SEGRE NUMBERS, A GENERALIZED KING FORMULA, AND LOCAL INTERSECTIONS

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Abstract. Let $\mathcal{J}$ be an ideal sheaf on a reduced analytic space X with zero set Z. We show that the Lelong numbers of the restrictions to Z of certain generalized Monge-Ampère products $(dd^c \log|f|^2)^k$, where f is a tuple of generators of $\mathcal{J}$, coincide with the so-called Segre numbers of $\mathcal{J}$, introduced independently by Tworzewski, Achilles-Manaresi, and Gaffney-Gassler. More generally we show that these currents satisfy a generalization of the classical King formula that takes into account fixed and moving components of Vogel cycles associated with $\mathcal{J}$. A basic tool is a new calculus for products of positive currents of Bochner-Martinelli type. We also discuss connections to intersection theory.

1. Introduction

Let X be a reduced analytic space of pure dimension n and let $\mathcal{J}$ be a coherent ideal sheaf on X. Given a point $x \in X$, Tworzewski, [24], Achilles–Manaresi, [1], and Gaffney–Gassler, [10], independently introduced numbers $e_0(\mathcal{J}, X, x), \dots, e_n(\mathcal{J}, X, x)$ that in a certain sense measure the singularities of $\mathcal{J}$ at x and that generalize the classical Hilbert-Samuel multiplicity. Following [10] we will call them *Segre numbers*; Tworzewski used the term *extended index of intersection* whereas Achilles-Manaresi used the term *multiplicity sequence*. The definition in [24] goes via a local variant of the Stückrad-Vogel construction, [23], introduced in [24, 17], and a closely related, also geometric, procedure is used in [10]. In [1] the definition is purely algebraic; it is based on Hilbert functions of bigraded rings. It is proved in [2] that the definitions in [24, 1, 10] yield the same numbers; see also [18]. In this paper we give a (semi-)global representation of these numbers as the Lelong numbers of certain positive closed currents, constructed from a tuple of generators of $\mathcal{J}$. This is part of our main result Theorem 1.1, which is a generalizion of King's formula, [11, 13], for these currents.

Let us first recall the definition in [24]. In that paper X is a subvariety of a smooth manifold Y and $\mathcal{J}$ is the pullback to X of the sheaf associated with a smooth submanifold $A \subset Y$. However, we find it advantageous to avoid any ambient space and instead consider an arbitrary coherent ideal sheaf $\mathcal{J} \to X$. A sequence $h = (h_1, h_2, \dots, h_n)$ in the local ideal $\mathcal{J}_x$ is called a *Vogel sequence of $\mathcal{J}$ at x* if there is a neighborhood $\mathcal{U} \subset X$ of x where the h_j are defined, such that

$$\operatorname{codim}\big[(\mathcal{U} \setminus Z) \cap (|H_1| \cap \ldots \cap |H_k|)\big] = k \text{ or } \infty,\ k = 1, \dots, n; \tag{1.1}$$

here Z is the (reduced) zero set of $\mathcal{J}$ and $|H_\ell|$ are the supports of the divisors H_ℓ defined by the h_ℓ. Notice that, possibly after shrinking $\mathcal{U}$, the common zero set $\{h = 0\}$ in $\mathcal{U}$ equals $Z \cap \mathcal{U}$ and that if $f_0, \dots, f_m$ generate $\mathcal{J}_x$, then any generic

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sequence of n linear combinations of the f_j is a Vogel sequence at x. Let $X_0 = X$ and let X_0^Z denote the irreducible components of X_0 that are contained in Z and let $X_0^{X\setminus Z}$ be the remaining components[1], so that

$$X_0 = X_0^Z + X_0^{X\setminus Z}.$$

By the Vogel condition (1.1), H_1 intersects $X_0^{X\setminus Z}$ properly. Set

$$X_1 = H_1 \cdot X_0^{X\setminus Z} \tag{1.2}$$

and decompose analogously X_1 into the components X_1^Z contained in Z and the remaining components $X_1^{X\setminus Z}$, so that $X_1 = X_1^Z + X_1^{X\setminus Z}$. Define inductively $X_{k+1} = H_{k+1} \cdot X_k^{X\setminus Z}$, X_{k+1}^Z, and $X_{k+1}^{X\setminus Z}$. Then

$$V^h := X_0^Z + X_1^Z + \cdots + X_n^Z$$

is the *Vogel cycle*[2] associated with the Vogel sequence h. Let V_k^h denote the components of V^h of codimension k, i.e., $V_k^h = X_k^Z$. Tworzewski defines the extended index of intersection as $\min_{\rm lex}(\mathrm{mult}_x V_0^h, \dots, \mathrm{mult}_x V_n^h)$, where the $\min_{\rm lex}$ is taken over all Vogel sequences h of $\mathcal{J}_x$.

Let us next recall the definition of Segre numbers in [10], where also so-called polar multiplicities are introduced. Let f be a tuple of generators $f_0, \dots, f_m$ of $\mathcal{J}_x$ and let h be a Vogel sequence of linear combinations $h_j = \alpha_j \cdot f = \alpha_j^0 f_0 + \cdots + \alpha_j^m f_m$; notice that any Vogel sequence is on this form for some choice of generators and α_j. It is proved in [10, Section 2], see also Section 6 below, that the multiplicities $\mathrm{mult}_x V_k^h$ and $\mathrm{mult}_x X_k^{X\setminus Z}$ are independent of α_j for generic choices of α_j and also independent of f, and these numbers are the *Segre numbers*, $e_k(x) = e_k(\mathcal{J}, X, x)$, and *polar multiplicities*, $m_k(x) = m_k(\mathcal{J}, X, x)$, respectively. Throughout we will use this definition of the list $e(x) = e(\mathcal{J}, X, x) = \big(e_0(x), \dots, e_n(x)\big)$ of Segre numbers. In Section 8 below we prove that the definitions in [24] and [10] coincide, i.e.,

$$e(x) = \min_{\rm lex} \big(\mathrm{mult}_x V_0^h, \dots, \mathrm{mult}_x V_n^h\big). \tag{1.3}$$

It is not clear to us whether this coincidence has been explicitly stated in the literature before. In [2], both notions are discussed and are proved to coincide with Achilles-Manaresi's multiplicity sequence for the restrictive class of sheaves considered in [24]. The equivalence (1.3) also follows from [21, Theorem 3.3] in combination with [10, Lemma 2.2].

We remark that both definitions above are local. Indeed, the Vogel condition (1.1) as well as the genericity of α_j depend on x, cf., Remark 1.4. Also the algebraic definition in [1] is local.

Let f be a tuple of generators of the ideal sheaf $\mathcal{J}$. For $k = 0, 1, 2, \dots, n$ we consider the closed positive currents M_k^f introduced in [4]. The current M_k^f coincides with $\mathbf{1}_Z (dd^c \log|f|^2)^k$, where $\mathbf{1}_Z$ is the characteristic function for Z and

$$(dd^c \log|f|^2)^k := \lim_{\epsilon\to 0} \big(dd^c \log(|f|^2 + \epsilon)\big)^k; \tag{1.4}$$

[1] In [10], X_0^Z is empty by assumption, but for us it is convenient not to exclude the possibility that $\mathcal{J}$ vanishes identically on some irreducible component of X.

[2] The notion Vogel cycle was introduced by Massey [16, 17]. For a generic choice of Vogel sequence the associated Vogel cycle coincides with the *Segre cycle* introduced by Gaffney and Gassler, [10], see Lemma 2.2 in [10].

for $k \le \operatorname{codim}\{f=0\}$ it is well-known that the definition (1.4) coincides with the standard one. Notice that $(dd^c \log|f|^2)^0 = 1$ and hence, $M_0^f = \mathbf{1}_Z$ is the current of integration over the components of X that are contained in Z; in particular, it vanishes unless $f \equiv 0$ on some irreducible component of X. See Section 4 for other expressions for M_k^f. Our main result is the following.

Theorem 1.1 (Generalized King's formula)**.** *Let X be a reduced analytic space of pure dimension n and let $\mathcal{J}$ be a coherent ideal sheaf on X generated by a tuple f of holomorphic functions. Let Z be the zero set of $\mathcal{J}$ and let Z_j^k be the distinguished varieties of $\mathcal{J}$ of codimension k. Then*

$$M_k^f = \mathbf{1}_Z (dd^c \log|f|^2)^k = \sum_j \beta_j^k [Z_j^k] + N_k^f =: S_k^f + N_k^f, \quad k = 0, \dots, n, \tag{1.5}$$

where the β_j^k are positive integers, the N_k^f are positive closed currents, the Lelong numbers $\ell_x(N_k^f)$ are nonnegative integers that only depend on the integral closure class of $\mathcal{J}$ at x, and the set where $\ell_x(N_k^f) \ge 1$ has codimension at least $k+1$. The Lelong number of M_k^f at x is the Segre number $e_k(\mathcal{J}, X, x)$. The polar multiplicity $m_k(\mathcal{J}, X, x)$ coincides with the Lelong number at x of the current $\mathbf{1}_{X\setminus Z}(dd^c \log|f|^2)^k$.

Here $[Z_j^k]$ denotes the Lelong current[3], i.e., the current of integration, associated with the variety Z_j^k. Recall that the *integral closure* of $\mathcal{J}_x$ consists of all holomorphic germs ϕ such that $|\phi| \le C|f|$ for some $C > 0$ at x. For the definition of the (Fulton-MacPherson) distinguished varieties Z_j^k at x associated to $\mathcal{J}_x$, see Section 7 below. It turns out that $S^f = \sum_k S_k^f = \sum_{jk} \beta_j^k Z_j^k$ is precisely the cycle that appears in all Vogel cycles obtained from generic (enough) Vogel sequences. It is called the *fixed* part in [10]. The remaining parts of the Vogel cycles vary with the Vogel sequence and are called the *moving* parts. Notice that (1.5) is the Siu decomposition, [20], of M_k^f.

Note that, contrary to the previous local definitions of Segre numbers, Theorem 1.1, gives a (semi-)global representation of the Segre numbers

$$e_k(\mathcal{J}, X, x) = \ell_x(M_k^f). \tag{1.6}$$

Moreover, the M_k^f are obtained as limits of explicit expressions in generators of $\mathcal{J}$.

It follows from Lemma 2.2 that $M_k^f = 0$ if $k < \operatorname{codim} Z$ and that $N_{\operatorname{codim} Z}^f = 0$. The case $k = \operatorname{codim} Z$ of (1.5) is precisely the classical King formula, [11, 13].

Remark 1.2. If $\mathcal{J}_x$ is generated by $p < n$ functions $f_0, \dots, f_{p-1}$, we will see that $M_k^f = 0$ for $k > p$ and hence $e_k = 0$ for $k > p$. However, M_k^g may be non-vanishing if g is another, larger, set of generators. If, in addition, $\operatorname{codim} Z_x = p$, i.e., $\mathcal{J}_x$ is a complete intersection, then $e_p(\mathcal{J}, X, x)$ is the only non-zero entry in $e(\mathcal{J}, X, x)$. This number is the classical intersection number of the proper intersection of the divisors of the p generators f_j, see, e.g., [7]. □

Corollary 1.3. *If $\mathcal{J}$ is the radical ideal of a variety Z of pure codimension p, then $M_p^f = [Z]$.*

[3]We will often identify a cycle with its corresponding Lelong current.

Remark 1.4. Assume that x is a point where $n_k(\mathcal{J}, X, x) \geq 1$ for some k and let V^h be a generic Vogel cycle so that $\mathrm{mult}_x V_k^h = e_k(\mathcal{J}, X, x)$. Then $V_k^h = S_k^f + W$ where the moving part W is a positive cycle of codimension k, such that $\mathrm{mult}_x W = n_k(\mathcal{J}, X, x)$. Since $n_k(\mathcal{J}, X, y) \geq 1$ only on a set of codimension $\geq k+1$, at most points y on V_k^h we have that $e_k(\mathcal{J}, X, y) = \mathrm{mult}_y(S_k^f)$ and hence $\mathrm{mult}_y V_k^h > e_k(\mathcal{J}, X, y)$. As soon as there is a moving part at x it is thus impossible to find a Vogel cycle that represents the Segre numbers in a whole neighborhood of x. □

A fundamental ingredient in this paper is a current calculus described in Sections 3 to 5. It gives an expedient analytic approach to Vogel cycles; for instance, it becomes a straightforward matter to form mean values of (the Lelong currents of) such cycles. The currents M_k^f are in fact such mean values, see Section 6; this is the intuitive idea behind Theorem 1.1. The current calculus is fundamental for the proof of Theorem 1.1, which is given in Section 7, and it makes it possible to provide a proof of (1.3) in our slightly more general setting of a general sheaf $\mathcal{J}$ than what was considered in [24], see Section 8. Our current calculus is also useful for concrete computations of Segre numbers, see Section 11. In Section 9 we prove a certain invariance property of Segre numbers. The motivation in [24] for introducing these numbers was to develop a new intersection theory. In Section 10 we discuss some local aspects of connections to intersection theory. Our technique to form new currents by averaging Vogel cycles will be the starting point in a forthcoming paper where we will study a kind of global intersection products.

Remark 1.5. This paper is a shortened and slightly elaborated version of [5]; in that paper can be found, additionally, a discussion of global intersections in the sense of Tworzewski and various examples. □

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2. Preliminaries

Let us fix some notation. Throughout this paper X is a reduced analytic space of pure dimension n and $\mathcal{J}$ is a coherent ideal sheaf on X. Given a tuple $f = (f_0, \dots, f_m)$ of holomorphic functions on an analytic space we will use $\mathcal{J}(f)$ to denote the sheaf it generates. Similarly if $W \subset X$ is an analytic subset we will use $\mathcal{J}_W$ to denote the radical sheaf. We will denote the local ring of germs of holomorphic functions at x in X by $\mathcal{O}_{X,x}$. We say that a sequence $g_1, \dots, g_m$ of functions on an analytic space X is a *geometrically regular sequence* if $\mathrm{codim}\,\{g_1 = \dots = g_k = 0\} = k$ for $1 \leq k \leq m$. If X is smooth (or Cohen-Macaulay) a sequence is geometrically regular if and only if it is regular.

Though less natural at first sight it is often computationally more convenient to use regularizations based on analytic continuation rather than smooth regularizations as in (1.4). For instance, if h is a holomorphic function on X then

$$\lambda \mapsto \bar\partial |h|^{2\lambda} \wedge \frac{\partial \log |h|^2}{2\pi i},$$

a priori defined for $\mathrm{Re}\,\lambda \gg 0$, has a current-valued analytic continuation to a neighborhood of 0 and the value at 0 is the integration current associated to the divisor

defined by h. In general, if $\alpha(\lambda)$ is a current-valued function, defined in a neighborhood of 0, we let $\alpha(\lambda)|_{\lambda=0}$ denote the value at $\lambda = 0$.

2.1. **Positive currents.** Let $d^c = (4\pi i)^{-1}(\partial - \bar\partial)$ so that $dd^c = (2\pi i)^{-1}\bar\partial\partial$. We briefly recall some basic facts about positive currents, referring to [7, 8] for details. Let μ be a positive current of bidegree (k,k) defined in some open set $\Omega \subset \mathbb{C}^N$. Then μ has order zero, so that the restriction $\mathbf{1}_S\mu$ is well-defined for any Borel set $S \subset \Omega$. If in addition μ is closed and S is analytic, then the Skoda-El Mir theorem asserts that $\mathbf{1}_S\mu$ is closed as well. If μ is closed then one can define inductively

$$(dd^c \log|z-x|^2)^{j+1}\wedge\mu = dd^c\big(\log|z-x|^2 dd^c((\log|z-x|^2)^j\wedge\mu)\big),$$

$(dd^c \log|z-x|^2)^{N-k}\wedge\mu$ is a (N,N)-current. Its mass at x is the *Lelong number* $\ell_x(\mu)$ at x of μ, which depends semi-continuously on μ, in the sense that

$$\ell_x(\mu) \geq \limsup_{j\to\infty} \ell_x(\mu_j) \tag{2.1}$$

if $\mu_j \to \mu$. It follows that $x \mapsto \ell_x(\mu)$ is upper semi-continuous.

Lemma 2.1. *If μ is a closed positive (k,k)-current in $\Omega \subset \mathbb{C}^N$, then*[4]

$$\ell_0(\mu)[0] = \lim_{\lambda\to 0^+}\Big(\bar\partial|z|^{2\lambda}\wedge\frac{\partial|z|^2}{2\pi i|z|^2}\wedge(dd^c\log|z|^2)^{N-k-1}\wedge\mu\Big). \tag{2.2}$$

If $k = N$, then the right hand side of (2.2) shall be interpreted as

$$\lim_{\lambda\to 0^+}\left(1-|z|^{2\lambda}\right)\mu = \mathbf{1}_{\{0\}}\mu,$$

so Lemma 2.1 is trivially true in this case.

Sketch of proof. If ξ is a test function, then

$$\int (dd^c\log|z|^2)^{N-k}\wedge\mu\wedge\xi = \lim_{\lambda\to 0^+}\int\frac{|z|^{2\lambda}-1}{\lambda}\wedge(dd^c\log|z|^2)^{N-k-1}\wedge\mu\wedge dd^c\xi. \tag{2.3}$$

After an integration by parts, the right-hand side of (2.3) may be rewritten as

$$\lim_{\lambda\to 0^+}\int\bar\partial|z|^{2\lambda}\wedge\frac{\partial|z|^2}{2\pi i|z|^2}\wedge(dd^c\log|z|^2)^{N-k-1}\wedge\mu\wedge\xi + \lim_{\lambda\to 0^+}\int|z|^{2\lambda}(dd^c\log|z|^2)^{N-k}\wedge\mu\wedge\xi.$$

The second term is precisely the action of $\mathbf{1}_{\mathbb{C}^N\setminus\{0\}}(dd^c\log|z|^2)^{N-k}\wedge\mu$ on ξ, and consequently the point mass at 0 of $(dd^c\log|z|^2)^{N-k}\wedge\mu$ is the same as the point mass at 0 of the first term, which proves (2.2). □

[4] By slight abuse of notation we will write [0], instead of the formally more correct [{0}], to denote point evaluation at 0.

2.2. **Currents on an analytic space.** [5] Let X be a reduced analytic space of pure dimension n. Given a local embedding $i\colon X \hookrightarrow \mathbb{C}^N$, we let $\mathcal{E}_X$ be the sheaf of smooth forms on X, obtained from the sheaf of smooth forms in the ambient space, where two forms are identified if their pullbacks to X_{reg} coincide; it is well-known that this definition does not depend on the particular embedding. We say that μ is a *current on X of bidegree* (p,q) if it acts on test forms on X of bidegree $(n-p, n-q)$. Such currents μ are naturally identified with currents $\tau = i_*\mu$ of bidegree $(N-n+p, N-n+q)$ in the ambient space such that τ vanish on the kernel of i^*. Observe that the d-operator is well-defined on currents on X. If W is a subvariety of X of pure codimension $p \geq 0$, then

$$\phi \mapsto [W].\phi = \int_{W_{\rm reg}} \phi$$

is a closed (p,p)-current on X; this is the *current of integration* over W.

Recall that a current ν is *normal* if both ν and $d\nu$ have order zero. The following lemma follows immediately from the corresponding one in $\mathbb{C}^N$.

Lemma 2.2. *Suppose that μ is a normal current of bidegree (p,p) on X that has support on a subvariety W of codimension k. If $k > p$ then $\mu = 0$. If $k = p$ and μ is closed, then $\mu = \sum_j \alpha_j [W_j]$ for some numbers α_j, where W_j are the irreducible components of W of codimension p.*

It is readily checked that if we have a proper holomorphic mapping $\nu\colon X' \to X$ between analytic spaces, then the push-forward ν_* is well-defined on currents on X'.

Assume that μ is a positive closed current on the analytic space X. Fix $x \in X$ and let $i\colon X \hookrightarrow \mathbb{C}^N$ be a local embedding. We define the Lelong number $\ell_x(\mu)$ as $\ell_x(i_*\mu)$. After a suitable change of coordinates i can be factorized as $i = j \circ i'$, where $i'\colon X \to \mathbb{C}^M$ is a minimal embedding and j is the natural embedding $\mathbb{C}^M \to \mathbb{C}^M \times \mathbb{C}^{N-M}$. Since the Lelong number is invariant under holomorphic changes of coordinates, all minimal embeddings are equal up to a holomorphic change of variables, and $\ell_x(\tau) = \ell_x(j_*\tau)$, it follows that $\ell_x(\mu)$ is well-defined. Thus if Z is a subvariety of an analytic space X and we have an embedding $X \hookrightarrow \mathbb{C}^N$, then the number $\ell_x([Z])$ is independent of whether we consider $[Z]$ as the Lelong current of Z on X or on $\mathbb{C}^N$.

Recall that if Z is a variety in $\mathbb{C}^N$, then the *multiplicity* $\mathrm{mult}_x Z$ of Z at x coincides with the Lelong number $\ell_x([Z])$, see [7, Prop. 3.15.1.2]; here $\mathrm{mult}_x Z$ is defined as in [7, Ch. 2.11.1]. In particular, the Lelong number of the function 1, considered as a current on an analytic space X, at x is precisely $\mathrm{mult}_x X$.

The classical *Siu decomposition*, [20], of positive closed currents extends immediately to currents on our analytic space X. Let μ be a positive closed (p,p)-current on X; then there is a unique decomposition

$$\mu = \sum_i \beta_i [W_i] + N,$$

where W_i are irreducible analytic varieties of codimension p, $\beta_i \geq 0$, and, for each $\delta > 0$, the set where $\ell_x(N) \geq \delta$ is analytic and has codimension strictly larger than p.

[5] For a more detailed exposition of currents on an analytic space we refer to [12, Section 4.2].

2.3. **Cycles and Lelong currents.** Given an analytic cycle $Z = \sum \alpha_j W_j$, where W_j are varieties, we let $[Z] = \sum \alpha_j [W_j]$ be the associated Lelong current. We will often identify analytic cycles with their Lelong currents. We let $|Z|$ denote the *support* of Z, and we let $\mathbf{1}_Z$ mean $\mathbf{1}_{|Z|}$. If H is a Cartier divisor defined by (a germ of) a holomorphic function h, we will (sometimes) use the notation $[h]$ for $[H]$ and $\mathbf{1}_h$ for $\mathbf{1}_{|H|}$. Given an analytic cycle $Z = \sum \alpha_i W_i$ of pure dimension, the *multiplicity* of Z at x is defined as $\sum \alpha_i \mathrm{mult}_x W_i$ (this definition follows [10, p. 704]). It follows that

$$\mathrm{mult}_x Z = \ell_x([Z]).$$

If $Z = \sum_{k=0}^{n} Z_k$, where Z_k is an analytic cycle of codimension k we define

$$\mathrm{mult}_x Z := (\mathrm{mult}_x Z_0, \dots, \mathrm{mult}_x Z_n) \tag{2.4}$$

Throughout this paper all analytic cycles are effective, unless otherwise stated.

2.4. **Proper intersections.** Let Y be a complex manifold and let $Z_1, \dots, Z_r$ be (effective) analytic cycles in Y of pure codimensions p_j, $j = 1, \dots, r$, that intersect properly, i.e., the intersection V of their supports has codimension $p_1 + \cdots + p_r$. There is a well-defined cycle, called the (proper) intersection of the Z_j,

$$Z_r \cdots Z_1 = \sum m_j V_j, \tag{2.5}$$

where V_j are the irreducible components of V and m_j are certain positive integers. One can obtain these numbers m_j by defining the intersection number $i(x)$, algebraically or geometrically, at each fixed point x of V, and prove that $i(x)$ is generically constant on each V_j, see, e.g., [7]. However, by means of currents, (2.5) can be obtained in a more direct way: By an appropriate regularization one can define the wedge product $[Z_r] \wedge \cdots \wedge [Z_1]$, see, e.g., [7, 8], and this current indeed coincides with the Lelong current of $Z_r \cdots Z_1$. In particular, if the Z_j are (effective) divisors defined by holomorphic functions h_j, then the Lelong current of the intersection can be obtained explicitly as

$$[Z_r \cdots Z_1] = \lim_{\epsilon \to 0} \bigwedge dd^c \log(|h_j|^2 + \epsilon). \tag{2.6}$$

At each point x there is a well-defined intersection number

$$\epsilon(x) := \sum_j m_j \mathrm{mult}_x(V_j);$$

here $\mathrm{mult}_x(V_j)$ is the multiplicity of the variety V_j at x. The number $\epsilon(x)$ is precisely equal to the Lelong number $\ell_x([Z_r \cdots Z_1])$ of the positive closed current $[Z_r \cdots Z_1]$.

3. Multiplying a Lelong current by a Cartier divisor

In this section we will describe how the inductive construction of a Vogel cycle V^h can be expediently expressed as certain products of Lelong currents. Notice that the map $[W] \mapsto \mathbf{1}_Z [W]$ is linear. Notice also that if Z, Z' are analytic cycles in X, then

$$\mathbf{1}_{Z'}[Z] = [Z^{Z'}]; \tag{3.1}$$

recall that $Z^{Z'}$ denotes the irreducible components of Z that are contained in Z'. To see (3.1) we may assume that Z is irreducible. If $|Z|$ is contained in $|Z'|$, then $\mathbf{1}_{Z'}[Z] = [Z]$. Otherwise, $|Z| \cap |Z'|$ has higher codimension than $|Z|$, and thus $\mathbf{1}_{Z'}[Z]$ vanishes by Lemma 2.2. Notice that $\mathbf{1}_Z$ is 1 on the components of X that are contained in Z and 0 otherwise, i.e., it is the Lelong current of X^Z.

If h is a non-vanishing holomorphic function on (each irreducible component of) the analytic space Z, then $\log|h|^2$ is a well-defined $(0,0)$-current on Z. This is clear if Z is smooth and follows in general, e.g., by means of a smooth resolution $\widetilde{Z} \to Z$, cf., the proof below.

Lemma 3.1. *Let Z be an analytic cycle in X, h be a holomorphic function, and let u be a nonvanishing smooth function on X. Then*

$$\lambda \mapsto \bar\partial|uh|^{2\lambda} \wedge \frac{\partial \log|uh|^2}{2\pi i} \wedge [Z], \tag{3.2}$$

a priori defined when $\operatorname{Re}\lambda$ is large, has an analytic continuation to a half-plane $\operatorname{Re}\lambda > -\epsilon$. The value at $\lambda = 0$ is independent of u.

If h does not vanish identically on any irreducible component of (the support of) Z, then this value is equal to $dd^c(\log|h|^2\,[Z])$.

Notice that $v^\lambda := \bar\partial|uh|^{2\lambda} \wedge \partial\log|uh|^2/(2\pi i)$ has continuous coefficients when $\operatorname{Re}\lambda > 1$, so the product in (3.2) is then well-defined.

Proof. First assume that $Z = X = \mathbb{C}^N$ and h is a monomial $h = z_1^{a_1}\cdots z_N^{a_N}$. Then (3.2) is equal to

$$v^\lambda = \bar\partial|uz_1^{a_1}\cdots z_N^{a_N}|^{2\lambda} \wedge \frac{1}{2\pi i}\Big[\sum_1^N a_j\frac{dz_j}{z_j} + \frac{\partial|u|^2}{|u|^2}\Big].$$

One can check that the desired analytic continuation exists, and that the value at $\lambda = 0$ is the current $\sum_1^N a_j[z_j] = dd^c\log|h|^2$; in particular, it is independent of u.

Consider now the general case. By linearity, we may assume that Z is irreducible. If h vanishes identically on Z and $\operatorname{Re}\lambda$ is large, then $v^\lambda \wedge [Z] = 0$, and thus it trivially extends to $\lambda \in \mathbb{C}$. Assume that h does not vanish identically on Z. Let $i\colon Z \hookrightarrow X$ be an embedding and let $\pi\colon \widetilde{Z} \to Z$ be a smooth modification of Z such that π^*i^*h is locally a monomial; such a modification exists due to Hironaka's theorem on resolution of singularities. After a partition of unity we are back to the case above. It follows that $\pi^*i^*v^\lambda$ has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ for some $\epsilon > 0$ and thus $v^\lambda\wedge[Z] = i_*\pi_*(\pi^*i^*v^\lambda)$ has the desired analytic continuation. The value at $\lambda = 0$ is equal to

$$i_*\pi_*(dd^c\log|\pi^*i^*h|^2)$$

which proves the second statement, since $(\log|h|^2)[Z] = i_*\pi_*(\log|\pi^*i^*h|^2)$. □

Let H denote the Cartier divisor defined by h. We define $[H]\wedge[Z]$ as the value of (3.2) at $\lambda = 0$. According to the lemma it does not depend on the particular choice of h defining H. It follows from the definition that

$$[H]\wedge([Z_1]+[Z_2]) = [H]\wedge[Z_1] + [H]\wedge[Z_2] \tag{3.3}$$

and thus $[Z] \mapsto [H]\wedge[Z]$ is a linear operator on Lelong currents, cf., (3.1). However, in general it is *not* true that $([H_1]+[H_2])\wedge[Z] = [H_1]\wedge[Z]+[H_2]\wedge[Z]$ or $[H_1]\wedge[H_2] = [H_2]\wedge[H_1]$.

Since (3.2) is analytic by Lemma 3.1 it follows that $[H]\wedge\mathbf{1}_H[Z] = 0$, and so, using (3.3), we get

$$[H]\wedge[Z] = [H]\wedge\big(\mathbf{1}_H[Z] + \mathbf{1}_{X\setminus H}[Z]\big) = [H]\wedge\mathbf{1}_{X\setminus H}[Z].$$

Now, $\mathbf{1}_{X\setminus H}[Z]$ is the current of integration over $Z^{X\setminus H}$, i.e., the components of Z that are not contained in $|H|$. Since H and $Z^{X\setminus H}$ intersect properly it follows that $[H]\wedge\mathbf{1}_{X\setminus H}[Z] = [H\cdot Z^{X\setminus H}]$, cf. Section 2.4. Summarizing, we get the computation rules

$$[H]\wedge[Z] = [H]\wedge\mathbf{1}_{X\setminus H}[Z] = [H\cdot Z^{X\setminus H}]. \tag{3.4}$$

Remark 3.2. It is important to emphasize that $[H]\wedge[Z]$ is *not* the same as (the Lelong current associated with) the intersection $H\cdot Z$ in [9]. In fact, if Z is irreducible and contained in H, then $[H]\wedge[Z] = 0$, whereas in [9] the product is a cycle in Z of codimension 1 that is well-defined up to rational equivalence. □

Example 3.3. Let H_1 and H_2 be Cartier divisors and let $H = H_1 + H_2$. Then $[H_1]\wedge[H] = [H_1]\wedge[H_2]$ but $[H]\wedge[H_1] = 0$. Moreover $[H_1]\wedge\mathbf{1}_{H_1}[H] = [H_1]\wedge[H_1] = 0$ but $\mathbf{1}_{H_1}[H_1]\wedge[H] = \mathbf{1}_{H_1}[H_1]\wedge[H_2] = [H_1]\wedge[H_2]$. □

We can construct Vogel cycles, cf., Section 1, by inductively applying operators $\mathbf{1}_Z$ and $[H]\wedge$.

Proposition 3.4. *Let X be an analytic space of dimension n and let $h = (h_1,\dots,h_n)$ be a Vogel sequence of an ideal $\mathcal{J}$ with variety Z at $x\in X$, with corresponding divisors $H_1,\dots,H_n$. Then on X,*

$$[X_0] = 1, \quad [X_\ell] = [H_\ell]\wedge\cdots\wedge[H_1]\,,\ \ell = 1,\dots,n \tag{3.5}$$

and

$$[X_0^Z] = \mathbf{1}_Z, \quad [X_\ell^Z] = \mathbf{1}_Z[H_\ell]\wedge\cdots\wedge[H_1],\ \ell = 1,\dots,n. \tag{3.6}$$

In particular,

$$[V^h] = \mathbf{1}_Z + \mathbf{1}_Z[H_1] + \mathbf{1}_Z[H_2]\wedge[H_1] + \cdots + \mathbf{1}_Z[H_n]\wedge\cdots\wedge[H_1]. \tag{3.7}$$

If we consider X as embedded in some larger analytic space Y, then we have instead

$$[X_0] = [X], \quad [X_\ell] = [H_\ell]\wedge\cdots\wedge[H_1]\wedge[X],\ \ell = 1,\dots,n$$

and

$$[X_0^Z] = \mathbf{1}_Z[X], \quad [X_\ell^Z] = \mathbf{1}_Z[H_\ell]\wedge\cdots\wedge[H_1]\wedge[X],\ \ell = 1,\dots,n$$

Proof. In view of (3.1), (3.6) follows from (3.5). Using (3.3), we have, in view of (1.2), that

$$[X_1] = [H_1]\wedge[X_0^{X\setminus Z}] = [H_1]\wedge([X_0] - [X_0^Z]) = [H_1]$$

since $[H_1]\wedge[X_0^Z] = [H_1]\mathbf{1}_Z = 0$. One obtains (3.5) by induction. □

4. Bochner-Martinelli currents

Let $f = (f_0,\dots,f_m)$ be a tuple of holomorphic functions on X that generates $\mathcal{J}$ and let Z be the zero set of $\mathcal{J}$. For $\operatorname{Re}\lambda \gg 0$, let

$$M_0^{f,\lambda} := 1 - |f|^{2\lambda}$$

$$M_k^{f,\lambda} := \bar\partial|f|^{2\lambda}\wedge\frac{\partial\log|f|^2}{2\pi i}\wedge(dd^c\log|f|^2)^{k-1} \text{ if } k\geq 1,$$

and

$$M^{f,\lambda} := \sum_{k=0}^{\infty} M_k^{f,\lambda}, \tag{4.1}$$

where $|f|^2 = \sum_{j=0}^m |f_j|^2$. The sum in (4.1) is finite for degree reasons, and when $\operatorname{Re}\lambda \gg 0$, $M^{f,\lambda}$ is locally integrable. We will show that $\lambda \mapsto M_k^{f,\lambda}$ has a current-valued analytic continuation to $\operatorname{Re}\lambda > -\epsilon$, for some $\epsilon > 0$. We denote the value of $M_k^{f,\lambda}$ at $\lambda = 0$ by M_k^f and we write $M^f := \sum_k M_k^f$. The current M^f and its components M_k^f will be referred to as *Bochner-Martinelli currents*, cf. Remark 4.2 below.

A computation yields that

$$M_k^{f,\lambda} = \lambda \frac{i}{2\pi} \frac{\partial|f|^2 \wedge \bar\partial|f|^2}{|f|^{4-2\lambda}} \wedge (dd^c \log|f|^2)^{k-1}$$

which is positive when $\lambda > 0$, and thus M_k^f is a positive current. Note that M_0^f is the current of integration over the components of X, on which $f \equiv 0$. In particular, if f does not vanish identically on any component of X, then $M_0^f = 0$.

Let $\pi\colon \widetilde{X} \to X$ be a normal modification such that the pull-back ideal sheaf $\mathcal{J}\cdot\mathcal{O}_{\widetilde{X}}$ is principal; for instance one can take the normalization of the blow-up of X along $\mathcal{J}$. Then $\pi^* f = f^0 f'$ where f^0 is a section of the holomorphic line bundle $L \to \widetilde{X}$ corresponding to the exceptional divisor D_f of $\pi\colon \widetilde{X} \to X$, i.e., the divisor defined by $\mathcal{J}\cdot\mathcal{O}_{\widetilde{X}}$, and f' is a nonvanishing tuple of sections of L^{-1}. Let L be equipped with the metric defined by $|f^0|_L = |\pi^* f| = |f^0 f'|$, and let

$$\omega_f := dd^c \log|f'|^2; \tag{4.2}$$

here the right hand side is computed locally for any local trivialization of L^{-1}. Then $-\omega_f$ is the first Chern form of $(L, |\cdot|_L)$, and clearly $\omega_f \geq 0$.

Since $\log|\pi^* f|^2 = \log|f^0|^2 + \log|f'|^2$ it follows from the Poincare-Lelong formula that

$$dd^c \log|\pi^* f|^2 = [D_f] + \omega_f. \tag{4.3}$$

In particular, $\pi^*(dd^c \log|f|^2) = \omega_f$ outside $\pi^{-1}\{f = 0\}$. Therefore, for $\operatorname{Re}\lambda \gg 0$,

$$\pi^* M_0^{f,\lambda} = 1 - |f^0 f'|^{2\lambda} \tag{4.4}$$

$$\pi^* M_k^{f,\lambda} = (2\pi i)^{-1} \bar\partial |f^0 f'|^{2\lambda} \wedge \partial \log|f^0 f'|^2 \wedge \omega_f^{k-1},\ k \geq 1. \tag{4.5}$$

Now Lemma 3.1 asserts that $\lambda \mapsto \pi^* M_k^{f,\lambda}$ has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ and since $M_k^{f,\lambda} = \pi_*\pi^* M_k^{f,\lambda}$ for $\operatorname{Re}\lambda \gg 0$, it follows that $\lambda \mapsto M_k^{f,\lambda}$ has the desired analytic continuation. Moreover

$$M_0^f = M_0^{f,\lambda}|_{\lambda=0} = \pi_*(\pi^* M_0^{f,\lambda}|_{\lambda=0}) = \pi_*(\mathbf{1}_{D_f}) = \mathbf{1}_{\{f=0\}}\,. \tag{4.6}$$

$$M_k^f = M_k^{f,\lambda}|_{\lambda=0} = \pi_*(\pi^* M_k^{f,\lambda}|_{\lambda=0}) = \pi_*([D_f]\wedge\omega_f^{k-1}),\ k \geq 1. \tag{4.7}$$

Following for example [4] one can check that for $k \geq 1$,

$$M_k^f = \mathbf{1}_Z (dd^c \log|f|^2)^k \tag{4.8}$$

and

$$\mathbf{1}_{X\setminus Z}(dd^c \log|f|^2)^k = \pi_*(\omega_f^k).$$

It is not hard to see that $M_k^{f,\lambda}$ is locally integrable for $\operatorname{Re}\lambda > 0$ and that $M_k^{f,\lambda} \to M_k^f$ as measures when $\lambda \to 0^+$.

Remark 4.1. For future reference, let g be a tuple of holomorphic functions such that $|g| \sim |f|$, i.e., there exists $C \in \mathbb{R}$ such that $|f|/C \leq |g| \leq C|f|$, and let $\pi\colon \widetilde{X} \to X$ be a normal modification such that both $\mathcal{J}(f) \cdot \mathcal{O}_{\widetilde{X}}$ and $\mathcal{J}(g) \cdot \mathcal{O}_{\widetilde{X}}$ are principal. Then $|f^0 f'| \sim |g^0 g'|$ and since f' and g' are non-vanishing it follows that f^0 and g^0 define the same divisor on $\widetilde{X}$. Therefore the corresponding negative Chern forms ω_f and ω_g are dd^c-cohomologous, i.e., there is a global smooth function γ such that $dd^c\gamma = \omega_f - \omega_g$. □

By combining [4, Proposition 3.2] and [22, Corollary 4] it follows that

$$M_k^f = \lim_{\epsilon \to 0} \frac{\epsilon (dd^c|f|^2)^k}{(|f|^2+\epsilon)^{k+1}}. \tag{4.9}$$

Remark 4.2. Given a tuple f as above, associated residue currents of Bochner-Martinelli type were introduced in [19]. Let E be a trivial vector bundle with basis elements $e_0, \dots, e_m$ and consider $f = f_0 e_0 + \cdots + f_m e_m$ as a section of the dual bundle E^* with basis elements e_j^*. Following [3] one can define a residue current $R^f = R_0^f + \cdots + R_n^f$, where R_k^f is a current of bidegree $(0,k)$ with values in the exterior product $\Lambda^k E$, such that the coefficients in R^f are precisely the currents in [19]. It is proved in [4] that

$$M_k^f = R^f \cdot (df)^k/(2\pi i)^k k!,$$

where $\cdot$ denotes the natural contraction. For more details, see, e.g., [4]. □

5. Products of Bochner-Martinelli currents

Given tuples $f_1, \dots, f_r$ of holomorphic functions in X, we will give meaning to the product

$$M^{f_r} \wedge \cdots \wedge M^{f_1} \tag{5.1}$$

of Bochner-Martinelli currents. The construction is recursive. Assume that $M^{f_\ell} \wedge \cdots \wedge M^{f_1}$ is defined; it follows from the proof of Proposition 5.2 below that

$$\lambda \mapsto M^{f_{\ell+1},\lambda} \wedge M^{f_\ell} \wedge \cdots \wedge M^{f_1} \tag{5.2}$$

is holomorphic for $\operatorname{Re}\lambda > -\epsilon$, where $\epsilon > 0$. Set

$$M^{f_{\ell+1}} \wedge M^{f_\ell} \dots \wedge M^{f_1} := M^{f_{\ell+1},\lambda} \wedge M^{f_\ell} \wedge \dots \wedge M^{f_1}\big|_{\lambda=0}. \tag{5.3}$$

We define the products $M_{k_r}^{f_r} \wedge \cdots \wedge M_{k_1}^{f_1}$ in the analogous way so that

$$M^{f_r} \wedge \cdots \wedge M^{f_1} = \sum_{k_r,\dots,k_1 \geq 0} M_{k_r}^{f_r} \wedge \cdots \wedge M_{k_1}^{f_1}. \tag{5.4}$$

Notice that if the f_j are single functions, then $M^{f_j} = \mathbf{1}_{f_j} + [f_j]$ and

$$M^{f_r} \wedge \cdots \wedge M^{f_1} = (\mathbf{1}_{f_r} + [f_r]) \wedge \cdots \wedge (\mathbf{1}_{f_1} + [f_1]); \tag{5.5}$$

cf. Section 3.

Proposition 5.1. *If $h = (h_1, \dots, h_n)$ is a Vogel sequence of some ideal at x, then*

$$M^{h_n} \wedge \cdots \wedge M^{h_1} = [V^h].$$

Proof. Let Z be the zero set of the ideal generated by h. In light of Lemma 2.2,

$$\mathbf{1}_{h_\ell}\cdots\mathbf{1}_{h_{k+1}}[h_k]\wedge\cdots\wedge[h_1]=\mathbf{1}_Z[h_k]\wedge\cdots\wedge[h_1].$$

Thus, by (3.4), $[h_{\ell+1}]\wedge\mathbf{1}_{h_\ell}\cdots\mathbf{1}_{h_{k+1}}[h_k]\wedge\cdots\wedge[h_1]=0$. Hence, in view of (5.5),

$$M^{h_n}\wedge\cdots\wedge M^{h_1}=\sum_{k=0}^{n}\mathbf{1}_{h_n}\cdots\mathbf{1}_{h_{k+1}}[h_k]\wedge\cdots\wedge[h_1]=\sum_{k=0}^{n}\mathbf{1}_Z[h_k]\wedge\cdots\wedge[h_1]; \tag{5.6}$$

here we have used that $[h_n]\wedge\cdots\wedge[h_1]$ has support on Z. Now, Proposition 3.4 asserts that the right hand side of (5.6) is equal to $[V^h]$. □

In contrast to the definition of products of residue currents of Bochner-Martinelli type introduced in [25], the recursively defined products (5.1) are *not* commutative in general, not even if the tuples just consist of one single function. For instance, in $\mathbb{C}^2_{x,y}$ we have that $M^{xy}\wedge M^y=0$, whereas $M^y\wedge M^{xy}=[0]$, cf., Example 3.3. Various approaches to recursively defined products of residue currents are investigated in [14].

Proposition 5.2. *Let $f_1,\dots,f_r$ be tuples of holomorphic functions in X, with common zero set $Z=\{f_1=\ldots=f_r=0\}$. Then the current $M^{f_r}\wedge\cdots\wedge M^{f_1}$, defined by* (5.3)*, is positive and has support on Z.*

Let $\pi\colon\widetilde{X}\to X$ be a normal modification such that the sheaves $\mathcal{J}(f_\ell)\cdot\mathcal{O}_{\widetilde{X}}$ are principal for $\ell=1,\dots r$. As in Section 4, let D_{f_ℓ} and ω_{f_ℓ} be the corresponding divisors and negative Chern forms, respectively. Then

$$M^{f_r}_{k_r}\wedge\ldots\wedge M^{f_1}_{k_1}=\pi_*\big([D_{f_r}]\wedge\cdots\wedge[D_{f_1}]\wedge\omega_{f_r}^{k_r-1}\wedge\cdots\wedge\omega_{f_1}^{k_1-1}\big), \tag{5.7}$$

where, if $k_\ell=0$, the factor $[D_{f_\ell}]$ shall be replaced by $\mathbf{1}_{D_{f_j}}$ and the factor $\omega_{f_\ell}^{k_\ell-1}$ shall be removed.

Assume that $g_1,\dots g_r$ are tuples of holomorphic functions in X such that $|g_\ell|\sim|f_\ell|$ for $\ell=1,\dots,r$. Then there is a normal current T with support on Z such that

$$dd^cT=M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}-M^{g_r}_{k_r}\wedge\cdots\wedge M^{g_1}_{k_1}\,. \tag{5.8}$$

Proof. Iteratively using Lemma 3.1, the computation rules (3.4), and (4.4)–(4.7) we see that the desired analytic continuation of (5.2) exists and that (5.7) holds. It follows that $M^{f_r}_{k_r}\wedge\ldots\wedge M^{f_1}_{k_1}$ has its support contained in $\pi(|D_{f_r}|\cap\cdots\cap|D_{f_1}|)=Z$. Moreover $M^{f_r}_{k_r}\wedge\ldots\wedge M^{f_1}_{k_1}$ is the push-forward of a product of positive $(1,1)$-currents and positive forms, and hence it is positive.

To prove the last part, it suffices to change one of the f_ℓ to g_ℓ with $|g_\ell|\sim|f_\ell|$. First notice that then $M^{f_\ell}_0=\mathbf{1}_{f_\ell}=\mathbf{1}_{g_\ell}=M^{g_\ell}_0$. Let us then assume that $k_\ell\geq 1$, and that the modification π is chosen so that also $\mathcal{J}(g_\ell)\cdot\mathcal{O}_{\widetilde{X}}$ is principal. By Remark 4.1, there is a smooth global function γ on $\widetilde{X}$ such that $\omega_{f_\ell}-\omega_{g_\ell}=dd^c\gamma$ and thus we can find a smooth global form w such that $dd^cw=\omega_{f_\ell}^{k_\ell-1}-\omega_{g_\ell}^{k_\ell-1}$. Let

$$T:=\pi_*\big(\tau_r\wedge\cdots\wedge\tau_{\ell+1}\wedge[D_{f_\ell}]\wedge w\wedge\tau_{\ell-1}\wedge\cdots\wedge\tau_1\big),$$

where $\tau_j=\mathbf{1}_{D_{f_j}}$ if $k_j=0$ and $\tau_j=[D_{f_j}]\wedge\omega_{f_j}^{k_j-1}$ otherwise. Then T satisfies (5.8). Note that $\tau_r\wedge\cdots\wedge\tau_{\ell+1}\wedge[D_{f_\ell}]\wedge w\wedge\tau_{\ell-1}\wedge\cdots\wedge\tau_1$ is normal, and since normality is preserved under push-forward, so is T. □

We also define products of Bochner-Martinelli currents and Lelong currents. If $f_1, \dots, f_r$ are tuples of holomorphic functions in X and Z is an analytic subset of X, we define recursively $M^{f_1}\wedge[Z] := M^{f_1,\lambda}\wedge[Z]\big|_{\lambda=0}$, and

$$M^{f_{k+1}}\wedge\cdots\wedge M^{f_1}\wedge[Z] := M^{f_{k+1},\lambda}\wedge M^{f_k}\wedge\cdots\wedge M^{f_1}\wedge[Z]\big|_{\lambda=0}.$$

By arguments as in the proof of Proposition 5.2 one can prove that the desired analytic continuations exist, and thus $M^{f_r}\wedge\cdots\wedge M^{f_1}\wedge[Z]$ is well-defined. It is readily checked that if $i\colon Z \hookrightarrow X$, then, for any $k_1, ..., k_r \in \mathbb{N}$,

$$M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}\wedge[Z] = i_*[M^{i^*f_r}_{k_r}\wedge\cdots\wedge M^{i^*f_1}_{k_1}]. \tag{5.9}$$

Moreover, if $Z = Z' + Z''$, $Z'' \subset \{f_j = 0\}$, and $k_j > 0$, then one checks that

$$M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}\wedge[Z] = M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}\wedge[Z'], \tag{5.10}$$

cf. (the first equality of) (3.4).

For future reference, note that if f is a tuple of holomorphic functions on the analytic space X then

$$M^f = M^f\mathbf{1}_X = \sum_j M^f\mathbf{1}_{X_j}, \tag{5.11}$$

where X_j are the irreducible components of X.

Proposition 5.3. *Let $f_1, \dots, f_r$ be tuples of holomorphic functions in X and let ξ be a tuple of holomorphic functions such that $\{\xi = 0\} = \{x\}$, where $x \in X$. Then*

$$M^\xi\wedge M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1} = M^\xi_{n-k}\wedge M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1} = \alpha[x], \tag{5.12}$$

where $k = k_1 + \cdots + k_r$ and α is a non-negative integer. If ξ generates the maximal ideal at $x \in X$, then $\alpha = \ell_x\big(M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}\big)$.

Proof. By Proposition 5.2, $M^\xi_{n-k}\wedge M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}$ is positive and has support at x, and thus by Lemma 2.2 it is of the form $\alpha[x]$ for some non-negative α. Let $\pi : \widetilde{X} \to X$ be a normal modification such that $\mathcal{J}(f_\ell)\cdot\mathcal{O}_{\widetilde{X}}$ and $\mathcal{J}(\xi)\cdot\mathcal{O}_{\widetilde{X}}$ are principal. Let us use the notation from Section 4. Then, from (5.7), we see that α is an intersection number and hence an integer.

Now assume that ξ generates the maximal ideal at x and that $i\colon X \hookrightarrow \mathbb{C}^N$ is a local embedding such that $i(x) = 0$, so that $i_*[x] = [0]$. By the second part of Proposition 5.2 we may assume that $f_j = i^*F_j$ and $\xi = i^*z$ for some tuples F_j and the standard coordinate system $z = (z_1, \dots, z_N)$ in $\mathbb{C}^N$. Then

$$i_*(M^\xi_{n-k}\wedge M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}) = M^z_{n-k}\wedge M^{F_r}_{k_r}\wedge\cdots\wedge M^{F_1}_{k_1}\wedge[X], \tag{5.13}$$

cf. (5.9). By Lemma 2.1, the right hand side of (5.13) is precisely the Lelong number of $M^{F_r}_{k_r}\wedge\cdots\wedge M^{F_1}_{k_1}\wedge[X]$ at 0 in $\mathbb{C}^N$ times $[0]$. □

Proposition 5.4. *The Lelong number at x of $M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}$ is unchanged if we replace f_j by g_j such that $|f_j| \sim |g_j|$.*

Proof. It follows from the second part of Proposition 5.2, applied to $f_1, \dots, f_r, \xi$ since then T, which has bidegree $(n-1, n-1)$, must vanish by Lemma 2.2. □

One can replace all the evaluations in the definition of the product by one single evaluation in the following way; for the proof see [6].

Proposition 5.5. *Assume that μ_j are strictly positive integers such that $\mu_1 > \mu_2 > \ldots > \mu_r$. Then $\lambda \mapsto M_{k_r}^{f_r,\lambda^{\mu_r}} \wedge \cdots \wedge M_{k_1}^{f_1,\lambda^{\mu_1}}$ is holomorphic in a neighborhood of the half-axis $[0,\infty)$ in $\mathbb{C}$ and*

$$M_{k_r}^{f_r} \wedge \cdots \wedge M_{k_1}^{f_1} = M_{k_r}^{f_r,\lambda^{\mu_r}} \wedge \cdots \wedge M_{k_1}^{f_1,\lambda^{\mu_1}}\big|_{\lambda=0}\,. \tag{5.14}$$

In view of Proposition 5.1 we get

Corollary 5.6. *If $h_1,\ldots,h_n$ is a Vogel sequence of some ideal at some point x and μ_j are as in Proposition 5.5, then the Lelong current of the associated Vogel cycle is given as the value at $\lambda = 0$ of the function*

$$\lambda \mapsto \bigwedge_{k=1}^{n} M^{h_k,\lambda^{\mu_k}} = \bigwedge_{k=1}^{n} \big(1 - |h_k|^{2\lambda^{\mu_k}} + \bar\partial |h_k|^{2\lambda^{\mu_k}} \wedge \partial \log |h_k|^2/2\pi i\big)$$

6. Bochner-Martinelli currents and Vogel cycles

For a tuple $f = (f_0,\ldots,f_m)$ of holomorphic functions and $\beta = [\beta_0 : \ldots : \beta_m] \in \mathbb{P}^m$ we write $\beta \cdot f := \beta_0 f_0 + \cdots + \beta_m f_m$. Note that $M^{\beta\cdot f}$ only depends on $\beta \in \mathbb{P}^m$ and not on the choice of homogeneous coordinates. Our first result in this section relates Lelong numbers of Bochner-Martinelli currents to multiplicities of Vogel cycles.

Theorem 6.1. *Let $f = (f_0,\ldots,f_m)$ be a tuple of holomorphic functions in X, pick $x \in X$, and let $Z = \{f = 0\}$. Then for $k \geq 0$, and a generic choice of $\alpha = (\alpha_1,\ldots,\alpha_k) \in (\mathbb{P}^m)^k$,*

$$\ell_x\big(\mathbf{1}_Z [\alpha_k \cdot f] \wedge \cdots \wedge [\alpha_1 \cdot f]\big) = \ell_x(M_k^f)\,. \tag{6.1}$$

Here the current on the left hand side of (6.1) *should be interpreted as $\mathbf{1}_Z$ if $k = 0$.*

Assume that $f = (f_0,\ldots,f_m)$ generates the ideal sheaf $\mathcal{J}$. Then $\alpha_1 \cdot f,\ldots,\alpha_n \cdot f$ is a Vogel sequence for a generic choice of $\alpha = (\alpha_1,\ldots,\alpha_n) \in (\mathbb{P}^m)^n$. By Theorem 6.1 and Proposition 3.4,

$$\mathrm{mult}_x V_k^{\alpha\cdot f} = \ell_x(M_k^f) \tag{6.2}$$

and by Proposition 5.4 the right hand side only depends on $\mathcal{J}$ and not on the particular choice of generators f; in fact, it only depends on the integral closure of $\mathcal{J}$. Thus this gives an independent proof of Gaffney-Gassler's result, [10, Section 2], that the multiplicities of Vogel cycles V^h are independent of h for generic h, which guarantees that the Segre numbers are well-defined. Also, (1.6) immediately follows from (6.2).

Proof. Choose a normal modification $\pi\colon \widetilde{X} \to X$ such that $\mathcal{J}(f)\cdot\mathcal{O}_{\widetilde{X}}$ is principal; we will use the notation from Section 4. Assume moreover that the pullback of the maximal ideal at x is principal, and let D_ξ and ω_ξ be the corresponding divisor and Chern form, obtained from a tuple ξ that defines the maximal ideal at x.

Let W be any irreducible subvariety of $\widetilde{X}$. Since f' is nonvanishing on W it follows that for β outside a hypersurface in $\mathbb{P}^m$, the section $\beta \cdot f'$ is not vanishing identically on W. By induction it follows that there is a Zariski-open dense subset $A \subset (\mathbb{P}^m)^n$ such that for each $\alpha = (\alpha_1,\ldots,\alpha_n) \in A$, the sequence $\alpha_1 \cdot f',\ldots,\alpha_n \cdot f'$ is a geometrically regular sequence on each component of $\widetilde{X}$, $|D_f|$, $|D_\xi|$, and on the support of $[D_\xi]\wedge[D_f]$.

Since $\pi^*(\alpha_\ell \cdot f) = f^0\, \alpha_\ell \cdot f'$, we have that $[\alpha_\ell \cdot f] = \pi_*\big([D_f] + [\alpha_\ell \cdot f']\big)$ and if $\alpha \in A$, in light of (3.4), thus

$$[\alpha_2 \cdot f]\wedge[\alpha_1 \cdot f] = \pi_*\big([D_f]\wedge[\alpha_1 \cdot f'] + [\alpha_2 \cdot f']\wedge[\alpha_1 \cdot f']\big).$$

By induction,

$$[\alpha_k \cdot f]\wedge\cdots\wedge[\alpha_1 \cdot f] = \pi_*\big([D_f]\wedge[\alpha_{k-1} \cdot f']\wedge\cdots\wedge[\alpha_1 \cdot f'] + [\alpha_k \cdot f']\wedge\cdots\wedge[\alpha_1 \cdot f']\big), \tag{6.3}$$

and so

$$\mathbf{1}_Z[\alpha_k \cdot f]\wedge\cdots\wedge[\alpha_1 \cdot f] = \pi_*\big([D_f]\wedge[\alpha_{k-1} \cdot f']\wedge\cdots\wedge[\alpha_1 \cdot f']\big)\,. \tag{6.4}$$

Here we have used that $\mathbf{1}_{D_f}[\alpha_k \cdot f']\wedge\cdots\wedge[\alpha_1 \cdot f']$ vanishes by Lemma 2.2, and that

$$\mathbf{1}_Z\,(\pi_*\tau) = \pi_*(\mathbf{1}_{D_f}\tau). \tag{6.5}$$

For $k = 0, 1$, (6.1) follows from (4.6), (4.7) and (6.4); in fact, the currents in (6.1) coincide in these cases. Let us now assume that $k \geq 2$. We claim that there is a normal current $\mathcal{A}_k$ such that

$$dd^c\mathcal{A}_k = [D_\xi]\wedge\omega_\xi^{n-k-1}\wedge[D_f]\wedge\big(\omega_f^{k-1} - [\alpha_{k-1} \cdot f']\wedge\cdots\wedge[\alpha_1 \cdot f']\big). \tag{6.6}$$

For $\ell = 1, \dots, k$, $\log|\alpha_\ell \cdot f'|^2$ defines a singular metric on L^{-1} with first Chern form $[\alpha_\ell \cdot f']$, cf., (4.2), and thus $[\alpha_\ell \cdot f']$ is dd^c-cohomologous to ω_f. More precisely, $c_\ell := \log(|f'|^2/|\alpha_\ell \cdot f'|^2)$ is a global current on $\widetilde{X}$ and $\omega_f - [\alpha_\ell \cdot f'] = dd^c c_\ell$. Now, let

$$\mathcal{A}_k := [D_\xi]\wedge\omega_\xi^{n-k-1}\wedge[D_f]\wedge\sum_{\ell=1}^{k-1}\omega_f^{k-\ell-1}\wedge c_\ell\wedge[\alpha_{\ell-1} \cdot f']\wedge\cdots\wedge[\alpha_1 \cdot f']\,.$$

Then $\mathcal{A}_k$ is normal. Since $\alpha_\ell \cdot f'$ does not vanish identically on any irreducible component of (the support of) $[D_\xi]\wedge[D_f]\wedge[\alpha_{\ell-1} \cdot f']\wedge\cdots\wedge[\alpha_1 \cdot f']$ it follows from Lemma 3.1 and the discussion after the proof of that lemma that (6.6) holds. From (the proof of) Proposition 5.3 and (6.4) we get

$$dd^c\pi_*(\mathcal{A}_k) = \big(\ell_x(M_k^f) - \ell_x(\mathbf{1}_Z[\alpha_k \cdot f]\wedge\cdots\wedge[\alpha_1 \cdot f])\big)[x]. \tag{6.7}$$

On the other hand, $\pi_*\mathcal{A}_k$ is a normal $(n-1, n-1)$-current, and so since it has support at x, it vanishes according to Lemma 2.2. □

Our next result concerns mean values of Bochner-Martinelli currents. In particular, it says that M^f can be represented as a mean value of Vogel cycles.

Theorem 6.2. *Assume that $f = (f_0, \dots, f_m)$ is a tuple of holomorphic functions on X. Then*

$$M_k^f = \int_{\alpha\in(\mathbb{P}^m)^k} \mathbf{1}_Z[\alpha_k \cdot f]\wedge\cdots\wedge[\alpha_1 \cdot f] \tag{6.8}$$

where $Z = \{f = 0\}$. Moreover, if $\nu \geq \min(m+1, n+1)$, then

$$M^f = \int_{\alpha=(\alpha_1,\dots,\alpha_\nu)\in(\mathbb{P}^m)^\nu} M^{\alpha_\nu\cdot f}\wedge\cdots\wedge M^{\alpha_1\cdot f}. \tag{6.9}$$

For the proof we will use the following lemma which is a simple variant of Crofton's formula that should be well-known so we omit the proof, see also [5].

Lemma 6.3. *If ϕ is a non-vanishing holomorphic $(m+1)$-tuple on X, then, in the sense of currents,*

$$\int_{\beta\in\mathbb{P}^m} [\beta\cdot\phi]d\sigma(\beta) = dd^c\log|\phi|^2,$$

where $d\sigma$ is the normalized Fubini-Study metric.

Proof of Theorem 6.2. We use the same notation as in the proof of Theorem 6.1. In view of Lemma 6.3 and (4.2) we have that

$$\int_{\beta\in\mathbb{P}^m} [\beta\cdot f']d\sigma(\beta) = \omega_f. \tag{6.10}$$

Since all currents are positive we can apply Fubini's theorem and get (6.8) from (6.4) by repeated use of (6.10), cf., (4.7).

We now prove (6.9). By (4.6) and (4.7),

$$M^{\alpha_\ell\cdot f} = M_0^{\alpha_\ell\cdot f} + M_1^{\alpha_\ell\cdot f} = \mathbf{1}_{\alpha_\ell\cdot f} + [\alpha_\ell\cdot f].$$

As in the proof of Proposition 5.1 we get, for generic $(\alpha_1,\dots,\alpha_\nu)\in(\mathbb{P}^m)^\nu$, that

$$M^{\alpha_\nu\cdot f}\wedge\cdots\wedge M^{\alpha_1\cdot f} = \sum_{j=0}^{\nu} \mathbf{1}_Z[\alpha_j\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f] + \mathbf{1}_{X\setminus Z}[\alpha_\nu\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f].$$

Moreover, it follows from (6.3) and (6.4) that

$$\mathbf{1}_{X\setminus Z}[\alpha_\nu\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f] = \pi_*\big([\alpha_\nu\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']\big).$$

Hence, using (6.8) and Lemma 6.3, we conclude that

$$\int_{\alpha=(\alpha_1,\dots,\alpha_\nu)\in(\mathbb{P}^m)^\nu} M^{\alpha_\nu\cdot f}\wedge\cdots\wedge M^{\alpha_1\cdot f} \;=\; M^f \;+\; \pi_*(dd^c\log|f'|^2)^\nu \;=\; M^f;$$

indeed, $(dd^c\log|f'|^2)^\nu = 0$ since $\nu\geq\min(m+1,n+1)$. □

By arguments as in the proof of Theorem 6.2 one can check that

$$\int_{\alpha\in(\mathbb{P}^m)^k} \mathbf{1}_{X\setminus Z}[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f] = \mathbf{1}_{X\setminus Z}(dd^c\log|f|^2)^k. \tag{6.11}$$

7. Proof of the generalized King formula (Theorem 1.1)

Let X and $\mathcal{J}$ be as in Theorem 1.1 and let Z be the variety of $\mathcal{J}$. The *(Fulton-MacPherson) distinguished varieties* associated with $\mathcal{J}$ are defined in the following way, cf., [9]: Let $\nu\colon X^+\to X$ be the normalization of the blow-up of X along $\mathcal{J}$ and let E be the exceptional divisor of ν. Then $Z_j\subset X$ is a distinguished variety if it is the image under ν of an irreducible component of E. Let Z_j^k be the distinguished varieties of codimension k. Also, we define the irreducible components of X contained in Z to be distinguished varieties (of codimension 0).

Let us first consider the case $k=0$. By (5.11) we may assume that X is irreducible. Then either $\mathcal{J}=(0)$ or Z is a proper subvariety of X. In the first case $M_0^f = M_0^0 = \mathbf{1}_X$ and if h is a Vogel sequence of $\mathcal{J}$, then necessarily $h=(0,\dots,0)$ and so $V^h = V_0^h = X$. In the second case $M_0^f = 0$ and if h is a Vogel sequence of $\mathcal{J}$, then $V_0^h = X_0^Z = 0$, since $X\not\subset Z$. It follows that Theorem 1.1 holds for $k=0$.

Next, consider the case $k\geq 1$. Let $\pi\colon\widetilde{X}\to X$ be a normal modification such that $\mathcal{J}\cdot\mathcal{O}_{\widetilde{X}}$ is principal. We use the notation from Section 4, so that $M_k^f = \pi_*([D]\wedge\omega_f^{k-1})$,

where $D = D_f$. Moreover, we let D^k denote the components of D that are mapped to sets of codimension k in X. Note that $D = D^p + \ldots + D^n$, if $p = \operatorname{codim} Z$.

If $\ell > k$, then $\pi_*([D^\ell]\wedge\omega_f^{k-1})$ is a positive closed (k,k)-current with support on a variety of codimension $\ell > k$, and hence it must vanish in view of Lemma 2.2. Thus

$$M_k^f = S_k^f + N_k^f, \tag{7.1}$$

where

$$S_k^f = \pi_*\big([D^k]\wedge\omega_f^{k-1}\big), \quad N_k^f = \pi_*\Big(\sum_{\ell<k}[D^\ell]\wedge\omega_f^{k-1}\Big). \tag{7.2}$$

Note that $M_k^f = 0$ for $k < p$ and $N_p^f = 0$. We claim that (7.1) is the Siu decomposition of M_k^f, cf., Section 2.2. By Lemma 2.2, S_k^f is the Lelong current of a cycle of codimension k, so it is enough to show that N_k^f does not carry any mass on varieties of codimension k. Let $W \subset X$ be such a variety. By (6.5),

$$\mathbf{1}_W\pi_*([D^\ell]\wedge\omega_f^{k-1}) = \sum_j \pi_*(\mathbf{1}_{\pi^{-1}W}[D_j^\ell]\wedge\omega_f^{k-1}), \tag{7.3}$$

where D_j^ℓ are the irreducible components of D^ℓ. Then, since $\ell < k$, $\pi^{-1}(W)$ does not contain any component D_j^ℓ, thus each term in the right hand side of (7.3) vanishes, and thus the claim follows.

Since (7.1) is the Siu decomposition of M_k^f, it follows that S_k^f is independent of $\pi : \widetilde{X} \to X$. If we take π to be the normalization of the blow-up of $\mathcal{J}$, we see that the Z_j^k in (1.5) has to be among the distinguished varieties of $\mathcal{J}$. By Proposition 5.3 (for $r = 1$), the Lelong number of M_k^f is an integer at each point, and since the Lelong number of N_k^f generically vanishes on each Z_j^k, we conclude that the β_j^k and $\ell_x(N_k^f)$ are integers. That $\ell_x(N_k^f)$ is an integer can also be seen directly by copying the proof of Proposition 5.3. Moreover, cf., Proposition 5.4, β_j^k and $\ell_x(N_k^f)$ only depend on the integral closure of $\mathcal{J}$ at x.

We shall now see that the coefficients β_j^k of the distinguished varieties are, in fact, ≥ 1, following the proof of Corollary 5.4.19, in [15]. The blow-up $\pi_{\mathcal{J}}\colon \operatorname{Bl}_{\mathcal{J}}X \to X$ of X along $\mathcal{J}$ can be seen as the subvariety of $X \times \mathbb{P}_t^m$ defined by the equations $t_j f_k - t_k f_j = 0$, where $0 \leq j < k \leq m$. Moreover, the line bundle associated with the exceptional divisor is the pullback of $\mathcal{O}_{\mathbb{P}_t^m}(-1)$ from $\mathbb{P}^m$ to $\operatorname{Bl}_{\mathcal{J}}X$, so $\omega_t = dd^c\log|t|^2$ represents minus its first Chern class. This form is strictly positive on the fibers of $\pi_{\mathcal{J}}$, and since the normalization $X^+ \to \operatorname{Bl}_{\mathcal{J}}X$ is a finite map, the pullback ω of ω_t to X^+ remains strictly positive on the fibers of $\nu\colon X^+ \to X$ as well. Let E_j be one of the irreducible component of the exceptional divisor of ν. We conclude that $\nu_*([E_j]\wedge\omega^{k-1})$ is a positive integer times $[Z_j^k]$, where $Z_j^k := \nu(E_j)$. On the other hand, this current is unaffected if we replace ω by ω_f since these two forms are first Chern forms of the same line bundle. It follows that $\beta_j^k \geq 1$.

We saw in the discussion after Theorem 6.1 that $\ell_x(M_k^f)$ is equal to the k:th Segre number of $\mathcal{J}$ at x. Next, we show that the fixed Vogel components of $\mathcal{J}$ are precisely the S_k^f. Fix a point $x \in X$. As in proof of Theorem 6.1 we can construct, for $k \geq 1$ and a generic $\alpha \in (\mathbb{P}^m)^n$, a normal current $\mathcal{A}_k$ with support on $|D^k|$ such that

$$dd^c\mathcal{A}_k = [D^k]\wedge([\alpha_{k-1}\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f'] - \omega_f^{k-1}).$$

Now $\pi_*\mathcal{A}_k$ is a normal $(k-1,k-1)$-current with support on $\bigcup_j Z_j^k$, and thus it vanishes by Lemma 2.2. It follows that $\pi_*\big([D^k]\wedge[\alpha_{k-1}\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']\big) = S_k^f$ and hence S_k^f occurs in a generic Vogel cycle at x, meaning that S_k^f is a fixed Vogel cycle. On the other hand, the cycles

$$\pi_*\Big(\sum_{\ell<k}[D^\ell]\wedge[\alpha_{k-1}\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']\Big) \tag{7.4}$$

must be moving. Indeed, by (the proof of) Theorem 6.2, taking mean values of (7.4) over all $\alpha\in(\mathbb{P}^m)^k$, we get the current N_k^f, which carries no mass on any variety of codimension k, as seen above.

By arguments as in the proof of Theorem 6.1 one shows that for a generic choice of $\alpha\in(\mathbb{P}^m)^k$,

$$\ell_x(\mathbf{1}_{X\setminus Z}[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f]) = \ell_x(\mathbf{1}_{X\setminus Z}(dd^c\log|f|^2)^k) \tag{7.5}$$

cf. (6.11). However, it follows from Proposition 3.4 that the left hand side of (7.5) is equal to $m_k(x)$. This concludes the proof of Theorem 1.1.

Remark 7.1. One can see more directly that only the distinguished varieties occur in S_k^f if S_k^f is defined by (7.2) from an arbitrary normal modification $\pi\colon\widetilde{X}\to X$. To begin with, π factors over ν, i.e., there exists a modification $\widetilde{\nu}\colon\widetilde{X}\to X^+$ such that $\pi=\nu\circ\widetilde{\nu}$. If ω_+ is the form associated with $\mathcal{J}\cdot\mathcal{O}_{X^+}$ in X^+, then $\tilde{\nu}^*\omega_+=\omega_f$.

Let D_j^k be an irreducible component of the divisor D^k. Since $|D_j^k|\subset\pi^{-1}(Z)$, it follows that $\widetilde{\nu}(|D_j^k|)$ is contained in one of the components E_j of E in X^+. If $\widetilde{\nu}(|D_j^k|)$ has codimension ≥ 1 in E_j, then $\widetilde{\nu}_*([D_j^k]\wedge\omega_f^{k-1}) = (\widetilde{\nu}_*[D_j^k])\wedge\omega_+^{k-1}$ vanishes by Lemma 2.2. Hence $\pi_*([D_j^k]\wedge\omega^{k-1}) = \nu_*\tilde{\nu}_*([D_j^k]\wedge\omega^{k-1})$ vanishes unless $\widetilde{\nu}(|D_j^k|)=E_j$, in which case $\pi(|D_j^k|)$ is a distinguished variety. □

Assume that $f_0,\dots,f_{p-1}$ is a regular sequence. From the theory for proper intersections we know that

$$[f_{p-1}]\wedge\cdots\wedge[f_0] = \sum\beta_j[Z_j]$$

where Z_j are the irreducible components of $Z=\{f=0\}$, and that the intersection only depends on the ideal generated by the f_j, cf., Remark 1.2. In particular, the right hand side is unaffected if we replace f_j by $\alpha_j\cdot f$ for generic α_j. From (6.8) we conclude that

$$M_p^f = [f_{p-1}]\wedge\cdots\wedge[f_0]. \tag{7.6}$$

Proof of Corollary 1.3. If Z is smooth, then locally there are coordinates (z,w) so that $Z=\{w_1=\cdots=w_p=0\}$. In view of (7.6) we have that

$$M_p^w = [w_p]\wedge\cdots\wedge[w_1] = [Z],$$

and hence $\ell_x(M_p^w)=1$ for $x\in Z$. If Z is reduced and f generates the ideal $\mathcal{J}_Z$, therefore $\ell_x(M_p^f)=1$ for all smooth points $x\in Z$ in view of Proposition 5.4. From Theorem 1.1 we know that $M_p^f=\sum\beta_j[Z_j]$. Since the smooth points are dense, we conclude that $\beta_j=1$ for each j. □

8. The minimality property

Recall that the *lexicographical order* on $\mathbb{R}^N$ is a total order, defined by $(x_1,\dots,x_N) \le_{\rm lex} (y_1,\dots,y_N)$ if there is an $1 \le \ell \le N$ such that $x_i = y_i$ for $i \le \ell$ and $x_\ell < y_\ell$. We let $\min_{\rm lex}$ denote the minimum with respect to the lexicographical order. We will now give a proof that Tworzewski's extended index of intersection coincides with the list of Segre numbers. More precisely we will prove:

Theorem 8.1. *Let $\mathcal{J}$ be a coherent ideal sheaf on X and let $e(x)$ be the list of associated Segre numbers at x. Then* (1.3) *holds, where the* $\min_{lex}$ *is taken over all Vogel sequences h of ideals with the same integral closure as $\mathcal{J}_x$.*

Moreover, if f is a tuple of generators of $\mathcal{J}$ (or any ideal with the same integral clousure as $\mathcal{J}$) then it suffices to take the $\min_{lex}$ *over all Vogel sequences of the form $\alpha\cdot f = (\alpha_1\cdot f,\cdots,\alpha_n\cdot f)$, where $\alpha = (\alpha_1,\dots,\alpha_n) \in (\mathbb{P}^m)^n$.*

As mentioned in the introduction, Theorem 8.1 is known in the case when $\mathcal{J}$ is the pullback to X of the radical sheaf of a smooth manifold A in some ambient space. For the proof of Theorem 8.1 we will need the following result; if Z is smooth this is precisely Theorem 3.4 in [24].

Proposition 8.2. *Assume that $(W_j)_{j\in\mathbb{N}}$ and W are subvarieties of X of pure dimension such that $\lim_{j\to\infty}[W_j] = [W]$ as currents on X. Let Z be a fixed subvariety of X, let x be a fixed point in Z, and assume that*

$$\ell_x(\mathbf{1}_Z[W]) \le \ell_x(\mathbf{1}_Z[W_j]). \tag{8.1}$$

for all j. Then there is a neighborhood U of x in X, in which $\lim_{j\to\infty}(\mathbf{1}_Z[W_j]) = \mathbf{1}_Z[W]$ and $\lim_{j\to\infty}(\mathbf{1}_{X\setminus Z}[W_j]) = \mathbf{1}_{X\setminus Z}[W]$.

Proof. Since the currents $[W_j]$ are positive and locally uniformly bounded, so are the currents $\mathbf{1}_Z[W_j]$. Thus, there is a subsequence $(\mathbf{1}_Z[W_{j_k}])_{k\in\mathbb{N}}$ converging to a positive closed current with support on $W\cap Z$. By Lemma 2.2 this current is the integration current $[V]$ for some effective cycle V (with possibly real coefficients). Since $[W_j] - \mathbf{1}_Z[W_j]$ is positive, so is $[W]-[V] = \lim_k([W_{j_k}] - \mathbf{1}_Z[W_{j_k}])$, and since $|V|\subset|Z|$, it follows that

$$[V] = \mathbf{1}_Z[V] \le \mathbf{1}_Z[W]. \tag{8.2}$$

By (8.1) and semicontinuity, (2.1), we have that

$$\ell_x(\mathbf{1}_Z[W]) \le \limsup_k(\ell_x(\mathbf{1}_Z[W_{j_k}])) \le \ell_x([V]) \le \ell_x(\mathbf{1}_Z[W]).$$

Thus $\ell_x(\mathbf{1}_Z[W]) = \ell_x([V])$, and combined with (8.2) and the fact that V and W are effective cycles, it follows that $[V] = \mathbf{1}_Z[W]$ in some neighborhood of x.

Since each subsequence of $(\mathbf{1}_Z[W_j])_{j\in\mathbb{N}}$ has a subsequence that tends to $\mathbf{1}_Z[W]$, it follows that $\lim_{j\to\infty}(\mathbf{1}_Z[W_j]) = \mathbf{1}_Z[W]$. The last statement follows by complementarity. □

Proof of Theorem 8.1. Since each Vogel sequence h can be realized as $\alpha\cdot f$ for some choice of f and α, it is easy to check that the first statement follows from the second one. Let f be a tuple of generators of $\mathcal{J}$. By definition, $e(x) = \mathrm{mult}_x V^{\alpha\cdot f}$ for almost all α, and thus it is enough to prove that $e(x) \le_{\rm lex} \min_{\rm lex}\mathrm{mult}_x V^{\alpha\cdot f}$ if $\alpha\cdot f$ is a Vogel sequence.

Suppose that $e(x) \not\le_{\rm lex} \min_{\rm lex}\mathrm{mult}_x V^{\alpha\cdot f}$. Then there is an r and a Vogel sequence $\alpha\cdot f$ such that $e_k(x) = \mathrm{mult}_x V_k^{\alpha\cdot f}$ for $k \le r-1$ but $\mathrm{mult}_x V_r^{\alpha\cdot f} < e_r(x)$. Since $\alpha\cdot f$

is a Vogel sequence of $\mathcal{J}$ for a generic choice of α, we can choose $(\alpha^j)_{j\in\mathbb{N}}$ in $(\mathbb{P}^m)^n$ such that $(\alpha^j)_{j\in\mathbb{N}} \to \alpha$ and such that $\alpha^j \cdot f$ is a Vogel sequence of $\mathcal{J}$ for each j, and moreover, by Theorem 6.1, such that $\mathrm{mult}_x V^{\alpha^j\cdot f} = e(x)$. It then follows that, for $k \le r$,

$$\ell_x(\mathbf{1}_Z[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f]) \le e_k(x) = \ell_x(\mathbf{1}_Z[\alpha_k^j\cdot f]\wedge\cdots\wedge[\alpha_1^j\cdot f]). \tag{8.3}$$

We claim that

$$\lim_{j\to\infty}[\alpha_k^j\cdot f]\wedge\cdots\wedge[\alpha_1^j\cdot f] = [\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f] \tag{8.4}$$

for $k \le r$. For instance by [7, Chapter 2, Corollary 12.3.4], (8.4) holds for $k = 1$. Assume now that it holds for $k < r$. Then by (8.3) and Proposition 8.2,

$$\lim_{j\to\infty}(\mathbf{1}_{X\setminus Z}[\alpha_k^j\cdot f]\wedge\cdots\wedge[\alpha_1^j\cdot f]) = \mathbf{1}_{X\setminus Z}[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f]. \tag{8.5}$$

Since $\alpha^j\cdot f$ and $\alpha\cdot f$ are Vogel sequences, the currents in (8.5) intersect properly with $[\alpha_{k+1}^j\cdot f]$ and $[\alpha_{k+1}\cdot f]$, respectively. In light of [7, Chapter 2, Corollary 12.3.4] or [24, Theorem 3.6], (8.4) holds for $k+1$, and the claim follows by induction.

Proposition 8.2 and (8.3) imply that

$$\lim_{j\to\infty}(\mathbf{1}_Z[\alpha_r^j\cdot f]\wedge\cdots\wedge[\alpha_1^j\cdot f]) = \mathbf{1}_Z[\alpha_r\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f]. \tag{8.6}$$

By semicontinuity, (2.1), the Lelong number of the right hand side of (8.6) is greater than or equal to the Lelong number of $\mathbf{1}_Z[\alpha_r^j\cdot f]\wedge\cdots\wedge[\alpha_1^j\cdot f]$. Thus, $\mathrm{mult}_x V_r^{\alpha\cdot f} \ge e_r(x)$, which gives a contradiction. Hence $\min_{\mathrm{lex}}\mathrm{mult}_x V^{\alpha\cdot f} = e(x)$. □

Given a positive closed current v, we define $\ell_x(v) := (\ell_{x0}, ..., \ell_{xn})$, where ℓ_{xk} denotes the Lelong number at x of the component of v of bidegree (k,k). If v and w are positive and closed, we let $v \le_x w$ mean that $\ell_x(v) \le_{\mathrm{lex}} \ell_x(w)$, and $v =_x w$ means that $\ell_x(v) = \ell_x(w)$. Observe that if h is a Vogel sequence of an ideal $\mathcal{J}_x$, then the zero sets of h and $\mathcal{J}_x$ coincide. If $f_1,\dots,f_n$ is Vogel sequence of an ideal $\mathcal{J}_x$, then in view of Theorems 1.1 and 8.1 and Proposition 5.1, $M^f \le_x M^{f_n}\wedge\ldots\wedge M^{f_1}$. In fact we have:

Proposition 8.3. *Let $f_1,\dots,f_s$ be a sequence of elements in $\mathcal{O}_{X,x}$ and let $f = (f_1,\dots,f_s)$. Then*

$$M^f \le_x M^{f_s}\wedge\ldots\wedge M^{f_1}. \tag{8.7}$$

Proof. Let $Z := \{f = 0\}$. In order to prove (8.7), we proceed by induction on the number s of functions. Clearly (8.7) holds for $s = 1$, so assume that it holds for $s-1$ instead of s. Let $\widetilde{f} := (f_2,\dots,f_s)$. By (5.11) we may assume that X is irreducible and that f_1 does not vanish identically on X, so that $M^{f_1} = M_1^{f_1} = [f_1]$; otherwise $M^{f_1} = M_0^{f_1} = \mathbf{1}_X$ and $M^f = M^{\widetilde{f}}$ and we are back in the case $s-1$.

Let $[W] := [f_1]$, and let $i_{W_j} : W_j \hookrightarrow X$ be the irreducible components of $W^{X\setminus Z}$. Theorem 6.1 asserts that for a generic choice of $\alpha \in (\mathbb{P}^{s-2})^{n-1}$, $\alpha\cdot\widetilde{f}$ is a Vogel sequence[6] of $\mathcal{J}(i_{W_j}^*\widetilde{f})$ and $M^{\alpha_{n-1}\cdot\widetilde{f}}\wedge\cdots\wedge M^{\alpha_1\cdot\widetilde{f}} =_x M^{\widetilde{f}}$ on each W_j, so that

[6] If $s = 2$, then $\tilde{f} = f_2$ and $\mathbb{P}^0$ should be interpreted as $\{1\}$.

$M^{\alpha_{n-1}\cdot\tilde{f}}\wedge\cdots\wedge M^{\alpha_1\cdot\tilde{f}}\wedge[W^{X\setminus Z}] =_x M^{\tilde{f}}\wedge[W^{X\setminus Z}]$, cf. the discussion after Lemma 2.2. By the induction hypothesis

$$M^{\tilde{f}}\wedge[W^{X\setminus Z}] \leq_x M^{f_s}\wedge\cdots\wedge M^{f_2}\wedge[W^{X\setminus Z}].$$

In view of (5.9) and (5.10), since $[f_1] = [W^Z] + [W^{X\setminus Z}]$ and $\tilde{f}$ vanishes on Z, we get

$$M^{\alpha_{n-1}\cdot\tilde{f}}\wedge\cdots\wedge M^{\alpha_1\cdot\tilde{f}}\wedge M^{f_1} \leq_x M^{f_s}\wedge\cdots\wedge M^{f_2}\wedge M^{f_1}. \tag{8.8}$$

For a generic choice of α, the sequence $f_1, \alpha_1\cdot\tilde{f}, \dots, \alpha_{n-1}\cdot\tilde{f}$ is a Vogel sequence of $\mathcal{J}(f)$. Thus, by Theorem 8.1,

$$M^f \leq_x M^{\alpha_{n-1}\cdot\tilde{f}}\wedge\cdots\wedge M^{\alpha_1\cdot\tilde{f}}\wedge M^{f_1}. \tag{8.9}$$

Combining (8.8) and (8.9), we get (8.7).

□

9. An invariance property

We have the following invariance property of Segre numbers. In the setting when $\mathcal{J}$ is the pullback to X of the radical sheaf of a smooth manifold in some ambient space this was proved in [21, Section 5].

Proposition 9.1. *Let $\mathcal{J}$ be an ideal sheaf on an analytic space X, let $\mathcal{J}'$ be the pullback of $\mathcal{J}$ to $X\times\mathbb{C}_w$ under the projection $(x,w)\mapsto x$, and let $i\colon X\hookrightarrow X\times\mathbb{C}_w$ be the embedding $x\mapsto(x,0)$. Then*

$$e_k(\mathcal{J}', X\times\{0\}, i(x)) = e_k(\mathcal{J}, X, x) \tag{9.1}$$

and

$$e_{k+1}(\mathcal{J}'+(w), X\times\mathbb{C}_w, i(x)) = e_k(\mathcal{J}, X, x). \tag{9.2}$$

For the proof we will use the following invariance of Bochner-Martinelli currents.

Lemma 9.2. *Let f be a tuple of holomorphic functions on X and let $i : X\hookrightarrow X\times\mathbb{C}_w$ be the embedding $x\mapsto(x,0)$. Then $M_0^{(f,w)} = 0$ and*

$$M_{k+1}^{(f,w)} = i_*M_k^f, \quad k\geq 0. \tag{9.3}$$

Moreover, if $W\subset X$ is an analytic variety,

$$M_k^{f\otimes 1}\wedge[W\times\{0\}] = i_*(M_k^f\wedge[W]). \tag{9.4}$$

If we consider X as embedded in some larger analytic space X' and $i\colon X'\to X'\times\mathbb{C}_w$, $x\mapsto(x,0)$, then (9.3) reads

$$M_{k+1}^{(f,w)}\wedge[X\times\mathbb{C}_w] = i_*(M_k^f\wedge[X]).$$

In particular, if $f=0$,

$$M_1^w\wedge[X\times\mathbb{C}_w] = i_*[X] = [X\times\{0\}]. \tag{9.5}$$

Proof. Let z be local coordinates on X. Since $(z,w)\mapsto(f(z),w)$ does not vanish identically on $X\times\mathbb{C}_w$, it follows that $M_0^{(f,w)} = 0$.

Let us now prove (9.3). First consider the case when $k=0$. By (5.11) we may assume that X is irreducible. Then either $f\equiv 0$ on X or the zero set of f has at least codimension 1 in X. In the first case

$$M_1^{(f,w)} = M_1^w = [w] = i_*1 = i_*M_0^0 = i_*M_0^f.$$

In the latter case the zero set of (f,w) has at least codimension 2 on $X\times\mathbb{C}_w$, and and so both sides of (9.3) vanish by Lemma 2.2. Thus (9.3) holds for $k=0$.

Next let $\pi\colon \widetilde{X}\to X$ be a smooth modification such that $\mathcal{J}\cdot\mathcal{O}_{\widetilde{X}}$ is principal and moreover f^0 is locally a monomial; use the notation from Section 4. Observe that then $\pi\otimes \mathrm{id}_w\colon \widetilde{X}\times\mathbb{C}_w\to X\times\mathbb{C}_w$ is a smooth modification with the same properties. It follows that it is enough to prove (9.3) in case X is smooth, $\mathcal{J}=(f^0)$ is principal and f^0 is (in local coordinates) a monomial.

In light of Section 4 we thus have to show that

$$(2\pi i)^{-1}\bar\partial(|f|^2+|w|^2)^\lambda\wedge\partial\log(|f|^2+|w|^2)\wedge(dd^c\log(|f|^2+|w|^2))^k \tag{9.6}$$

is equal to $[f^0]\wedge(dd^c\log|f'|)^{k-1}\wedge[w]=i_*M^f_k$ when $\lambda=0$. Indeed, at $\lambda=0$, (9.6) is equal to $M^{(f,w)}_{k+1}$. Note that (9.6) is locally integrable for $\operatorname{Re}\lambda>0$. Moreover, if $\operatorname{Re}\lambda<1$, it is integrable in the w-direction and thus acts on forms that are just bounded in the w-direction. Since $M^{(f,w)}_{k+1}$ is of order zero and $\operatorname{supp}M^{(f,w)}_{k+1}\subset\{w=0\}$, it follows that to check the action of $M^{(f,w)}_{k+1}$ on test forms, it is enough to consider forms $\xi(z,w)=\tilde\xi(z)$, where $\tilde\xi(z)$ is any test form in X. However, after the (generically $1-1$) change of variables $f^0\omega=w$, so that $|f|^2+|w|^2=|f^0|^2(|f'|^2+|\omega|^2)$, the action of (9.6) on ξ is equal to

$$(2\pi i)^{-1}\int_{z,\omega}\bar\partial|f^0|^{2\lambda}(|f'|^2+|\omega|^2)^\lambda\wedge\partial\log|f^0|^2\wedge(dd^c\log(|f'|^2+|\omega|^2))^k\wedge\tilde\xi(z).$$

Taking $\lambda=0$, we get

$$\int_z[f^0]\wedge\tilde\xi(z)\wedge\int_\omega(dd^c\log(|f'|^2+|\omega|^2))^k. \tag{9.7}$$

One can check that the inner integral in (9.7) is equal to $(dd^c\log|f'|^2)^{k-1}$, which proves (9.3). Finally we prove (9.4). Let $j:W\hookrightarrow X$. Then, using (5.9),

$$M^{f\otimes 1}\wedge[W\times\{0\}]=i_*j_*M^{j^*i^*f\otimes 1}=i_*j_*M^{j^*f}=i_*M^f\wedge[W].$$

□

Proof of Proposition 9.1. Since the pullback of $\mathcal{J}'$ to $X\simeq X\times\{0\}$ is just $\mathcal{J}$, (9.1) should be clear. More formally: Let f be a tuple that defines the ideal sheaf $\mathcal{J}$ in X. Then $f\otimes 1$ defines $\mathcal{J}'$ in $X\times\mathbb{C}_w$ and

$$e_k(\mathcal{J}',X\times\{0\},x)=\ell_x(M^{f\otimes 1}_k\wedge[X\times\{0\}])=\ell_x(M^f_k\wedge[X])=e_k(\mathcal{J},X,x),$$

where we have used (9.4) for the second equality. This proves (9.1).

To see (9.2) notice that f,w defines $\mathcal{J}+(w)$ in $X\times\mathbb{C}_w$. Thus, using (5.9) and (9.3) we have

$$e_{k+1}(\mathcal{J}'+(w),X\times\mathbb{C}_w,(x,0))=\ell_x(M^{(f,w)}_{k+1}\wedge[X\times\mathbb{C}_w])=\ell_x(M^f_k\wedge[X])=e_k(\mathcal{J},X,x).$$

□

10. Local intersection numbers

Tworzewski's original motivation for introducing the extended index of intersection was to understand intersection theory in the nonproper case. Let $Z_1,\dots,Z_r$ be subvarieties of a smooth manifold Y that do not necessarily intersect properly.

A standard procedure to define an intersection product $Z_1 \cdots Z_r$ is to give some reasonable meaning to the intersection

$$\Delta \cdot Z_1 \times \cdots \times Z_r, \tag{10.1}$$

where $i\colon Y \simeq \Delta \to Y \times \cdots \times Y$ is the diagonal in $Y \times \cdots \times Y$. In this way one is reduced to the case of two varieties one of which is smooth.

Now assume that A, Z are subvarieties of Y, that A is smooth, and (initially) that Z has pure dimension. Let $\mathcal{J}_A$ denote the radical sheaf of A, and also, for simplicity, the pullback of $\mathcal{J}_A$ to Z. Following Tworzewski we define *local intersection numbers*

$$g_\ell(A, Z, x) := e_{\dim Z - \ell}(\mathcal{J}_A, Z, x) \tag{10.2}$$

at x. If $Z = \sum_j \alpha_j Z_j$, where the Z_j are pure dimensional, we set $g_\ell(A, Z, x) := \sum_j \alpha_j g_\ell(A, Z_j, x)$. The change of indices is made so that ℓ corresponds to the generic multiplicity of components of *dimension* ℓ of Vogel cycles. We will use the notation $A \circ Z$ for these lists of local intersection numbers, i.e.,

$$A \circ Z(x) = (g_{\dim Z}(A, Z, x), \dots, g_1(A, Z, x), g_0(A, Z, x)) \tag{10.3}$$

The local *multiplicities of intersection* of general varieties $Z_1, \dots, Z_r$ are then

$$\epsilon_\ell(Z_1, \dots, Z_r; x) = g_\ell(\Delta, Z_1 \times \cdots \times Z_r, (x, \dots, x)), \tag{10.4}$$

cf. [24, Section 6]. We will write

$$Z_1 \diamond \cdots \diamond Z_r(x) = (\epsilon_\nu(Z_1, \dots, Z_r; x), \dots, \epsilon_1(Z_1, \dots, Z_r; x), \epsilon_0(Z_1, \dots, Z_r; x)), \tag{10.5}$$

where ν is the dimension of the set-theoretical intersection $Z_1 \cap \cdots \cap Z_r$.

Note that the product $Z_1 \diamond \cdots \diamond Z_r$ by definition is commutative. It is also independent of the manifold Y in the following sense: If $\iota : Y \to \widetilde{Y}$ is an embedding of Y in a larger manifold $\widetilde{Y}$, then

$$Z_1 \diamond \cdots \diamond Z_r(x) = \iota(Z_1) \diamond \cdots \diamond \iota(Z_r)\big(\iota(x)\big). \tag{10.6}$$

This follows from Proposition 9.1, see also [21, Section 5].

The next result, which relates the two local intersections (10.3) and (10.5), we have not found in the literature.

Proposition 10.1. *Assume that A, Z are subvarities of a manifold Y, and that A is smooth. Then*

$$A \diamond Z = A \circ Z.$$

In particular, if A and B are smooth submanifolds of Y, since $A \diamond B$ is commutative, it follows that $A \circ B = B \circ A$.

Proof. Fix x in Y. We may assume, without loss of generality, that $Y = \mathbb{C}^N$. Choose local coordinates $z = (z', z'')$ on $\mathbb{C}^N$ so that $A = \{z' = 0\}$, and local coordinates (z, w) on $\mathbb{C}^N \times \mathbb{C}^N$. We will show that

$$\epsilon_j(A, Z; x) = e_{\dim A + \dim Z - j}(\mathcal{J}_\Delta, Z \times A, x) = \ell_x(M^{z-w}_{\dim A + \dim Z - j} \wedge [Z \times A]), \tag{10.7}$$

cf. (5.9), coincides with

$$g_j(A, Z, x) = e_{\dim Z - j}(\mathcal{J}_A, Z, x) = \ell_x(M^{z'}_{\dim Z - j} \wedge [Z]). \tag{10.8}$$

Note that $M^{z-w}_k \wedge [Z \times A] = M^{(z', z''-w'')}_k \wedge [Z \times \{w' = 0\}]$. Let (z', z'', w', η''), where $\eta'' = z'' - w''$, be new coordinates on $\mathbb{C}^N \times \mathbb{C}^N$. Then (9.3) implies that $M^{(z', \eta'')}_{k + \dim A} \wedge [Z \times \{w' = 0\}] = i_* M^{z'}_k \wedge [Z \times \{w' = 0\}]$, where $i : \mathbb{C}^{2N - \dim A}_{z', z'', w'} \hookrightarrow \mathbb{C}^{2N}_{z', z'', w', \eta''}$.

Moreover, by (9.4), $M_k^{z'}\wedge[Z\times\{w'=0\}] = j_*M_k^{z'}\wedge[Z]$, where $j : \mathbb{C}^N_{z',z''} \hookrightarrow \mathbb{C}^{2N-\dim A}_{z',z'',w'}$. Hence

$$M^{z-w}_{\dim A+\dim Z-j}\wedge[Z\times A] = i_*j_*M^{z'}_{\dim Z-j}\wedge[Z]$$

and thus (10.7) is equal to (10.8). □

Example 10.2. If Z_j intersect properly, then also (10.1) is a proper intersection and it is well-known that $Z_1\cdots Z_r$ coincides with the intersection (10.1) (after identifying $Y\simeq\Delta$). It follows that

$$\epsilon_\ell(Z_1,\dots,Z_r;x) = \mathrm{mult}_x(Z_1\cdots Z_r)$$

for $x\in|Z_1\cdots Z_r|$ and $\ell=\dim(Z_1\cdots Z_r)$, and 0 otherwise, cf. [24, Theorem 6.5]. In particular, if Z is a subvariety of the smooth manifold A, in view of (10.6),

$$A\diamond Z(x) = (\mathrm{mult}_x Z, 0,\dots,0). \tag{10.9}$$

□

In the nonproper case the classical intersection product $Z_1\cdots Z_r$ in the sense of Fulton, [9], cannot represent the local intersection multiplicities in any reasonable sense. Indeed, in general $Z_1\cdots Z_r$ is a cycle of codimension $\mathrm{codim}\, Z_1+\cdots+\mathrm{codim}\, Z_r$, determined modulo rational equivalence on $Z_1\cap\cdots\cap Z_r$.

Tworzewski, [24, Sections 5,6], proves that there is unique cycle $Z_1\bullet\cdots\bullet Z_r$, that he calls the *intersection product*, such that

$$\sum_k \mathrm{mult}_x(Z_1\bullet\cdots\bullet Z_r)_k = \sum_\ell \epsilon_\ell(Z_1,\dots,Z_r;x)$$

for each point x, where the index k denotes the component of dimension k. By definition this product respects the (sum of the) local intersection multiplicities, but it does not respect Bezout's formula in general. For instance, the self-intersection in $\mathbb{P}^2$ of any smooth curve C is just C itself, and thus $\deg(C\bullet C)\neq(\deg C)^2$ unless C is a line.

In a forthcoming paper we will introduce, in the case $Y=\mathbb{P}^n$, a global current that represents, at each point, the local intersection multiplicities, and respects Bezout's formula, in a reasonable sense. It is obtained as the mean value of various Vogel sequences, based on global variants of the ideas in Section 6 above.

11. Examples

Let us start by some computations of Bochner-Martinelli currents and Segre numbers. Our first example illustrates that the currents M^f in general depend on the set of generators f although the Lelong numbers only depend on (the integral closure of) the ideal generated by f, cf. Remark 1.2.

Example 11.1. Let us consider the primary ideal (x) in $\mathbb{C}^2_{x,y}$. We know from Corollary 1.3 and Remark 1.2, respectively, that $M_1^x=[x]$ and that $M_2^x=0$. Let us now consider the pair (x,xy) of generators for the same ideal. We first consider the Vogel cycles obtained from generic linear combinations of these generators. Since $Z=\{x=0\}$ and $[\alpha_0x+\alpha_1xy]=[x(\alpha_0+\alpha_1y)]=[x]+[\alpha_0+\alpha_1y]$ we have that

$$\mathbf{1}_Z[\alpha_0x+\alpha_1xy] = [x]$$

as expected, since $[x]$ must be a fixed component in any Vogel cycle. A simple computation yields that

$$\mathbf{1}_Z[\beta_0 x + \beta_1 xy]\wedge[\alpha_0 x + \alpha_1 xy] = [x]\wedge[\alpha_0 + \alpha_1 y]$$

for generic choices of α and β, cf. Section 3. Thus the component of the Vogel cycle of codimension 2 is non-vanishing for generic α, β. Taking mean values over $\mathbb{P}^1$ we get, cf., Theorem 6.2,

$$M_2^{x,xy} = [x]\wedge\frac{dy\wedge d\bar{y}}{\pi(1+|y|^2)^2}.$$

Here we have used that with the generic parametrization $\mathbb{C} \ni t \mapsto [-t, 1] \in \mathbb{P}^1$, we have

$$\int_{[\alpha]\in\mathbb{P}^1} [\alpha_0 + \alpha_1 y]d\sigma(\alpha) = \int_{t\in\mathbb{C}} [y-t]\wedge\frac{dt\wedge d\bar{t}}{\pi(1+|t|^2)^2} = \frac{dy\wedge d\bar{y}}{\pi(1+|y|^2)^2}.$$

□

Next we will discuss a simple example where a moving component occurs.

Example 11.2. Consider the tuple $f = t_3(t_1, t_2, t_3) = t_3 t$ in $X = \mathbb{C}^3_t$, with zero set $Z = \{t_3 = 0\}$. We will compute the Segre numbers $e_k(0) = e_k(\mathcal{J}(f), \mathbb{C}^3, 0)$, $k = 0, 1, 2, 3$. Let $\alpha \cdot f$ be a Vogel sequence of $\mathcal{J}(f)$ at 0 of the form $\alpha_1 \cdot f, \dots, \alpha_3 \cdot f$. Let us compute the corresponding Vogel cycle $V^{\alpha\cdot f}$. First note that $X_0^{X\setminus Z} = X_0 = X$. Thus, by Proposition 3.4,

$$[X_1] = M_1^{t_3(\alpha_1\cdot t)} = [t_3] + [\alpha_1 \cdot t] = [X_1^Z] + [X_1^{X\setminus Z}].$$

Furthermore, using (3.3) and (3.4), we get

$$[X_2] = M_1^{t_3(\alpha_2\cdot t)}\wedge M_1^{t_3(\alpha_1\cdot t)} = [t_3]\wedge[\alpha_1\cdot t] + [\alpha_2\cdot t]\wedge[\alpha_1\cdot t] = [X_2^Z] + [X_2^{X\setminus Z}]$$

and

$$[X_3] = M_1^{t_3(\alpha_3\cdot t)}\wedge M_1^{t_3(\alpha_2\cdot t)}\wedge M_1^{t(\alpha_1\cdot t)} = ([t_3] + [\alpha_3\cdot t])\wedge[\alpha_2\cdot t]\wedge[\alpha_1\cdot t] = 2[0] = [X_3^Z],$$

for a generic α. Hence

$$[V^h] = [V_1^h] + [V_2^h] + [V_3^h] = [t_3] + [t_3]\wedge[\alpha_1\cdot t] + 2[0]$$

and, in particular,

$$e_0(0) = 0,\ e_1(0) = 1,\ e_2(0) = 1,\ e_3(0) = 2.$$

Observe that V_1^h and V_3^h are fixed, whereas V_2^h is moving. A computation, using Theorem 6.2 and Lemma 6.3, yields

$$M_0^f = 0,\ M_1^f = [t_3],\ M_2^f = [t_3]\wedge dd^c\log(|t_1|^2 + |t_2|^2),\ M_3^f = 2[0].$$

□

The following simple lemma is useful for computations.

Lemma 11.3. *Let X and X' be two analytic spaces of dimension n, let $\tau\colon X' \to X$ be a holomorphic map, and let f be a tuple of holomorphic functions on X. Assume that τ is proper, surjective, and generically r to 1. Then*

$$rM_k^f = \tau_* M_k^{\tau^* f}. \tag{11.1}$$

Moreover, if ξ is a tuple that defines the maximal ideal at $x \in X$, then the Segre numbers at x associated with $\mathcal{J} = \mathcal{J}(f)$ on X are given by

$$e_k(x) = \frac{1}{r}\int_{X'} M_{n-k}^{\tau^*\xi} \wedge M_k^{\tau^* f}. \tag{11.2}$$

Proof. Since $\tau^* M_k^{f,\lambda} = M_k^{\tau^* f,\lambda}$ if $\operatorname{Re}\lambda \gg 0$, we have that then

$$\int_X M_k^{f,\lambda}\wedge\psi = \frac{1}{r}\int_{X'} M_k^{\tau^* f,\lambda}\wedge\tau^*\psi$$

for test forms ψ. Taking analytic continuations to $\lambda = 0$, we get (11.1). In view of Proposition 5.5 we have

$$e_k(x) = \ell_x M_k^f = \int_X M_{n-k}^{\xi,\lambda}\wedge M_k^{f,\lambda^2}\big|_{\lambda=0} = \frac{1}{r}\int_{X'} M_{n-k}^{\tau^*\xi,\lambda}\wedge M_k^{\tau^* f,\lambda^2}\big|_{\lambda=0} = \frac{1}{r}\int_{X'} M_{n-k}^{\tau^*\xi}\wedge M_k^{\tau^* f}.$$

□

In particular, it follows from Lemma 11.3 that

$$\operatorname{mult}_x X = \int_X M_n^{\xi} = \frac{1}{r}\int_{X'} M_n^{\tau^*\xi}. \tag{11.3}$$

Example 11.4. Let r, s be relatively prime integers and consider the cusp $X = \{z_1^r - z_2^s = 0\}$ in $\mathbb{C}^2_z$. Since we have the parametrization $\tau\colon t \mapsto (t^s, t^r)$ of X, using (11.3) we get

$$\operatorname{mult}_0 X = \int_X M_1^{(z_1,z_2)} = \int_{\mathbb{C}_t} M_1^{(t^s,t^r)} = \int_{\mathbb{C}_t} M_1^{t^{\min(s,r)}} = \min(s,r).$$

This multiplicity is of course well-known, and can be computed in various other ways. □

We will now proceed with some computations of local intersection numbers.

Example 11.5. Let $X = \{x_2 x_1^m - x_3^2 = 0\} \subset \mathbb{C}^3_x$, where $m \geq 1$, and let $A = \{x_2 = x_3 = 0\}$. Since A is smooth and contained in X, and X is smooth outside the origin in $\mathbb{C}^3$, we must have that $A \diamond X(x) = (\operatorname{mult}_x A, 0, \dots, 0)$ for $x \neq 0$, cf., (10.9).

We shall now compute the local intersection numbers at 0. To this end we consider a generic Vogel sequence of $\mathcal{J}_A$ on X at the origin and compute the corresponding Vogel cycle. Let H_1 be a generic hyperplane that contains A, defined by $h_1 = \alpha x_2 - x_3$. Then $X_1 = H_1 \cdot X$ is the curve $\{x_2 x_1^m - (\alpha x_2)^2 = 0, \alpha x_2 - x_3 = 0\}$. It follows that X_1^A is equal to A, whereas $X_1^{Z\setminus A}$ is the curve $\{x_1^m - \alpha^2 x_2 = 0, \alpha x_2 - x_3 = 0\}$. Next, let $h_2 = \beta x_2 - x_3$. Then $X_2 = H_2 \cdot X_1^{X\setminus A}$ is the cycle $\{x_3 = x_2 = 0,\ x_1^m = 0\}$. Since its support is contained in A, it is equal to X_2^A and it has order m at the origin. We conclude that $V^h = A + m[0]$. Thus $\epsilon_k(A, X, 0)$ is equal to 1 when $k = \dim A = 1$ and m when $k = 0$.

As an illustration, let us also compute $\epsilon_0(A, X, 0) = e_2(\mathcal{J}_A, X, 0)$ as the Lelong number of a certain Bochner-Martinelli current. Notice that $\tau : (t_1, t_2) \mapsto (t_1^2, t_2^2, t_1^m t_2)$ is a surjective, generically $2-1$, mapping $\mathbb{C}^2_t \to X$. If $i\colon X \hookrightarrow \mathbb{C}^3$ is the

inclusion map we have by Lemma 11.3 that

$$e_2(\mathcal{J}_A, X, 0) = \ell_x(M_2^{(x_2,x_3)} \wedge [X]) = \int_{\mathbb{C}^3} M_0^{(x_1,x_2,x_3)} \wedge M_2^{(x_2,x_3)} \wedge [x_2 x_1^m - x_3^2] =$$
$$\int_X M_0^{(i^*x_1, i^*x_2, i^*x_3)} \wedge M_2^{(i^*x_2, i^*x_3)} = \frac{1}{2} \int_{\mathbb{C}^2_{t_1,t_2}} M_0^{(t_1^2, t_2^2, t_1^m t_2)} \wedge M_2^{(t_2^2, t_1^m t_2)}.$$

According to Theorem 6.2, $M_2^{(t_2^2, t_1^m t_2)}$ is the mean value of all

$$[(\beta t_2 - t_1^m)t_2] \wedge [(\alpha t_2 - t_1^m)t_2]$$

for generic choices of $\alpha, \beta \in \mathbb{C}$. For generic α, β, using the new variables $v_1 = t_1, v_2 = \alpha t_2 - t_1^m$, we get

$$[\beta t_2 - t_1^m] \wedge [\alpha t_2 - t_1^m] = [\beta' v_2 - \alpha' v_1^m] \wedge [v_2] = [v_1^m] \wedge [v_2] = m[0]$$

for some $\alpha', \beta' \in \mathbb{C}$. Since $[(\alpha t_2 - t_1^m)t_2] = [t_2] + [\alpha t_2 - t_1^m]$, by (3.3) and (3.4), we thus have that

$$[(\beta t_2 - t_1^m)t_2] \wedge [(\alpha t_2 - t_1^m)t_2] = \big([\beta t_2 - t_1^m] + [t_2]\big) \wedge [\alpha t_2 - t_1^m] = 2m[0].$$

Now, $M_0^{(t_1^2, t_2^2, t_1^m t_2)} = \mathbf{1}_{(0,0)}$, so $\epsilon_0(A, X, 0) = m$ as expected. □

The following example is related to Example 11.2 above.

Example 11.6. The mapping $\gamma\colon \mathbb{C}^3_t \to \mathbb{C}^6_z$ defined by

$$(t_1, t_2, t_3) \mapsto \gamma(t) = (t_1, t_2, t_3 t_1, t_3 t_2, t_3^2, t_3^3)$$

is proper and injective, so that $X := \gamma(\mathbb{C}^3)$ is a subvariety of $\mathbb{C}^6$. Let $A = \{z_3 = z_4 = z_5 = z_6 = 0\}$. Then A is smooth and contained in X and, since X is smooth outside 0, it follows from (10.9) that $A \diamond X(x) = (\mathrm{mult}_x A, 0, \dots, 0)$ for $x \neq 0$. We want to determine the local intersection numbers $\epsilon_{3-k}(A, X, 0) = e_k(\mathcal{J}_A, X, 0)$ at 0. Since $\mathcal{J}_A$ has codimension 1 in X, $e_0(0) = 0$. Moreover, by King's formula, $M_1^{z_3,z_4,z_5,z_6}$ is a Lelong current on X, and at a point $x \neq 0$ we know that the Lelong number is 1 if $x \in A$ and 0 otherwise. We conclude that $M_1^{z_3,z_4,z_5,z_6} = [A]$, and hence $e_1(1) = \mathrm{mult}_0 A = 1$. By Lemma 11.3,

$$e_k(A, X, 0) = \int_{\mathbb{C}^3_t} M_{3-k}^{\gamma^* z} \wedge M_k^{\gamma^*(z_3,z_4,z_5,z_6)} = \int_{\mathbb{C}^3_t} M_{3-k}^{(t_1,t_2,t_3^2)} \wedge M_k^{t_3(t_1,t_2,t_3)},$$

where we have used that the ideal $\gamma^* z$ is generated by t_1, t_2, t_3^2, that the ideal $\gamma^* \mathcal{J}_A$ is generated by $t_3(t_1, t_2, t_3)$, and that $e_k(A, X, 0)$ only depends on the ideals. In light of Example 11.2 thus,

$$e_2(A, X, 0) = \int_{\mathbb{C}^3_t} M_1^{(t_1,t_2,t_3^2)} \wedge [t_3] \wedge dd^c \log|t|^2 = \int_{\mathbb{C}^3_t} M_1^{(t_1,t_2)} \wedge [t_3] \wedge dd^c \log|t|^2$$
$$= \int_{\mathbb{C}^2_{(t_1,t_2)}} M_1^{t_1,t_2} \wedge dd^c \log|t'|^2 = \ell_0(dd^c \log|t'|^2) = 1,$$

where $t' = (t_1, t_2)$. To see the last equality, in view of Theorem 6.1, one can replace $dd^c \log|t|^2$ by a generic hyperplane $[\alpha \cdot t]$. In a similar way one concludes that $e_3(A, X, 0) = 2$. □

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