\}$ whose corresponding least common multiple $m_I := \text{lcm}_{i \in I} m_i$ is unique.

5. PROOF OF THEOREM 1.1

We will prove a slightly more general version of Theorem 1.1. If N is a complete intersection ideal $(z_1^{b_1}, \dots, z_n^{b_n})$, by Example 4.2, $\text{hull}(N)$ is the polyhedral cell complex consisting of the faces of an oriented $(n-1)$ -simplex Δ , with vertices labeled by $z_1^{b_1}, \dots, z_n^{b_n}$. In particular, $\text{hull}(N)_{n-1}$ consists of only the simplex Δ .

Theorem 5.1. *Let M be an Artinian monomial ideal in $S = \mathbf{C}[z_1, \dots, z_n]$. Assume that $\mathbf{F}_X$ is a cellular resolution of S/M such that the underlying labeled polyhedral cell complex X refines the hull complex of a complete intersection ideal $N = (z_1^{b_1}, \dots, z_n^{b_n})$, i.e., the $(n-1)$ -simplex Δ with vertices labeled by $z_1^{b_1}, \dots, z_n^{b_n}$. Then the associated residue current R^X has one entry R_σ for each $(n-1)$ -dimensional face σ of X , and*

$$R_\sigma = \text{sgn}(\sigma, \Delta) \bar{\partial} \left[\frac{1}{z_n^{\alpha_n}} \right] \wedge \cdots \wedge \bar{\partial} \left[\frac{1}{z_1^{\alpha_1}} \right],$$

where $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is the label of σ .

Theorem 1.1 corresponds to the case when X equals $\text{hull}(M)$; the refinement is given by Proposition 4.3, and the orientation of $\text{hull}(M)$ is implicitly assumed to be such that $\text{sgn}(\sigma, \Delta) = 1$ for each $\sigma \in \text{hull}(M)_{n-1}$.

Proposition 5.2. *Let X and Y be oriented labeled polyhedral cell complexes such that X refines Y , and let $E_\bullet, \varphi_\bullet$ and $F_\bullet, \psi_\bullet$ be the corresponding vector bundle complexes. For $k \geq -1$ let $a_k : F_k \rightarrow E_k$ be the mapping*

$$a_k : e_\sigma \mapsto \sum_{\sigma' \subset \sigma} \text{sgn}(\sigma', \sigma) \frac{m_\sigma}{m_{\sigma'}} e_{\sigma'}, \quad (5.1)$$

where the sum is over all $\sigma' \in X_k$ that satisfy $\sigma' \subset \sigma \in Y_k$. Then the a_k are holomorphic and the diagram (2.4) commutes.

We let X and N be as in Theorem 5.1, and $Y = \text{hull}(N)$. Since $\dim X = \dim Y = n-1$, the complexes $E_\bullet, \varphi_\bullet$ and $F_\bullet, \psi_\bullet$ end at level $n-1$. Thus, identifying E_{-1} and F_{-1} and taking Proposition 5.2 for granted, (2.5) yields

$$R^X = R^E = a_{n-1} R^F = \sum_{\sigma \subset \Delta} \text{sgn}(\sigma, \Delta) \frac{m_\Delta}{m_\sigma} \bar{\partial} \left[\frac{1}{z_n^{b_n}} \right] \wedge \cdots \wedge \bar{\partial} \left[\frac{1}{z_1^{b_1}} \right] e_\sigma \otimes e_\emptyset^*;$$

here we have used (4.2) for the last equality. Since $|X| = |Y| = \Delta$, the sum is over all $\sigma \in X_k$, and since $m_\Delta = z_1^{b_1} \cdots z_n^{b_n}$ the coefficient of $e_\sigma \otimes e_\emptyset^*$ is just

$$\text{sgn}(\sigma, \Delta) \bar{\partial} \left[\frac{1}{z_n^{\alpha_n}} \right] \wedge \cdots \wedge \bar{\partial} \left[\frac{1}{z_1^{\alpha_1}} \right],$$

where $z_1^{\alpha_1} \cdots z_n^{\alpha_n} = m_\sigma$. This concludes the proof of Theorem 5.1.

Proof of Proposition 5.2. Since X refines Y as a labeled polyhedral cell complex, each $m_\sigma/m_{\sigma'}$ in (5.1) is holomorphic and thus the a_k are holomorphic.

To show that (2.4) commutes, we first consider the case $k \geq 1$. Pick $\sigma \in Y_k$. Then

$$e_\sigma \xrightarrow{\psi_k} \sum_{\tau \subset \sigma} \operatorname{sgn}(\tau, \sigma) \frac{m_\sigma}{m_\tau} e_\tau \xrightarrow{a_{k-1}} \sum_{\tau \subset \sigma} \sum_{\tau' \subset \tau} \operatorname{sgn}(\tau, \sigma) \operatorname{sgn}(\tau', \tau) \frac{m_\sigma}{m_{\tau'}} e_{\tau'}. \quad (5.2)$$

Here the first sum is over the facets $\tau \in Y_{k-1}$ of σ , and the second sum is over the faces $\tau' \in X_{k-1}$ that are contained in τ . Moreover

$$e_\sigma \xrightarrow{a_k} \sum_{\sigma' \subset \sigma} \operatorname{sgn}(\sigma', \sigma) \frac{m_\sigma}{m_{\sigma'}} e_{\sigma'} \xrightarrow{\varphi_k} \sum_{\sigma' \subset \sigma} \sum_{\tau' \subset \sigma'} \operatorname{sgn}(\sigma', \sigma) \operatorname{sgn}(\tau', \sigma') \frac{m_\sigma}{m_{\tau'}} e_{\tau'}. \quad (5.3)$$

Now the first sum is over the faces $\sigma' \in X_k$ that are contained in σ , whereas the second sum is over the facets $\tau' \in X_{k-1}$ of σ' .

Let X^σ be the k -dimensional subcomplex of faces of X that are contained in σ and consider $\tau' \in X_{k-1}^\sigma$. Note that X being a refinement of Y means that X^σ is a polyhedral subdivision of σ . Assume that τ' is contained in a facet τ of σ . Since $\dim \tau' = k-1 = \dim \tau$, there is a unique such τ , and thus the coefficient of $e_{\tau'}$ (in the rightmost expression) in (5.2) equals $\operatorname{sgn}(\tau, \sigma) \operatorname{sgn}(\tau', \tau) \frac{m_\sigma}{m_{\tau'}}$. Moreover, τ' is contained in the boundary of $|X^\sigma|$ and thus by Lemma 3.1 there is a unique $\sigma' \in X_k^\sigma$ such that $\tau' \subset \sigma'$. Therefore the coefficient of $e_{\tau'}$ (in the rightmost expression) in (5.3) is $\operatorname{sgn}(\sigma', \sigma) \operatorname{sgn}(\tau', \sigma') \frac{m_\sigma}{m_{\tau'}}$. By Lemma 3.3 these coefficients coincide.

If τ' is not contained in any facet τ of σ , then clearly the coefficient of $e_{\tau'}$ in (5.2) is zero. Also, then τ' is not contained in the boundary of X^σ , and thus by Lemma 3.1, τ' is a facet of exactly two faces $\sigma'_1, \sigma'_2 \in X_k^\sigma$. Hence the coefficient of $e_{\tau'}$ in (5.3) is

$$(\operatorname{sgn}(\sigma'_1, \sigma) \operatorname{sgn}(\tau', \sigma'_1) + \operatorname{sgn}(\sigma'_2, \sigma) \operatorname{sgn}(\tau', \sigma'_2)) \frac{m_\sigma}{m_{\tau'}},$$

which by Lemma 3.4 vanishes. Since the sums in (5.2) and (5.3) are only over $\tau', \sigma' \in X$ that are in X^σ , it follows that $a_{k-1} \circ \psi_k(e_\sigma) = \varphi_k \circ a_k(e_\sigma)$.

For $k = 0$, pick a vertex $\sigma \in Y_0$. Since X is a polyhedral subdivision of Y and σ is a vertex, the only $\sigma' \in X_0$ with $\sigma' \subset \sigma$ is $\sigma' = \sigma$. Thus $\varphi_0 \circ a_0(e_\sigma) = \varphi_0(m_\sigma/m_{\sigma'} e_{\sigma'}) = m_\sigma e_\emptyset$. Note that a_{-1} maps $e_\emptyset$ to $e_\emptyset$. Thus $a_{-1} \circ \psi_0(e_\sigma) = m_\sigma e_\emptyset$.

We conclude that $a_{k-1} \circ \psi_k = \varphi_k \circ a_k$ for $k \geq 0$; in other words, the diagram (2.4) commutes. $\square$

6. COMPARISON TO PREVIOUS RESULTS

In [27] the current $R = R^X$ constructed from a cellular resolution $\mathbf{F}_X$ of an Artinian monomial ideal M was computed up to multiplicative

constants; Proposition 3.1 in [27] asserts that R has one entry R_σ for each face $\sigma \in X_{n-1}$, which is of the form

$$R_\sigma = c_\sigma \bar{\partial} \left[\frac{1}{z_n^{\alpha_n}} \right] \wedge \cdots \wedge \bar{\partial} \left[\frac{1}{z_1^{\alpha_1}} \right] \quad (6.1)$$

for some $c_\sigma \in \mathbf{C}$, where $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is the label of σ . The main novelty in this paper, except for the new proof, is that we show that $c_\sigma = 1$ (or -1 , depending on the orientation of X) and thus give a complete description of R .

Let $\text{ann } R \subset \mathcal{O}_0^n$ denote the *annihilator ideal* of R , i.e., the ideal of germs of holomorphic functions ξ at $0 \in \mathbf{C}^n$ that satisfy $R\xi = 0$. Note that $\text{ann } R_\sigma = (z_1^{\alpha_1}, \dots, z_n^{\alpha_n}) =: \mathfrak{m}^\alpha$. A monomial ideal of this form is said to be *irreducible*. Each monomial ideal M can be written as finite intersection of irreducible ideals; this is called an *irreducible decomposition* of M . Since one has to annihilate each R_σ in order to annihilate R , Theorem 5.1 implies that, provided X is a polyhedral subdivision of Δ ,

$$\text{ann } R = \bigcap_{\sigma \in X_{n-1}} \mathfrak{m}^{\alpha_\sigma},$$

which gives an irreducible decomposition of $\text{ann } R = M$. Here α_σ is the multidegree of the label of σ . If $\mathbf{F}_X$ is a minimal resolution of M this decomposition is *irredundant* in the sense that no intersectand can be omitted. Each monomial ideal has a unique (monomial) irredundant irreducible decomposition.

Using that R satisfies the duality principle and results [9, Theorem 3.7] and [18, Theorem 5.42] about irreducible decompositions, in [27], we could in some cases determine which c_σ are nonzero. If M is a generic monomial ideal, Theorem 3.3 in that paper says that c_σ is nonzero if and only if σ is in the Scarf complex Δ_M (which is a subcomplex of any cellular resolution of M), and if $\mathbf{F}_X$ is a minimal resolution of M each c_σ is nonzero by Theorem 3.5 in [27]. Let us look at an example where these theorems do not apply.

Example 6.1. Consider the ideal $M = (z_1^2, z_1 z_2, z_1 z_3, z_2^2, z_2 z_3, z_3^2) \subset S = \mathbf{C}[z_1, z_2, z_3]$, i.e., the square of the maximal ideal at 0 in S . The hull complex of M is a refinement of the 2-simplex Δ with the vertices labeled by z_1^2, z_2^2, z_3^2 , see Figure 6.1.

There are four faces $\sigma_1, \dots, \sigma_4$ in $\text{hull}_2(M)$ with labels $m_{\sigma_1} = z_1^2 z_2 z_3$, $m_{\sigma_2} = z_1 z_2^2 z_3$, $m_{\sigma_3} = z_1 z_2 z_3^2$, and $m_{\sigma_4} = z_1 z_2 z_3$. By Theorem 5.1, the current R therefore has four entries: three entries of the form $R_{\sigma_\ell} = \pm \bar{\partial}[1/z_k^2] \wedge \bar{\partial}[1/z_j] \wedge \bar{\partial}[1/z_i]$ for $\ell = 1, 2, 3$, corresponding to the three corner triangles in $\text{hull}(M)$, and one component $R_{\sigma_4} = \bar{\partial}[1/z_3] \wedge \bar{\partial}[1/z_2] \wedge \bar{\partial}[1/z_1]$.

The hull resolution is not a minimal resolution of S/M . In particular, M is not generic. By arguing as in the proofs of Theorems 3.3 and 3.5 in [27], using that R satisfies the duality principle and that

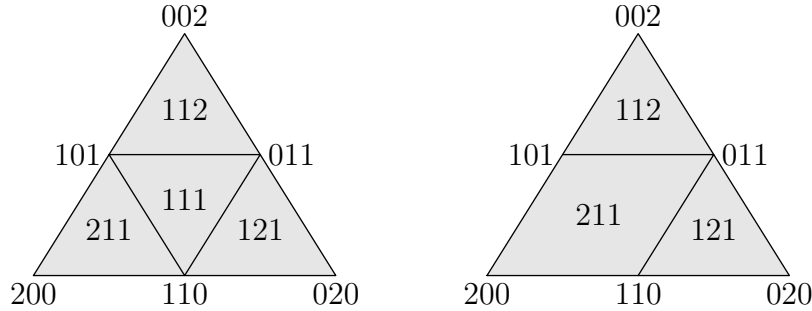


FIGURE 6.1. The hull complex of the ideal M in Example 6.1 (labels on vertices and 2-faces) (left) and the cell complex of a minimal free resolution of M (right).

$M = (z_1^2, z_2, z_3) \cap (z_1, z_2^2, z_3) \cap (z_1, z_2, z_3^2)$ is the irredundant irreducible decomposition of M , one can conclude that first three c_{σ_j} in (6.1) are non-zero, but not that c_{σ_4} is.

A minimal resolution of S/M is obtained by removing one of the edges of the inner triangle in $\text{hull}(M)$, see, e.g., [18, Example 3.19]. The cell complex X of one such resolution is depicted in Figure 6.1. Note that X is a refinement of Δ (although different from $\text{hull}(M)$) so that Theorem 5.1 applies; the corresponding residue current consists of the three entries R_{σ_1} , R_{σ_2} , and R_{σ_3} above. $\square$

In [27] the current R is computed as the push-forward of a current on a toric log-resolution of M . The computations are inspired by [26], where Bochner-Martinelli residue currents, cf. Example 2.1, of monomial ideals are computed, and they become quite involved. The coefficients c_σ appear as certain integrals in the log-resolution and seem to be hard to compute in general. The proof of Theorem 5.1 given here is more direct and much less technical than in [27].

It would be interesting to investigate whether the comparison formula for residue currents could be used also to compute Bochner-Martinelli residue currents. In [26] it was shown that if M is an Artinian monomial ideal, the Bochner-Martinelli current R_{BM}^M of (a monomial sequence of generators of) M is a vector-valued current with entries of the form (6.1), for certain exponents α . In some cases we can compute the coefficients c_σ , e.g., if $n = 2$ and each minimal generator of the monomial ideal M is a vertex of the so-called Newton polytope of M ; the coefficients are then equal to ± 1 , see [28, Section 4.2].

If $E_\bullet, \varphi_\bullet$ is the Koszul complex of M and $F_\bullet, \psi_\bullet$ is the Koszul complex of a complete intersection ideal $(z_1^{\beta_1}, \dots, z_n^{\beta_n})$ contained in M , it is not hard to explicitly find mappings a_k so that the diagram (2.4) commutes. Indeed, let $m_1, \dots, m_r$ be a minimal set of generators of M , ordered so that $m_j = z_j^{\alpha_j}$ for $j = 1, \dots, n$; note that there are such generators since M is Artinian. Identify the set of generators with a section $\sum m_j e_j$ of a

(trivial) rank r bundle $\tilde{E}$. Similarly identify $z_1^{\beta_1}, \dots, z_n^{\beta_n}$ with a section $\sum z_j^{\beta_j} \epsilon_j$ of a rank n bundle $\tilde{F}$ and construct the Koszul complexes $E_\bullet, \varphi_\bullet$ and $F_\bullet, \psi_\bullet$ as in Example 2.1. Now we can choose $a_{k-1} : \Lambda^k \tilde{F}^* \rightarrow \Lambda^k \tilde{E}^*$ as the mapping $a_{k-1} : \epsilon_{i_1}^* \wedge \dots \wedge \epsilon_{i_k}^* \mapsto z_{i_1}^{\beta_{i_1} - \alpha_{i_1}} \dots z_{i_k}^{\beta_{i_k} - \alpha_{i_k}} \epsilon_{i_1}^* \wedge \dots \wedge \epsilon_{i_k}^*$. Theorem 3.2 in [15] then gives a formula relating the currents $R^E = R_{BM}^M$ and R^F , the latter given by (4.2). However, when M is not a complete intersection and thus E does not end at level $n-1$, the formula relating the currents is more involved than (2.5); there appears an extra term, which seems hard to compute in general, see [15, Equation (3.2)].

6.1. Non-Artinian monomial ideals. In [27] we also computed residue currents (up to nonvanishing factors) associated with cellular resolutions of non-Artinian monomial ideals.

The method in this paper is not as well adapted to resolutions of non-Artinian ideals. First, to be able to use the simple form (2.5) of the comparison formula for residue currents it is important that M is Cohen-Macaulay. Second, even if M is Cohen-Macaulay, there is in general no such natural (resolution of an) ideal to compare with as the monomial complete intersection ideal $N = (z_1^{b_1}, \dots, z_n^{b_n})$ in the Artinian case.

Example 6.2. Let M be the ideal $M = (z_1 z_2, z_1 z_3, z_2 z_3)$ in $S = \mathbf{C}[z_1, z_2, z_3]$. Then

$$0 \longrightarrow S^{\oplus 2} \xrightarrow{\begin{bmatrix} -z_3 & 0 \\ z_2 & -z_2 \\ 0 & z_1 \end{bmatrix}} S^{\oplus 3} \xrightarrow{\begin{bmatrix} z_1 z_2 & z_1 z_3 & z_2 z_3 \end{bmatrix}} S \quad (6.2)$$

is a free resolution of M . Let $E_\bullet, \varphi_\bullet$ be the corresponding vector bundle complex. Next, let f be the regular sequence $f_1 = z_1 z_2, f_2 = (z_1 + z_2) z_3$, and let $F_\bullet, \psi_\bullet$ be the Koszul complex of f . Then it is not hard to explicitly find the morphisms a_1, a_0 , and a_{-1} . Since the ideals M and (f_1, f_2) are Cohen-Macaulay we may apply the comparison formula (2.5). A computation gives

$$R^E = \frac{1}{z_1} \bar{\partial} \frac{1}{z_3} \wedge \bar{\partial} \frac{1}{z_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{z_2} \bar{\partial} \frac{1}{z_3} \wedge \bar{\partial} \frac{1}{z_1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{z_3} \bar{\partial} \frac{1}{z_2} \wedge \bar{\partial} \frac{1}{z_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Observe that R is not symmetric in z_1 and z_2 , although the ideal M is. This is, however, not too surprising, since the resolution (6.2) is not symmetric in z_1 and z_2 . $\square$

A general strategy for computing the residue current associated with the resolution $E_\bullet, \varphi_\bullet$ of a (monomial) Cohen-Macaulay ideal M of codimension p is to look for a regular sequence $f_1, \dots, f_p$ contained in M and then apply the comparison formula (2.5) to $E_\bullet, \varphi_\bullet$ and the Koszul

complex $F_\bullet, \psi_\bullet$ of f . One way of finding such a regular sequence is to consider p sufficiently generic linear combinations $f_1, \dots, f_p$ of the generators of M , as was done in Example 6.2. However, when the f_j are not monomials the computation of the current $R^F = R_{CH}^f$ can become much more involved. Also, although the complex $F_\bullet, \psi_\bullet$ is simple, it may be hard to find the morphism a_k in general.

If $E_\bullet, \varphi$ is a resolution of a non-Cohen-Macaulay ideal, the comparison formula in [15] is more involved than (2.5). For computations of residue currents in this case, see [15, Section 5].

7. RELATIONS TO FUNDAMENTAL CYCLES

Our original motivation for computing the coefficients c_σ of the entries (6.1) of R^X was that we wanted to understand the current

$$D\varphi \circ R := D\varphi_0 \circ \dots \circ D\varphi_{p-1} \circ R, \quad (7.1)$$

when $R = R^E$ is the residue current associated with a resolution (2.3) of an ideal sheaf $\mathfrak{a}$ of codimension p and D is the connection on $\text{End} E$ induced by connections on $E = \bigoplus E_k$.

Let $\mathfrak{a}$ be a complete intersection ideal, defined by a regular sequence $f_1, \dots, f_p$ and let (2.2) be the Koszul complex of f_j , see Example 2.1, equipped with the trivial metrics so that D is the trivial connection d . Then (7.1) equals $p!$ times the current

$$R_{CH}^f \wedge df_1 \wedge \dots \wedge df_p = (2\pi i)^p [\mathfrak{a}], \quad (7.2)$$

where $[\mathfrak{a}]$ is the current of integration along the *fundamental cycle* of $\mathfrak{a}$. The equality (7.2) was proved in [10]. Recall that for an Artinian ideal $\mathfrak{a} \subseteq \mathcal{O}_0^n$, the fundamental cycle of $\mathfrak{a}$ is $[\mathfrak{a}] = m[0]$, where $m = \dim_{\mathbb{C}} \mathcal{O}_0^n / \mathfrak{a}$ is the *geometric multiplicity* of $\mathfrak{a}$. For an arbitrary ideal $\mathfrak{a}$, with irreducible components Z_i (i.e., irreducible components of the radical ideal of $\mathfrak{a}$), the fundamental cycle of $\mathfrak{a}$ is $[\mathfrak{a}] = \sum m_i [Z_i]$ where m_i are the geometric multiplicities of $\mathfrak{a}$ along Z_i . The geometric multiplicity m_i of $\mathfrak{a}$ along Z_i can be defined as the geometric multiplicity of the Artinian ideal $\mathfrak{a} + \mathfrak{b}$, where $\mathfrak{b}$ is the ideal of a generic smooth variety transversal to Z_i . For more details regarding fundamental cycles, see [14, Section 1.5].

Using the comparison formula for residue currents from [15], we recently managed to prove that

$$D\varphi \circ R = p!(2\pi i)^p [\mathfrak{a}] \quad (7.3)$$

for any resolution (2.3) of any equidimensional ideal (i.e., all minimal primes are of the same dimension) $\mathfrak{a} \subset \mathcal{O}_0^n$, thus generalizing (7.2). This factorization of the fundamental cycle is closely related to a result by Lejeune-Jalabert, [17], who proved a cohomological version of (7.3) for Cohen-Macaulay ideals, and it will be the subject of the forthcoming paper [16].

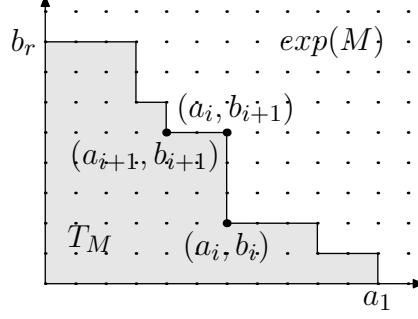


FIGURE 7.1. The staircase T_M of an Artinian monomial ideal in $\mathbf{C}^2$. The lattice points above T_M are the exponents $\exp(M)$ of monomials in M .

For the residue current associated with the hull resolution of a generic Artinian monomial ideal we can give an alternative proof of (7.3) (with the trivial connection d) using Theorem 1.1. In fact, we get a refinement of (7.3): For each permutation $s_1, \dots, s_n$ of $1, \dots, n$,

$$\frac{\partial f_1}{\partial z_{s_1}} dz_{s_1} \circ \dots \circ \frac{\partial f_n}{\partial z_{s_n}} dz_{s_n} \circ R = c_n (2\pi i)^n [\mathbf{a}], \quad (7.4)$$

where $c_n = (-1)^{n^2} \cdot (-1)^{\frac{n(n-1)}{2}}$. For an explanation of why the constant c_n appears in the right hand side of (7.4), but not in (7.3), see [16]. We will show how this works when $n = 2$. For $n \geq 3$, the computation of $d\varphi$ gets more involved; the general case will therefore be treated in the separate paper [29].

First, let us describe the geometric multiplicity $\dim_{\mathbf{C}} \mathcal{O}_0^n / M$ of a monomial ideal $M \subset \mathcal{O}_0^n$. Let $\mathbf{R}_+$ denote the nonnegative real numbers and let T_M be the *staircase* $\mathbf{R}_+^n \setminus \bigcup_{z^\alpha \in M} \{\alpha + \mathbf{R}_+^n\}$ of M . If M is Artinian, then T_M is a bounded set in $\mathbf{R}_+^n$. The name staircase is motivated by the shape of T_M . If $n = 2$ each Artinian monomial ideal M is of the form $M = (z^{a_1} w^{b_1}, \dots, z^{a_r} w^{b_r})$ for some integers $a_1 > \dots > a_r = 0$ and $0 = b_1 < \dots < b_r$. Then T_M looks like a staircase with inner corners (a_j, b_j) and outer corners (a_j, b_{j+1}) , see Figure 7.1. In general there is an “inner corner” α for each minimal generator z^α of M and one “outer corner” α for each intersectand $\mathbf{m}^\alpha$ in the irredundant irreducible decomposition. If M is generic, there is a one-to-one correspondence between faces $\sigma \in \text{hull}(M)_{n-1}$, with labels $m_\sigma = z^{\alpha_\sigma}$, and outer corners α in T_M . The points in $\mathbf{Z}^n \cap T_M$ are precisely the exponents of monomials that are not in M . In other words, $\mathcal{O}_0^n / M = \text{span}_{\mathbf{C}} \{z^\alpha \mid \alpha \notin T_M\}$. It follows that $\dim_{\mathbf{C}} \mathcal{O}_0^n / M = \text{Vol}(T_M)$, where Vol is the usual Euclidean volume in $\mathbf{R}^n$.

Now assume that $n = 2$, and that M is an Artinian ideal, minimally generated by $z^{a_i} w^{b_i}$, $a_1 > \dots > a_r = 0$ and $0 = b_1 < \dots < b_r$. Then $\text{hull}(M)$ is one-dimensional, with one vertex v_i for each generator $z^{a_i} w^{b_i}$

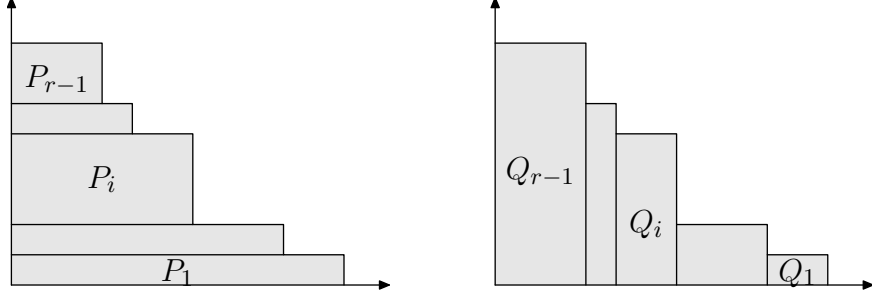


FIGURE 7.2. Partitions of T_M as rectangles P_i and rectangles Q_i .

and one edge σ_i , with label $z^{a_i}w^{b_{i+1}}$, for each outer corner (a_i, b_{i+1}) in T_M . The mappings in $\mathbf{F}_{\text{hull}(M)}$ are given by $\varphi_0 : e_{v_i} \mapsto z^{a_i}w^{b_i}e_\emptyset$ and $\varphi_1 : e_{\sigma_i} \mapsto z^{a_i-a_{i+1}}e_{v_{i+1}} - w^{b_{i+1}-b_i}e_{v_i}$ and by Theorem 1.1,

$$R = R^{\text{hull}(M)} = \sum_{i=1}^{r-1} \bar{\partial} \left[\frac{1}{w^{b_{i+1}}} \right] \wedge \bar{\partial} \left[\frac{1}{z^{a_i}} \right] e_{\sigma_i} \otimes e_\emptyset^*.$$

Let us compute $\frac{\partial \varphi_0}{\partial z} dz \circ \frac{\partial \varphi_1}{\partial w} dw \circ R$. Note that

$$\frac{\partial \varphi_0}{\partial z} dz = \sum_{i=1}^r a_i z^{a_i} w^{b_i} \frac{dz}{z} e_{v_i}^* \otimes e_\emptyset$$

and

$$\frac{\partial \varphi_1}{\partial w} dw = - \sum_{i=1}^{r-1} (b_{i+1} - b_i) w^{b_{i+1}-b_i} \frac{dw}{w} e_{\sigma_i}^* \otimes e_{v_i},$$

so that

$$-\frac{\partial \varphi_0}{\partial z} dz \circ \frac{\partial \varphi_1}{\partial w} dw = \sum_{i=1}^{r-1} a_i (b_{i+1} - b_i) z^{a_i} w^{b_{i+1}} \frac{dz}{z} \wedge \frac{dw}{w} e_{\sigma_i}^* \otimes e_\emptyset.$$

Let $P_i = \{x \in T_M \mid 0 \leq x_1 < a_i, b_i \leq x_2 < b_{i+1}\}$ for $i = 1, \dots, r-1$. Then the P_i form a partition of T_M , cf. Figure 7.2 and, in particular, $\text{Vol}(T_M) = \sum \text{Vol}(P_i)$. Note that $\text{Vol}(P_i) = a_i(b_{i+1} - b_i)$. Hence (identifying $e_\emptyset^* \otimes e_\emptyset$ with 1)

$$\begin{aligned} -\frac{\partial \varphi_0}{\partial z} dz \circ \frac{\partial \varphi_1}{\partial w} dw \circ R &= \sum_{i=1}^{r-1} \text{Vol}(P_i) z^{a_i} w^{b_{i+1}} \frac{dz}{z} \wedge \frac{dw}{w} \wedge \bar{\partial} \left[\frac{1}{w^{b_{i+1}}} \right] \wedge \bar{\partial} \left[\frac{1}{z^{a_i}} \right] = \\ &= \sum_{i=1}^{r-1} \text{Vol}(P_i) \bar{\partial} \left[\frac{1}{w} \right] \wedge \bar{\partial} \left[\frac{1}{z} \right] \wedge dz \wedge dw = (2\pi i)^2 \text{Vol}(T_M)[0], \end{aligned}$$

so we have proved (7.4) (for $z_{s_1} = z$ and $z_{s_2} = w$).

By similar arguments we get that $-\frac{\partial \varphi_0}{\partial w} dw \circ \frac{\partial \varphi_1}{\partial z} dz \circ R = \sum_{i=1}^{r-1} \text{Vol}(Q_i) (2\pi i)^2 [0]$, where $Q_i = \{x \in T_M \mid a_{i+1} \leq x_1 < a_i, 0 \leq x_2 < b_{i+1}\}$ for $i = 1, \dots, r-1$, see Figure 7.2. Again, the rectangles Q_i form a partition

of T_M and thus (7.4) holds also for this permutation ($z_{s_1} = w$ and $z_{s_2} = z$) of the variables. To conclude, we have proved (7.3) for hull resolutions of monomial ideals in dimension 2 with $D = d$.

For a generic Artinian monomial ideal $M \subset \mathcal{O}_0^n$, $n \geq 3$ one can analogously define cuboids $P_{\alpha,s}$, where α is an outer corner of T_M and s is a permutation $s_1, \dots, s_n$ of $1, \dots, n$, such that for a fixed s , $\{P_{\alpha,s}\}_\alpha$ defines a partition of T_M and moreover

$$\frac{\partial \varphi_0}{\partial z_{s_1}} dz_{s_1} \circ \dots \circ \frac{\partial \varphi_{n-1}}{\partial z_{s_n}} dz_{s_n} = \sum_{\sigma \in \text{hull}(M)_{n-1}} \text{Vol}(P_{\alpha_{\sigma,s}}) z^{\alpha_{\sigma}} \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n} e_{\sigma}^* \otimes e_{\emptyset}.$$

Together with Theorem 1.1 this proves (7.4) and thus (7.3) in this case. However, for $n \geq 3$, the construction of the $P_{\alpha,s}$ is more delicate than that of P_i and Q_i , see [29].

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