ON THE EFFECTIVE MEMBERSHIP PROBLEM FOR POLYNOMIAL IDEALS

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Abstract. We discuss the possibility of representing elements in polynomial ideals in $\mathbb{C}^N$ with optimal degree bounds. Classical theorems due to Macaulay and Max Noether say that such a representation is possible under certain conditions on the variety of the associated homogeneous ideal. We present some variants of these results, as well as generalizations to subvarieties of $\mathbb{C}^N$.

Dedicated to the memory of Mikael Passare

1. Introduction

Let V be an algebraic subvariety of $\mathbb{C}^N$ of pure dimension n and let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^N$. We are interested in finding solutions to the polynomial division problem

$$F_1Q_1 + \cdots + F_mQ_m = \Phi \tag{1.1}$$

on V with degree estimates, provided Φ is in the ideal (F_j) on V. By a result of Hermann, [18], if $\deg F_j \leq d$, there are polynomials Q_j such that $\deg(F_jQ_j) \leq \deg \Phi + C(d,N)$, where $C(d,N)$ is like $2(2d)^{2^N-1}$ for large d and thus doubly exponential. It is shown in [24] (see also [10, Example 3.9]) that in general this estimate cannot be substantially improved.

If one imposes conditions on V and F_j one can, however, obtain much sharper estimates. The following two results in $\mathbb{C}^n$ are classical.

If $F_1, \ldots, F_m$ are polynomials in $\mathbb{C}^n$ of degrees $d_1 \geq \ldots \geq d_m$ with no common zeros even at infinity and Φ is any polynomial, then one can solve (1.1) *with $deg\,(F_jQ_j) \leq \max(deg\,\Phi, d_1 + \ldots + d_{n+1} - n)$.*

If $F_1, \ldots, F_n$ are polynomials in $\mathbb{C}^n$ such that their common zero set is discrete and does not intersect the hyperplane at infinity, and Φ belongs to the ideal (F_j), then one can find polynomials Q_j such that (1.1) *holds and $deg\,(F_jQ_j) \leq deg\,\Phi$.*

The first theorem is due to Macaulay, [23], and the second one is Max Noether's AF+BG theorem, [25], originally stated for $n = 2$. Noether's result is clearly optimal.

In this paper we present extensions of these results to the case of more general varieties $V \subset \mathbb{C}^N$, and also generalizations in which we relax the condition on (the zero set of) the F_j. It grew out of our paper [9], in which we extended to the singular setting a framework for solving polynomial ideal membership problems with residue techniques introduced in [3] and further developed in [5, 30, 31], see below. The proofs in this paper follow the same setup. However, at least some of the results also admit algebraic proofs, see Remark 6.2.

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Throughout we will let X denote the closure of V in $\mathbb{P}^N$, and $\operatorname{reg} X$ the *regularity* of X, see Section 4 for the definition. For each F_j we let f_j denote the induced section of $\mathcal{O}(\deg F_j)|_X$.

We begin with an extension of Macaulay' theorem to singular varieties; this can easily be proved by standard arguments, cf. Remark 6.2.

Theorem 1.1. *Let V be an algebraic subvariety of $\mathbb{C}^N$, with closure X in $\mathbb{P}^N$, and let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^N$ of degrees $d_1 \geq \ldots \geq d_m$. Assume that f_j have no common zeros on X. Then for each polynomial Φ in $\mathbb{C}^N$ there are polynomials Q_j such that* (1.1) *holds and*

$$\mathit{deg}\,(F_jQ_j) \leq \max(\mathit{deg}\,\Phi, d_1 + \cdots + d_{n+1} - (n+1) + \operatorname{reg} X).$$

If X is smooth, then $\operatorname{reg} X \leq (n+1)(\deg X - 1) + 1$; this is Mumford's bound, see, e.g., [22, Example 1.8.48]. If X is Cohen-Macaulay in $\mathbb{P}^N$ (and N is minimal) then $\operatorname{reg} X \leq \deg X - (N - n)$, see, [17, Corollary 4.15]. In particular, if $V = \mathbb{C}^n$ so that $X = \mathbb{P}^n$, then $\operatorname{reg} X = 1$; thus we get back Macaulay's theorem. For a discussion of bounds on $\operatorname{reg} X$ for a general X, see, e.g., [10, Section 3].

Let Z^f denote the common zero set of $f_1, \ldots, f_m$ in X. Moreover, let $X_\infty := X \setminus V$. For smooth varieties we have the following version of Max Noether's theorem.

Theorem 1.2. *Let V be an algebraic subvariety of $\mathbb{C}^N$ of dimension n such that its closure X in $\mathbb{P}^N$ is smooth, and let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^N$ of degrees $d_1 \geq \ldots \geq d_m$. Assume that $m \leq n$, that*

$$\operatorname{codim}(Z^f \cap V) \geq m, \tag{1.2}$$

and that Z^f has no irreducible component contained in X_∞. If Φ is a polynomial that belongs to the ideal (F_j) on V, then there is a representation (1.1) *with*

$$\mathit{deg}\,(F_jQ_j) \leq \max(\mathit{deg}\,\Phi, d_1 + \cdots + d_m - m + \operatorname{reg} X). \tag{1.3}$$

If in addition X is Cohen-Macaulay in $\mathbb{P}^N$ one can choose Q_j so that

$$\mathit{deg}\,(F_jQ_j) \leq \mathit{deg}\,\Phi. \tag{1.4}$$

Remark 1.3. If X is Cohen-Macaulay it suffices that V is smooth to obtain (1.4). $\square$

For $V = \mathbb{C}^n$ Theorem 1.2 appeared in [3, Theorem 1.2].

For a general X, in order to have a Max Noether theorem, we need the common zero set of the f_j not to intersect the singular locus of X too badly. To make this statement more precise we need to introduce what we call the *intrinsic BEF-varieties*

$$X^{n-1} \subset \cdots \subset X^1$$

of $X \subset \mathbb{P}^N$. These are the sets where the mappings in a locally free resolution of $\mathcal{O}^{\mathbb{P}^N}/\mathcal{J}_X$ do not have optimal rank. They are intrinsically defined subvarieties of X that are contained in $X^0 := X_{\text{sing}}$. The codimension of X^ℓ is at least $\ell + 1$, and if X is locally Cohen-Macaulay X^ℓ is empty for $\ell \geq 1$, see Sections 2.3 and 2.5.

Theorem 1.4. *Let V be an algebraic subvariety of $\mathbb{C}^N$ of dimension n, with closure X in $\mathbb{P}^N$, and let F_j be as in Theorem 1.2. Assume that Z^f satisfies* (1.2)*, that Z^f has no irreducible component contained in X_∞, and moreover that*

$$\operatorname{codim}(Z^f \cap X^\ell) \geq m + \ell + 1, \qquad \ell \geq 0. \tag{1.5}$$

If Φ is a polynomial that belongs to the ideal (F_j) on V, then there is a representation (1.1) *such that* (1.3) *holds. If in addition X is Cohen-Macaulay in $\mathbb{P}^N$, and $m \leq n$, we can choose Q_j such that* (1.4) *holds.*

Notice that (1.5) forces that either $Z^f \cap X_{\text{sing}} = \emptyset$ or $m < n$. If X is smooth, then (1.5) is vacuous, and thus Theorem 1.2 follows immediately from Theorem 1.4. If only V is smooth but X is Cohen-Macaulay, then by the assumption on Z^f $\operatorname{codim}(Z^f \cap X_\infty) \geq m+1$ and since $X^0 \subset X_\infty$, (1.5) is satisfied. This proves the claim in Remark 1.3.

Next we will present some generalizations of Theorem 1.4 where we relax the hypotheses on the common zero set Z^f of the f_j. First, we drop the size hypothesis (1.2) on $Z^f \cap V$. We then still get an estimate of the form (1.3) but the second entry on the right hand side is now replaced by a constant that depends on F_j in a more involved manner. The condition that Z^f has no irreducible component at infinity should now be understood as that the ideal sheaf $\mathcal{J}_f$ over X generated by the sections $f_1, \ldots, f_m$ has no associated variety, in the sense of [28], contained in X_∞, see Section 3. This means that at each $x \in X_\infty$, $(\mathcal{J}_f)_x$ has no (varieties of) associated prime ideals contained in X_∞. Let J_f be the homogeneous ideal in $\mathbb{C}[z_0, \ldots, z_N]$ associated with $\mathcal{J}_f$, and let $\operatorname{reg} J_f$ be the *regularity* of J_f, cf. Section 4.

Theorem 1.5. *Let V be an algebraic subvariety of $\mathbb{C}^N$, with closure X in $\mathbb{P}^N$, and let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^N$. Assume that $\mathcal{J}_f$ has no associated variety contained in X_∞. Then there is a constant $\beta = \beta(X, F_1, \ldots, F_m)$ such that if $\Phi \in (F_j)$, then there is a representation* (1.1) *on V with*

$$\deg(F_j Q_j) \leq \max(\deg \Phi, \beta). \tag{1.6}$$

If $V = \mathbb{C}^N$, one can take $\beta = \operatorname{reg} J_f$.

Conversely, if there is an associated prime of $\mathcal{J}_f$ contained in X_∞, then there is no β such that one can solve (1.1) *with* (1.6) *for all Φ in (F_j).*

In [27] Shiffman computed the regularity of a zero-dimensional homogeneous polynomial ideal J_f to be $\leq d_1 + \ldots + d_{n+1} - n$. Using this he obtained (the first part of) Theorem 1.5 for $V = \mathbb{C}^N$ and $\dim Z^f = 0$ with $\beta = \operatorname{reg} J_f = d_1 + \cdots + d_{n+1} - n$, i.e., the same bound as in Macaulay's theorem, see [27, Theorem 2(iv)]. Theorem 1.5 can thus be seen as a generalization of Shiffman's result.

The estimate (1.6) is clearly sharp if $\deg \Phi \geq \beta$. If the ideal sheaf $\mathcal{J}_f$ is locally Cohen-Macaulay, for instance locally a complete intersection, then there are no embedded primes of $\mathcal{J}_f$, and so the hypothesis that $\mathcal{J}_f$ has no associated variety at infinity just means that no irreducible component of Z^f is contained in X_∞. Thus we get back the hypothesis in Theorems 1.2 and 1.4.

Next, let us instead relax the condition that Z^f has no irreducible components at infinity. If the degrees of F_j are $\leq d$, we let $\tilde{f}_j$ denote the section of $\mathcal{O}(d)|_X$ corresponding to F_j. We let $Z^{\tilde{f}}$ be the common zero set of $\tilde{f}_1, \ldots, \tilde{f}_m$ and $\mathcal{J}_{\tilde{f}}$ the coherent analytic sheaf over X generated by the $\tilde{f}_j$. Moreover, we let c_∞ be the maximal codimension of the so-called *(Fulton-MacPherson) distinguished varieties* of $\mathcal{J}_{\tilde{f}}$ that are contained in X_∞, see Section 5.1. If there are no distinguished varieties contained in X_∞, then we interpret c_∞ as $-\infty$. Note that it is not sufficient that $Z^{\tilde{f}} \cap V = Z^{\tilde{f}}$, since there may be embedded distinguished varieties contained in X_∞.

It is well-known that the codimension of a distinguished variety cannot exceed the number m, see, e.g., Proposition 2.6 in [15], and thus $c_\infty \le \mu$, where

$$\mu := \min(m, n).$$

Theorem 1.6. *Let V be an algebraic subvariety of $\mathbb{C}^N$, with closure X in $\mathbb{P}^N$, and let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^N$ of degree $\le d$. Assume that $Z^{\tilde{f}}$ satisfies*

$$\operatorname{codim}(Z^{\tilde{f}} \cap X) \ge m \tag{1.7}$$

and

$$\operatorname{codim}(Z^{\tilde{f}} \cap X^\ell) \ge m + \ell + 1, \qquad \ell \ge 0. \tag{1.8}$$

If Φ is a polynomial that belongs to (F_j) on V, then there is a representation (1.1) *on V with*

$$deg\,(F_j Q_j) \le \max(deg\,\Phi + \mu d^{c_\infty} deg\,X, (d-1)\min(m, n+1) + \operatorname{reg} X). \tag{1.9}$$

If in addition X is locally Cohen-Macaulay in $\mathbb{P}^N$ and $m \le n$, then we can choose Q_j such that

$$deg\,(F_j Q_j) \le deg\,\Phi + m d^{c_\infty} deg\,X.$$

Note that for most choices of F_j and Φ the first entry in (1.9) is much larger than the second entry. For instance this is true for all Φ if $c_\infty \ge 2$ and d is large enough. In particular, if $X = \mathbb{P}^n$, so that $\operatorname{reg} X = 1$, and $c_\infty \ge 2$, the first entry is the largest for all d.

For $X = \mathbb{P}^n$ Theorem 1.6 is due to the first author and Götmark, [5, Theorem 1.3]. In the case when $\deg F_j = d$, so that $\tilde{f}_j = f_j$, Theorem 1.6 generalizes Theorems 1.1 – 1.4, see Remark 6.3.

Example 1.7. If the F_j have no common zeros on V, then Theorem 1.6 gives a solution to

$$F_1 Q_1 + \cdots + F_m Q_m = 1$$

with $\deg(F_j Q_j) \le \mu d^\mu \deg X$ if d is large enough. Except for the annoying factor μ we then get back is Jelonek's optimal effective Nullstellensatz, [20]. $\square$

Note that the estimates of $\deg(F_j Q_j)$ in the theorems above hold for representations of *all* Φ in (F_j). If one, instead of adding conditions on V and F_j, imposes further conditions on Φ, then Hermann's degree estimate for solutions to (1.1) can also be essentially improved. Theorem 1.1 in our recent paper [9] asserts that for any $V \subset \mathbb{C}^N$ there is a number μ_0 such that if $F_1, \ldots, F_m$ are polynomials in $\mathbb{C}^N$ of degree $\le d$ and Φ is a polynomial such that $|\Phi| \le C|F|^{\mu+\mu_0}$ locally on V, where $|F|^2 = |F_1|^2 + \cdots + |F_m|^2$, then one can solve (1.1) with

$$\deg(F_j Q_j) \le \max\big(\deg \Phi + (\mu + \mu_0) d^{c_\infty} \deg X, (d-1)\min(m, n+1) + \operatorname{reg} X\big). \tag{1.10}$$

The statement that $|\Phi| \le C|F|^{\mu+\mu_0}$ implies that there is a representation (1.1) is a direct consequence of Huneke's uniform Briançon-Skoda theorem, [12, 19], and thus the degree estimate (1.10) can be seen as a global effective Briançon-Skoda-Huneke theorem.

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2. Residue currents

We will briefly recall some residue theory. For more details we refer to [9] and the references therein.

2.1. **Currents on a singular variety.** If nothing else is mentioned X will be a reduced subvariety of $\mathbb{P}^N$ of pure dimension n. The sheaf $\mathcal{C}_{\ell,k}$ of currents of bidegree (ℓ, k) on X is by definition the dual of the sheaf $\mathcal{E}_{n-\ell,n-k}$ of smooth $(n-\ell, n-k)$-forms on X. If $i\colon X \to \mathbb{P}^N$ is an embedding of X, then $\mathcal{E}_{n-\ell,n-k}$ can be identified with the quotient sheaf $\mathcal{E}^{\mathbb{P}^N}_{n-\ell,n-k}/\mathrm{Ker}\, i^*$, where $\mathrm{Ker}\, i^*$ is the sheaf of forms ξ on $\mathbb{P}^N$ such that $i^*\xi$ vanish on X_{reg}. It follows that the currents τ in $\mathcal{C}_{\ell,k}$ can be identified with currents $\tau' = i_*\tau$ on $\mathbb{P}^N$ of bidegree $(N-n+\ell, N-n+k)$ that vanish on $\mathrm{Ker}\, i^*$.

Given a holomorphic function f on X, we write $1/f$ for the *principal value distribution*, defined for instance as $\lim_{\epsilon\to 0} \chi(|f|^2/\epsilon)(1/f)$, where $\chi(t)$ is the characteristic function of the interval $[1,\infty)$ or a smooth approximand of it, or as the analytic continuation of $\lambda \to |f|^{2\lambda}(1/f)$ to $\lambda = 0$. It is readily checked that $f(1/f) = 1$ as distributions and that the *residue current* $\bar\partial(1/f)$ satisfies $f\bar\partial(1/f) = 0$. We will need the fact that

$$v^\lambda |f|^{2\lambda} \frac{1}{f}\Big|_{\lambda=0} = \frac{1}{f} \tag{2.1}$$

if v is a strictly positive smooth function; cf. [1, Lemma 2.1].

2.2. **Pseudomeromorphic currents.** The notion of pseudomeromorphic currents on manifolds was introduced in [8]. A slightly extended version appeared in [6]: A current on X is *pseudomeromorphic* if it is (the sum of terms that are) the push-forward under (a composition of) modifications, projections, and open inclusions of currents of the form

$$\frac{\xi}{s_1^{\alpha_1}\cdots s_{n-1}^{\alpha_{n-1}}} \wedge \bar\partial \frac{1}{s_n^{\alpha_n}},$$

where s is a local coordinate system and ξ is a smooth form with compact support, see, e.g., [6] for details.

Pseudomeromorphic currents in many respects behave like positive closed currents. For example they satisfy the *dimension principle: If τ is a pseudomeromorphic current on X of bidegree $(*, p)$ that has support on a variety of codimension $> p$, then $\tau = 0$.*

Also, pseudomeromorphic currents allow for multiplication with characteristic functions of constructible sets so that ordinary computational rules hold. If τ is a pseudomeromorphic current on X and V is a subvariety of X, then the natural restriction of τ to the open set $X \setminus V$ has a canonical extension $\mathbf{1}_{X\setminus V}\tau := |h|^{2\lambda}\tau|_{\lambda=0}$, where h is any holomorphic tuple such that $\{h = 0\} = V$. It follows that $\mathbf{1}_V\tau := \tau - \mathbf{1}_{X\setminus V}\tau$ is a pseudomeromorphic current with support on V. Note that if α is a smooth form, then $\mathbf{1}_V\alpha\wedge\tau = \alpha\wedge\mathbf{1}_V\tau$ and if W are W' are constructible sets, then

$$\mathbf{1}_W\mathbf{1}_{W'}\tau = \mathbf{1}_{W\cap W'}\tau. \tag{2.2}$$

Moreover, if $\pi\colon \widetilde{X} \to X$ is a modification, $\tilde\tau$ is a pseudomeromorphic current on $\widetilde{X}$, and $\tau = \pi_*\tilde\tau$, then

$$\mathbf{1}_V\tau = \pi_*\big(\mathbf{1}_{\pi^{-1}V}\tilde\tau\big) \tag{2.3}$$

for any subvariety $V \subset X$. If W is a subvariety of X and $\mathbf{1}_V \tau = 0$ for all subvarieties $V \subset W$ of positive codimension we say that τ has the *the standard extension property*, SEP with respect to W, see [11].

Recall that a current is *semi-meromorphic* if it is the quotient of a smooth form and a holomorphic function. Following [6] we say that a current τ is *almost semi-meromorphic* in X if there is a modification $\pi\colon \widetilde{X} \to X$ and a semi-meromorphic current $\tilde{\tau}$ such that $\tau = \pi_* \tilde{\tau}$.

2.3. **Residue currents associated with Hermitian complexes.** Consider a complex of Hermitian holomorphic vector bundles over a complex manifold Y of dimension n,

$$0 \to E_M \xrightarrow{f^M} \dots \xrightarrow{f^3} E_2 \xrightarrow{f^2} E_1 \xrightarrow{f^1} E_0 \to 0, \tag{2.4}$$

that is pointwise exact outside an analytic variety $Z \subset Y$ of positive codimension p. Suppose that the rank of E_0 is 1. In [2, 7] was associated to (2.4) a $\bigoplus \operatorname{Hom}(E_0, E_k)$-valued pseudomeromorphic current $R = R^f$; it has support on Z and in a certain sense it measures the lack of exactness of the associated sheaf complex of holomorphic sections

$$0 \to \mathcal{O}(E_M) \xrightarrow{f^M} \dots \xrightarrow{f^3} \mathcal{O}(E_2) \xrightarrow{f^2} \mathcal{O}(E_1) \xrightarrow{f^1} \mathcal{O}(E_0). \tag{2.5}$$

Proposition 2.1. *If ϕ is a holomorphic section of E_0 such that $R\phi = 0$, then $\phi \in \operatorname{Im} f^1$. Moreover, if*

$$H^{k-1}(Y, \mathcal{O}(E_k)) = 0, \quad 1 \le k \le \min(M, n+1), \tag{2.6}$$

then there is a global holomorphic section q of E_1 such that $f^1 q = \phi$.

We also have the *duality principle: If* (2.5) *is exact, i.e., if it is a locally free resolution of the sheaf $\mathcal{O}(E_0)/\operatorname{Im} f^1$, then $R\phi = 0$ if and only if $\phi \in \operatorname{Im} f^1$.*

As in [9] we will refer to a (locally) free resolution (2.5) of $\mathcal{O}(E_0)/\mathcal{J}$ together with Hermitian metrics on the corresponding vector bundles as a *Hermitian (locally) free resolution.*

Let us look at the construction of R in a special case; see, e.g., [9] for more details and the general case. Let R_k denote the component of R that takes values in $\operatorname{Hom}(E_0, E_k)$.

Example 2.2 (The Koszul complex). Given Hermitian line bundles $S \to Y$ and $L_1, \dots, L_m \to Y$ and a tuple f of holomorphic sections $f_1, \dots, f_m$ of $L_1, \dots, L_m$, respectively, let (2.4) be the (twisted) Koszul complex of f: Let E^j be disjoint trivial line bundles with basis elements e_j, let $E = L_1^{-1} \otimes E^1 \oplus \dots \oplus L_m^{-1} \otimes E^m$, and identify f with a section $f = \sum f_j e_j^*$ of E^*, where e_j^* are the dual basis elements. Moreover, let

$$E_0 = S, \quad E_k = S \otimes \Lambda^k E,$$

and let all f^k in (2.4) be interior multiplication δ_f by the section f.

The current associated with the Koszul complex was introduced in [1]; we will briefly recall the construction. Let σ be the section of E over $Y \setminus Z$ with pointwise minimal norm such that $f \cdot \sigma = \delta_f \sigma = 1$, i.e.,

$$\sigma = \sum_j \frac{f_j^* e_j}{|f|^2},$$

where f_j^* is the section of L_j^{-1} of minimal norm such that $f_j f_j^* = |f_j|^2_{L_j}$, and $|f|^2 = |f_1|^2_{L_1} + \cdots + |f_m|^2_{L_m}$. Then R_k equals the analytic continuation to $\lambda = 0$ of

$$R_k^\lambda = R_k^{f,\lambda} := \bar\partial |f|^{2\lambda} \wedge \sigma \wedge (\bar\partial\sigma)^{k-1}. \tag{2.7}$$

Here the exterior product is with respect to the exterior algebra over $E \oplus T^*(Y)$ so that $d\bar z_j \wedge e_\ell = -e_\ell \wedge d\bar z_j$ etc; in particular, $\bar\partial\sigma$ is a form of even degree.

If $m = 1$, then σ is just $(1/f_1)e_1$ and $R = \bar\partial(1/f_1)\wedge e_1$. In general, the coefficients of R are the Bochner-Martinelli residue currents introduced by Passare-Tsikh-Yger [26]. The sheaf complex associated with the Koszul complex is exact if and only if f is a *complete intersection*, i.e., $\operatorname{codim} Z^f = m$. In this case one can prove that (the coefficient of) $R = R_m$ coincides with the classical *Coleff-Herrera residue current* $\bar\partial(1/f_1)\wedge \cdots \wedge \bar\partial(1/f_m)$.

□

Since, in light of the above example, R generalizes the classical Coleff-Herrera residue current (as well as the Bochner-Martinelli residue currents), we say that R is the *residue current* associated with the Hermitian complex (2.4).

The construction of R in general involves the minimal inverse σ_k of each f^k in (2.4); R is defined as the analytic continuation to $\lambda = 0$ of a regularization R^λ which generalizes (2.7). The component R_k is of the form $\bar\partial |f|^{2\lambda} \wedge \sigma_k \bar\partial \sigma_{k-1} \cdots \bar\partial \sigma_1|_{\lambda=0}$; see, e.g., [7] for a precise interpretation of this. It follows that outside the set Z_k where f^k does not have optimal rank,

$$R_k = \alpha_k R_{k-1}, \tag{2.8}$$

where α_k is a smooth $\operatorname{Hom}(E_{k-1}, E_k)$-valued $(0,1)$-form. If (2.5) is exact, these sets are independent of the resolution; we call them *BEF varieties* (which is an acronym for Buchsbaum-Eisenbud-Fitting, cf. [9]) and denote them $Z_k^{\text{bef}} = Z_k^{\text{bef}}(\mathcal{J}_f)$. The Buchsbaum-Eisenbud theorem asserts that $\operatorname{codim} Z_k^{\text{bef}} \geq k$; more precisely it says that the complex (2.5) is exact if and only if the codimension of the set where f_k does not have optimal rank is $\geq k$, see, e.g., [17, Theorem 3.3]. If $\mathcal{J}_f$ has pure codimension p, then $\operatorname{codim} Z_k^{\text{bef}} \geq k+1$ for $k > p$, see [16, Corollary 20.14]. Also, note that if in addition X is locally Cohen-Macaulay, then $Z_k = \emptyset$ for $k > p$. The current R_k has bidegree $(0,k)$, and thus, by the dimension principle, $R_k = 0$ for $k < p$, and for degree reasons, $R_k = 0$ for $k > n$.

If the complex (2.4) is twisted by a Hermitian line bundle, the residue current R is not affected. This follows since the σ_k are not affected by the twisting.

2.4. **BEF-varieties on singular varieties.** Let $i : X \to Y$ be a (local) embedding of X of dimension n into a smooth manifold Y of dimension N. Note that if $\mathcal{J}_f$ is a coherent ideal sheaf on X, then $\mathcal{J}_f + \mathcal{J}_X$ is a well-defined sheaf on Y. Indeed, locally $\mathcal{J}_f$ is the pullback $i^*\widetilde{\mathcal{J}}_f$ of an ideal sheaf on Y and the sheaf $\widetilde{\mathcal{J}}_f + \mathcal{J}_X$ is independent of the choice of $\widetilde{\mathcal{J}}_f$. We define *kth BEF-variety* $Z_k^{\text{bef}}(\mathcal{J}_f)$ of $\mathcal{J}_f$ as $Z_{k+N-n}^{\text{bef}}(\mathcal{J}_f + \mathcal{J}_X)$, which clearly is a subvariety of X.

This definition is independent of the embedding i. To see this recall that (locally) i can be factorized as $X \xrightarrow{\iota} \Omega \to \Omega \times \mathbb{C}^r = Y$, where ι is a minimal embedding. From a locally free resolution of $\mathcal{O}^\Omega/\mathcal{J}$, where $\mathcal{J}$ is a coherent ideal sheaf over Ω, it is not hard to construct a locally free resolution of $\mathcal{O}^Y/(\mathcal{J} + \mathcal{J}_\Omega)$. By relating the sets where the mappings in these resolutions do not have have optimal rank one can show that the BEF-varieties of $\mathcal{J}$ are independent of i, cf. [4, Remark 4.6] and [9, Section 3].

2.5. **The structure form ω on a singular variety.** Now assume that X is as in Section 2.1, and let R be the residue current associated with a Hermitian free resolution $\mathcal{O}(E_\bullet), g^\bullet$ of the sheaf $\mathcal{J}_X$ of X, and let Ω be a global nonvanishing $(\dim \mathbb{P}^N, 0)$-form with values in $\mathcal{O}(N+1)$. It was shown in [6, Proposition 3.3] that there is a (unique) almost semi-meromorphic current $\omega = \omega_0 + \cdots + \omega_{n-1}$ on X, that is smooth on $X_{\rm reg}$ and such that

$$i_*\omega = R\wedge\Omega.$$

We say that ω is a *structure form* on X. Let E^ℓ denote the restriction of $E_{N-n+\ell}$ to X. Then the component ω_ℓ is an (n,ℓ)-form taking values in $\mathrm{Hom}\,(E^0, E^\ell)$. Moreover, let $X^0 = X_{\rm sing}$ and $X^\ell = X_{N-n+\ell}$, where X_j are the BEF-varieties of $\mathcal{J}_X$. In the language of the previous section X^ℓ is the ℓth BEF-variety of the zero sheaf. It follows from that section that the X^ℓ are independent of the embedding $i : X \to Y$ of X into a smooth manifold Y; we therefore call them the *intrinsic BEF-varieties of* X. In light of (2.8) there are almost semi-meromorphic forms α^ℓ, smooth outside X^ℓ, such that

$$\omega_\ell = \alpha^\ell \omega_{\ell-1}. \tag{2.9}$$

on X.

3. Gap sheaves and primary decomposition of sheaves

Recall that any ideal $\mathfrak{a}$ in a Noetherian ring A admits a *primary decomposition* (or *Noether-Lasker decomposition*), i.e., it can be written as $\mathfrak{a} = \bigcap \mathfrak{a}_k$, where $\mathfrak{a}_k$ is $\mathfrak{p}_k$-primary ($ab \in \mathfrak{a}_k$ implies $a \in \mathfrak{a}_k$ or $b^s \in \mathfrak{a}_k$ for some s and $\sqrt{\mathfrak{a}_k} = \mathfrak{p}_k$) for some prime ideal $\mathfrak{p}_k$. The primes in a minimal such decomposition are called the *associated primes* of $\mathfrak{a}$ and the set $\mathrm{Ass}(\mathfrak{a})$ of associated primes is independent of the primary decomposition.

Given a coherent subsheaf $\mathcal{J}$ of $\mathcal{O}^X$, Siu [28] gave a way of defining a "global" primary decomposition. Let us briefly recall his construction. First, for $p = 0, 1, \dots, \dim X$, let $\mathcal{J}_{[p]} \supset \mathcal{J}$ be the pth *gap sheaf (Lückergarbe)*, introduced by Thimm [29]: A germ $s \in \mathcal{O}_x$ is in $(\mathcal{J}_{[p]})_x$ if and only if there is a neighborhood U of x and a section $t \in \mathcal{J}(U)$ such that $s_x = t_x$ and $t_y \in \mathcal{J}_y$ for all $y \in U$ outside an analytic set of dimension at most p. It is not hard to see that $\mathcal{J}_{[p]}$ is a coherent sheaf, see [29], and that the set Y^p where $(\mathcal{J}_{[p]})_x \neq \mathcal{J}_x$ is an analytic variety of dimension at most p, see [28, Theorem 3]. The irreducible components of Y^p, $p = 0, 1, \dots, \dim X$, are called the *associated (sub)varieties* of $\mathcal{J}$. A coherent sheaf $\mathcal{J}$ is said to be *primary* if it has only one associated variety Y; we then say that $\mathcal{J}$ is Y*-primary.* Theorem 6 in [28] asserts that each coherent $\mathcal{J} \subset \mathcal{O}^X$ admits a decomposition

$$\mathcal{J} = \bigcap \mathcal{J}_i, \tag{3.1}$$

where there is one Y_i-primary intersectand $\mathcal{J}_i$ for each associated variety Y_i of $\mathcal{J}$. For a radical sheaf $\mathcal{J}_X$, the decomposition (3.1) corresponds to decomposing X into irreducible components.

By Theorem 4 in [28] if Y is an associated prime variety of $\mathcal{J}$, then at $x \in X$ the irreducible components $\mathrm{Ass}(\mathcal{J}_{Y_x})$ of Y_x are germs of varieties of associated primes of $\mathcal{J}_x$. Furthermore, if Y_x is (the variety of) an associated prime of $\mathcal{J}_x$, then Y_x is contained in Y_x^p for $p \geq \dim Y_x$. For fixed x we get that

$$\bigcup_{Y \in \mathrm{Ass}(\mathcal{J}), Y \ni x} \mathrm{Ass}(\mathcal{J}_{Y_x})$$

is a disjoint union of $\mathrm{Ass}(\mathcal{J}_x)$. Thus we have

Lemma 3.1. *The germ at x of $\mathcal{J}_{[p]}$ is precisely the intersection of the primary components of $\mathcal{J}_x$ that are of dimension $> p$.*

Given a subvariety Z of X, the *gap sheaf* $\mathcal{J}[Z] \supset \mathcal{J}$ is defined as follows: A germ $s \in \mathcal{O}_x$ is in $\mathcal{J}[Z]_x$ if and only if it extends to a section of $\mathcal{J}(U)$ for some neighborhood U of x, where $s_y \in \mathcal{J}_y$ for all $y \in U \setminus Z$. Note that $\mathcal{J}[Z]_x$ is the intersection of all components in a primary decomposition of $\mathcal{J}_x$ for which the associated varieties are not contained in Z. It is not hard to see that $\mathcal{J}[Z]$ is coherent, see [29]. Observe that $\mathcal{J}_{[p]} = \mathcal{J}[Y^p]$.

Remark 3.2. We claim that in fact

$$\mathcal{J}_{[p]} = \mathcal{J}[Z_{n-p}^{\mathrm{bef}}]. \tag{3.2}$$

To see this assume first that X is smooth. Then the (germs of) varieties of associated prime ideals of $\mathcal{J}$ of dimension $\leq p$ are precisely the (germs of) varieties of associated prime ideals that are contained in Z_{n-p}^{bef}, see, e.g., [16, Corollary 20.14]. Now (3.2) follows from Lemma 3.1.

For a general X, let $i : X \to Y$ be a local embedding of X into a manifold Y of dimension N and let $\widetilde{\mathcal{J}} = \mathcal{J} + \mathcal{J}_X$, cf. Section 2.4. It is not hard to verify that if $\mathfrak{a}$ is an ideal in $\mathcal{O}_x^X$ and $\tilde{\mathfrak{a}} := \mathfrak{a} + (\mathcal{J}_X)_x$ is the corresponding ideal in $\mathcal{O}_x^Y$ then $\mathfrak{a} = \cap \mathfrak{a}_k$ is a primary decomposition of $\mathfrak{a}$ if and only if $\tilde{\mathfrak{a}} = \cap \tilde{\mathfrak{a}}_k$ is a primary decomposition of $\tilde{\mathfrak{a}}$. Hence, in light of Lemma 3.1, $i^*\widetilde{\mathcal{J}}[V] = \mathcal{J}[V \cap X]$ and $i^*\widetilde{\mathcal{J}}_{[p]} = \mathcal{J}_{[p]}$. By the definition of BEF-varieties in Section 2.4, thus $i^*\widetilde{\mathcal{J}}[Z_{N-p}^{\mathrm{bef}}(\widetilde{\mathcal{J}})] = \mathcal{J}[Z_{N-p}^{\mathrm{bef}}(\widetilde{\mathcal{J}})] = \mathcal{J}[Z_{n-p}^{\mathrm{bef}}(\mathcal{J})]$, which proves (3.2) since $\widetilde{\mathcal{J}}_{[p]} = \widetilde{\mathcal{J}}[Z_{N-p}^{\mathrm{bef}}(\widetilde{\mathcal{J}})]$.

□

Given a residue current R constructed from a Hermitian locally free resolution of $\mathcal{O}^X/\mathcal{J}$ on a smooth X as in Section 2.3, in [8] we showed that the germ R_x of the current R at $x \in X$ can be written as $R_x = \sum R^{\mathfrak{p}}$, where the sum is over the associated primes of $\mathcal{J}_x$, and $R^{\mathfrak{p}}$ has support on the variety $V(\mathfrak{p})$ of $\mathfrak{p}$ and has the SEP with respect to $V(\mathfrak{p})$.

4. Resolutions of homogeneous ideals

Let $\mathcal{J}$ be a coherent ideal sheaf on $\mathbb{P}^N$. Then there is a locally free resolution $\mathcal{O}(E_\bullet^f), f^\bullet$, where E_k is a direct sum of line bundles $E_k = \bigoplus_i \mathcal{O}(-d_k^i)$ and $f^k = (f_{ij}^k)$ are matrices of homogeneous forms with $\deg f_{ij}^k = d_k^j - d_{k-1}^i$, see, e.g., [22, Ch.1, Example 1.2.21]. Let J denote the homogeneous ideal in the graded ring $\mathcal{S} = \mathbb{C}[z_0, \ldots, z_N]$ associated with $\mathcal{J}$, and let $\mathcal{S}(\ell)$ denote the module $\mathcal{S}$ where all degrees are shifted by ℓ. Then $\mathcal{O}(E_\bullet^f), f^\bullet$ corresponds to a free resolution

$$\ldots \to \oplus_i \mathcal{S}(-d_k^i) \to \ldots \to \oplus_i \mathcal{S}(-d_2^i) \to \oplus_i \mathcal{S}(-d_1^i) \to \mathcal{S} \tag{4.1}$$

of the module $\mathcal{S}/J$. Conversely, any such free resolution corresponds to a locally free resolution $\mathcal{O}(E_\bullet), f^\bullet$.

Recall that the *regularity* of a homogeneous module with a minimal graded free resolution (4.1) is defined as $\max_{k,i}(d_k^i - k)$, see, e.g., [17, Ch.4]. The *regularity* $\operatorname{reg} J$ of the ideal J equals $\operatorname{reg}(\mathcal{S}/J)+1$, cf. [17, Exercise 4.3].

If X is a subvariety of $\mathbb{P}^N$, then the *regularity* of X, $\operatorname{reg} X$, is defined as the regularity of J_X. Notice that if X has pure dimension, then the ideal J_X has pure dimension in $\mathcal{S}$; in particular the ideal associated to the origin is not an associated prime ideal. Theorem 20.14 in [16] thus implies that Z_0^{bef} is empty. Therefore the depth of $\mathcal{S}/J_X$ is at least 1, and hence a minimal free resolution of $\mathcal{S}/J_X$ has length $\leq N$. For such a resolution we thus get

$$\operatorname{reg} X = \max_{k\leq \min(M,N)} (d_k^i - k) + 1. \tag{4.2}$$

A global section of $\mathcal{O}(s)|_X \to X$ extends to a global section of $\mathcal{O}(s) \to \mathbb{P}^N$ as soon as $s \geq \operatorname{reg} X - 1$, see, e.g., [17, Chapter 4].

5. Division problems on singular varieties

Let $E_\bullet^g, g^\bullet$ be a complex that corresponds to a Hermitian free resolution of $\mathcal{O}^{\mathbb{P}^N}/\mathcal{J}_X$ as above, and let $E_\bullet^f, f^\bullet$ be an arbitrary Hermitian pointwise generically surjective complex over $\mathbb{P}^N$. Then the product current

$$R^f \wedge R^g := R^{f,\lambda}\wedge R^g|_{\lambda=0}$$

is well-defined on $\mathbb{P}^n$,

$$R^f\wedge\omega := R^{i^*f,\lambda}\wedge\omega|_{\lambda=0}$$

is a well-defined current on X, and $i_*(R^f\wedge\omega) = R^f\wedge R^g$, see [9, Section 2]. In particular, $R^f\wedge R^g$ and $R^f\wedge\omega$ only depend on the restriction of f to X, and thus these currents are well-defined even if f is only defined over X. Moreover $R^f\wedge R^g\phi = 0$ if and only if $R^f\wedge\omega i^*\phi = 0$. On X_{reg}, $R^f\wedge\omega$ is just the product of the current R^f and the smooth form ω.

The current $R^f\wedge R^g$ is related to the tensor product complex $E_\bullet^h, h^\bullet$, where

$$E_k^h = \bigoplus_{i+j=k} E_i^f \otimes E_j^g,$$

and $h = f+g$, cf. [9, Section 2.5], in a similar way as is the current R^h associated with this complex, see [4]. In particular, if ϕ is a section of $E_0^h = E_0^f\otimes E_0^g$ such that $R^f\wedge R^g\phi = 0$, one can locally solve $f^1q + g^1q' = \phi$. Moreover if (2.6) is satisfied for the product complex there is a global such section (q,q') of $E_1^h = E_1^f\otimes E_0^g \oplus E_0^f\otimes E_1^g$. In general, however, $R^f\wedge R^g$ does not coincide with R^h.

In fact, the definition of R^f in Section 2.3 works also when Y is singular. However, Proposition 2.1 and the duality principle do not hold in general, see, e.g., [21], and therefore R^f itself is not so well suited for division problems.

Example 5.1. Assume that $E_\bullet^f, f^\bullet$ is the Koszul complex generated by sections f_j of $L_j = \mathcal{O}(d_j)|_X$, where $X\subset\mathbb{P}^N$, twisted by $S = \mathcal{O}(\rho)$, as in Example 2.2, and that $E_\bullet^g, g^\bullet$ is a complex associated with a minimal Hermitian free resolution of $\mathcal{S}/J_X$ as in Section 4. Note that then E_ℓ^h is a direct sum of line bundles

$$\mathcal{O}(\rho - (d_{i_1} + \cdots + d_{i_\ell}) - d_{k-\ell}^i).$$

Recall that

$$H^k(\mathbb{P}^N, \mathcal{O}(\ell)) = 0 \qquad \text{if} \quad \ell \geq -N \quad \text{or} \quad k < N, \tag{5.1}$$

see, e.g., [13]. Thus (2.6) is satisfied if $\rho \geq d_{i_1} + \cdots + d_{i_\ell} + d^i_{N+1-\ell} - N$ for $\ell = 1, 2, \ldots, \min(m, n+1)$ and all choices of i and i_j. Notice that, cf., (4.2),

$$d^i_{N+1-\ell} - N = \big(d^i_{N+1-\ell} - (N+1-\ell)\big) + 1 - \ell \leq \operatorname{reg} X - \ell.$$

Hence (2.6) is satisfied if

$$\rho \geq d_1 + \cdots + d_{min(m,n+1)} - \min(m, n+1) + \operatorname{reg} X. \tag{5.2}$$

Summing up we have:

If ρ satisfies (5.2) *and ϕ is a section of $\mathcal{O}(\rho)$ on $\mathbb{P}^N$ such that $R^f{\wedge}R^g\phi = 0$ (or equivalently $R^f{\wedge}R^g i^*\phi = 0$) then there are global sections q_j of $\mathcal{O}(\rho - d_j)$ such that $f_1q_1 + \cdots + f_mq_m = \phi$ on X.*

If X is Cohen-Macaulay we may assume that $E^g_\bullet, g^\bullet$ ends at level $N - n$. If moreover $m \leq n$, then $E^h_\bullet, h^\bullet$ ends at level $\leq N$ and thus (2.6) is satisfied for any ρ.

□

Example 5.2. Let F_j be polynomials in $\mathbb{C}^N$, let $\hat{f}_j$ be the sections of $\mathcal{O}(\deg F_j) \to \mathbb{P}^N$ corresponding to F_j, and let $\mathcal{J}_{\hat{f}}$ be the ideal sheaf on $\mathbb{P}^N$ generated by the $\hat{f}_j$. Moreover, let $E^f_\bullet, f^\bullet$ and $E^g_\bullet, g^\bullet$ be complexes associated with minimal free resolutions of $\mathcal{J}_{\hat{f}}$ and $\mathcal{J}_X$ as in Section 4, where X is a subvariety of $\mathbb{P}^N$; say $E^f_k = \bigoplus \mathcal{O}(\delta^i_k)$ and $E^g_k = \bigoplus \mathcal{O}(d^i_k)$. Then E^h_k is a direct sum of line bundles $\mathcal{O}(-\delta^i_\ell - d^j_{k-\ell})$, and thus (2.6) is satisfied if $\rho \geq \delta^i_\ell + d^j_{N+1-\ell} - N$ for all i, j, ℓ, cf. Example 5.1. Notice that, in light of Section 4,

$$\delta^i_\ell + d^j_{N+1-\ell} - N = (\delta^i_\ell - \ell) + (d^j_{N+1-\ell} - (N+1-\ell)) + 1 \leq \operatorname{reg} J_{\hat{f}} + \operatorname{reg} X - 1,$$

where $J_{\hat{f}}$ is the homogeneous ideal associated with $\mathcal{J}_{\hat{f}}$. Thus (2.6) is satisfied if $\rho \geq \operatorname{reg} J_{\hat{f}} + \operatorname{reg} X - 1$.

Let $Z^{\hat{f}}_k$ and Z^g_ℓ be the BEF-varieties of $\mathcal{J}_{\hat{f}}$ and $\mathcal{J}_X$, respectively. Theorem 4.2 in [4] asserts that if

$$\operatorname{codim}(Z^{\hat{f}}_k \cap Z^g_\ell) \geq k + \ell, \tag{5.3}$$

then $R^f{\wedge}R^g\phi = 0$ if and only if $\phi \in \mathcal{J}_{\hat{f}} + \mathcal{J}_X = \mathcal{J}_f + \mathcal{J}_X$, where $\mathcal{J}_f$ is the sheaf on X generated by the restrictions f_j of $\hat{f}_j$, cf. Section 2.4. If moreover $\mathcal{J}_{\hat{f}}$ and $\mathcal{J}_X$ are both Cohen-Macaulay and the resolutions $\mathcal{O}(E^f_\bullet), f^\bullet$ and $\mathcal{O}(E^g_\bullet), g^\bullet$ have minimal length, then $R^f{\wedge}R^g = R^h$, see [4, Theorem 4.2]. □

5.1. **Distinguished varieties.** Let X be a subvariety of $\mathbb{P}^N$ and let $\tilde{f}_j$ be sections of $L = \mathcal{O}(d)|_X$. Moreover, let $\nu\colon X_+ \to X$ be the normalization of the blow-up of X along $\mathcal{J}_{\tilde{f}}$, and let $W = \sum r_j W_j$ be the exceptional divisor; here W_j are irreducible Cartier divisors. The images $Z_j := \nu(W_j)$ are called the *(Fulton-MacPherson) distinguished varieties* associated with $\mathcal{J}_{\tilde{f}}$, see, e.g., [22]. If we consider $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_m)$ as a section of $E^* := \oplus_1^m \mathcal{O}(-d)$, then $\nu^*\tilde{f} = \tilde{f}^0\tilde{f}'$, where $\tilde{f}^0$ is a section of the line bundle $\mathcal{O}(-W)$ and $\tilde{f}' = (\tilde{f}'_1, \ldots, \tilde{f}'_m)$ is a nonvanishing section of $\nu^*E^* \otimes \mathcal{O}(W)$, where $\mathcal{O}(W) = \mathcal{O}(-W)^{-1}$. Furthermore, $\omega_{\tilde{f}} := dd^c \log|\tilde{f}'|^2$ is a smooth first Chern form for $\nu^*L \otimes \mathcal{O}(W)$. We will use the geometric estimate

$$\sum r_j \deg_L Z_j \leq \deg_L X \tag{5.4}$$

from [15, Proposition 3.1], see also [22, (5.20)].

Let $R^{\tilde{f}}$ be the residue current associated with the Koszul complex of the $\tilde{f}_j$ as in Example 2.2 and consider the regularization (2.7) of $R^{\tilde{f}}$. Using the notation in Example 2.2, $\nu^*\sigma = (1/\tilde{f}^0)\sigma'$, where $1/\tilde{f}^0$ is a meromorphic section of $\mathcal{O}(W)$ and σ' is a smooth section of $\nu^*E \otimes \mathcal{O}(-W)$. It follows that

$$\nu^*(\sigma\wedge(\bar\partial\sigma)^{k-1}) = \frac{1}{(\tilde{f}^0)^k}\sigma'\wedge(\bar\partial\sigma')^{k-1},$$

and hence

$$\nu^* R_k^\lambda = \bar\partial|\tilde{f}^0\tilde{f}'|^{2\lambda}\wedge\frac{1}{(\tilde{f}^0)^k}\sigma'\wedge(\bar\partial\sigma')^{k-1} \text{ for } \operatorname{Re}\lambda >> 0,$$

when $k \geq 1$. Since $\tilde{f}'$ is nonvanishing, by (2.1) the value at $\lambda = 0$ is precisely

$$R_k^+ := \bar\partial\frac{1}{(\tilde{f}^0)^k}\wedge\sigma'\wedge(\bar\partial\sigma')^{k-1}. \tag{5.5}$$

Thus

$$\nu_* R_k^+ = R_k^{\tilde{f}}.$$

6. Proofs

Proof of Theorem 1.5. For $j = 1, \ldots, m$, let $\hat{f}_j$ be the $\deg F_j$-homogenization of the polynomial F_j, considered as a section of $\mathcal{O}(\deg F_j) \to \mathbb{P}^N$. Moreover let $g_1, \ldots, g_r$ be global generators of the ideal sheaf $\mathcal{J}_X$; assume they are sections of $\mathcal{O}(d_1), \ldots, \mathcal{O}(d_r)$, respectively. Let $\mathcal{J} = \mathcal{J}_{\hat{f}} + \mathcal{J}_X = \mathcal{J}_f + \mathcal{J}_X$. Then there is a locally free resolution $\mathcal{O}(E_\bullet^h), h^\bullet$ of $\mathcal{O}/\mathcal{J}$, where each E_k^h is a direct sum of line bundles $E_k = \bigoplus_i \mathcal{O}(-d_k^i)$ and in particular $E^1 = \bigoplus_1^m \mathcal{O}(-\deg F_j) \oplus_1^r \bigoplus \mathcal{O}(-d_k)$ and $h^1 = (f_1, \ldots, f_m, g_1, \ldots, g_r) =: f + g$, cf. Section 4. Let $R = R^h$ be the residue current associated with $E_\bullet^h, h^\bullet$.

Recall from Section 3 that for fixed $x \in X$, $R_x = \sum R^{\mathfrak{p}}$, where the sum is over $\operatorname{Ass}(\mathcal{J}_x)$ and where $R^{\mathfrak{p}}$ has the SEP with respect to $V(\mathfrak{p})$; in particular, $\mathbf{1}_{H_\infty}R^{\mathfrak{p}} = R^{\mathfrak{p}}$ if $V(\mathfrak{p}) \subset H_\infty$ and $\mathbf{1}_{H_\infty}R^{\mathfrak{p}} = 0$ otherwise. Thus

$$\mathbf{1}_{H_\infty}R_x = \sum_{\mathfrak{p}\in\operatorname{Ass}(\mathcal{J}_x), V(\mathfrak{p})\subset H_\infty} R^{\mathfrak{p}}. \tag{6.1}$$

In Remark 3.2 we saw that $\mathfrak{a} = \cap\mathfrak{a}_k$ is a primary decomposition of the ideal $\mathfrak{a}$ in $\mathcal{O}_x^X$ if and only if $\tilde{\mathfrak{a}} = \cap\tilde{\mathfrak{a}}_k$ is a primary decomposition of the ideal $\tilde{\mathfrak{a}} = \mathfrak{a} + (\mathcal{J}_X)_x$ in $\mathcal{O}_x^Y$. Thus, that $\mathcal{J}_f$ has no associated varieties contained in X_∞ implies that, for a fixed $x \in X$, $\mathcal{J}_x$ has no (varieties of) associated primes contained in the hyperplane H_∞ at infinity in $\mathbb{P}^N$. We conclude, in light of (6.1), that $\mathbf{1}_{H_\infty}R = 0$. If ϕ is any homogenization of Φ then $\mathbf{1}_{\mathbb{C}^N}R\phi = 0$ because of the duality principle and hence $R\phi = \mathbf{1}_{H_\infty}R\phi + \mathbf{1}_{\mathbb{C}^N}R\phi = 0$.

Assume that the complex $E_\bullet^h, h^\bullet$ ends at level M (by Hilbert's syzygy theorem we may assume that $M \leq N+1$) and let

$$\beta := \max_i d_{N+1}^i - N \text{ if } M = N+1 \quad \text{ and } \beta := 0 \text{ otherwise.} \tag{6.2}$$

If $\rho \geq \beta$ then (2.6) is satisfied for $E_\bullet^h, h^\bullet$ twisted by $\mathcal{O}(\rho)$ in light of (5.1) and thus by Proposition 2.1 there are global holomorphic sections $q = (q_j)$ of $\bigoplus\mathcal{O}(\rho - \deg F_j)$ and $q' = (q_k')$ of $\bigoplus\mathcal{O}(\rho - d_k)$ over $\mathbb{P}^N$ such that $\hat{f}q + gq' = \phi$. Indeed, recall from the end of Section 2.3 that R is also the residue current associated with the twisted complex. Dehomogenizing gives polynomials Q_j, Q'_j, and G_j in $\mathbb{C}^N$ such that

$$\sum F_j Q_j + \sum G_j Q'_j = \Phi$$

and where $\deg(F_j Q_j) \leq \rho$. Since the G_j vanish on V we get the desired solution to (1.1) on V, and thus the first part of Theorem 1.5 follows with β as in (6.2).

If $V = \mathbb{C}^N$, $\mathcal{O}_X$ should be interpreted as the zero sheaf. Then $E^h_\bullet, h^\bullet$ is a locally free resolution of $\mathcal{O}/\mathcal{J}_f$ and $\beta \leq \operatorname{reg} J_f$, cf. Section 4.

For the second part of Theorem 1.5, assume that $\mathcal{J}_f$ has an associated variety contained in X_∞. We are to prove that for arbitrarily large ℓ there is a polynomial $\Phi = \Phi_\ell$ of degree $\geq \ell$ in (F_j) on V for which one can not solve (1.1) with $\deg(F_j Q_j) \leq \deg \Phi_\ell$.

Let $L = \mathcal{O}(1)|_X$. The hypothesis on $\mathcal{J}_f$ then means that $\mathcal{J}_f[X_\infty]$ is strictly larger than $\mathcal{J}_f$. Therefore, since L is ample, for some large enough s_0 there is a global section ψ_0 of $L^{\otimes s_0} \to X$ such that ψ_0 is in $\mathcal{J}_f[X_\infty]$ but not in $\mathcal{J}_f$. Moreover we can find a global section ψ of $L^{\otimes s}$ for some $s \geq 1$ such that ψ does not vanish identically on any of the associated varieties of $\mathcal{J}_f$ that are contained in X_∞. We may assume that $s_0, s \geq \operatorname{reg} X - 1$, so that ψ_0 and ψ extend to global sections $\hat{\psi}_0$ and $\hat{\psi}$ of $\mathcal{O}(s_0)$ and $\mathcal{O}(s)$, respectively. Let Ψ_0 and Ψ be the corresponding dehomogenized polynomials in $\mathbb{C}^N$. For $\ell \geq 0$, let $\phi_\ell = \psi_0 \psi^\ell$ and $\Phi_\ell = \Psi_0 \Psi^\ell$. Since $\mathcal{J}_f[X_\infty]_x = (\mathcal{J}_f)_x$ for all $x \in V$, Φ_ℓ is in the ideal (F_j) on V, and thus we can solve (1.1) for $\Phi = \Phi_\ell$ on V. Assume that there is a solution to (1.1) with $\deg(F_j Q_j) \leq \rho_\ell$. Then there are sections q_j of $L^{\rho_\ell - \deg F_j}$ such that

$$\sum f_j q_j = z_0^{\rho_\ell - (s_0 + s\ell)} \phi_\ell$$

on X. Since ϕ_ℓ is not in $\mathcal{J}_f$ it follows that $\rho_\ell - (s_0 + s\ell) \geq 1$ and thus $\rho_\ell \geq 1 + (s_0 + s\ell) \geq 1 + \deg \Phi_\ell$. Since $\hat{\psi}$ does not vanish identically at X_∞, $\deg \Psi \geq 1$ and hence $\deg \Phi_\ell \geq \ell$. Hence we have found Φ_ℓ with the desired properties and the second part of Theorem 1.5 follows.

□

Remark 6.1. If $\mathcal{J}_{\hat{f}}$ and $\mathcal{J}_X$ are Cohen-Macaulay and the BEF-varieties of $\mathcal{J}_{\hat{f}}$ and $\mathcal{J}_X$ satisfy (5.3), then we can choose the complex $E^h_\bullet, h^\bullet$ in the above proof to be the tensor product of the complexes $E^f_\bullet, f^\bullet$ and $E^g_\bullet, g^\bullet$ corresponding to minimal resolutions of $\mathcal{J}_{\hat{f}}$ and $\mathcal{J}_X$, see Example 5.2. In this case, by Example 5.2, we get Theorem 1.5 for $\beta = \operatorname{reg} J_{\hat{f}} + \operatorname{reg} X - 1$.

□

The residue current technique in the preceding proof is convenient and makes it possible to carry out the proof within our general framework, but it is not crucial.

Remark 6.2 (The algebraic approach). Let us first sketch an algebraic proof of the first part of Theorem 1.5. We use the notation from the proof above. To begin with we have to prove that ϕ is in $\mathcal{J}$, which of course precisely corresponds to proving that $R\phi = 0$. Since (the restriction to V of) ϕ is in $\mathcal{J}_f$ on V it follows that $\phi_{x'}$ is in $\mathcal{J}$ outside H_∞. Since moreover $\mathcal{J} = \mathcal{O}^{\mathbb{P}^N}$ outside X, we have to prove that $\phi_x \in \mathcal{J}_x$ for each $x \in X_\infty$. At such a point x we have a minimal primary decomposition $\mathcal{J}_x = \cap_\ell \mathcal{J}^\ell_x$. Since $\mathcal{J}$ is coherent, $\mathcal{J} \subset \mathcal{J}^\ell$ in a neighborhood $\mathcal{U}$ of x, where $\mathcal{J}^\ell$ is the coherent sheaf defined by $\mathcal{J}^\ell_x$. Let Z^ℓ be the zero-set of $\mathcal{J}^\ell$. Since $\phi_{x'}$ is in $\mathcal{J}_{x'}$ for x' outside H_∞ it follows that $\phi_{x'}$ is in $\mathcal{J}^\ell_{x'}$ for $x' \in Z^\ell \setminus H_\infty$. Hence $\mathcal{F} := (\mathcal{J}^\ell + (\phi))/\mathcal{J}^\ell$ is a coherent sheaf in $\mathcal{U}$ with support on $Z^\ell \cap H_\infty$. Since by assumption $\mathcal{J}_f$ has no associated varieties contained in X_∞ it follows that $Z^\ell \cap H_\infty$ has positive codimension in Z^ℓ, cf. the proof of Theorem 1.5 above. Therefore, by the Nullstellensatz there is a holomorphic function h, not vanishing identically on Z^ℓ such that $h\mathcal{F} = 0$. In particular, $h_x\phi_x \in \mathcal{J}^\ell_x$. Since h_x is not in the radical of $\mathcal{J}^\ell_x$ and $\mathcal{J}^\ell_x$ is primary it follows that $\phi_x \in \mathcal{J}^\ell_x$. We conclude that $\phi_x \in \mathcal{J}_x$. Notice that the last arguments above can be thought of as an algebraic version of the SEP-argument in the proof of Theorem 1.5 above.

Next we would like to use that $\phi \in \mathcal{J}$ to conclude that there is a global holomorphic solution to $hq = \phi$. By a partition of unity, using that $E^h_\bullet, h^\bullet$ is exact, one can glue local such solutions together to obtain a global smooth solution to $(h - \bar\partial)\psi = \phi$, cf. [9, Section 4]. By solving a certain sequence of $\bar\partial$-equations in $\mathbb{P}^N$ we can modify ψ to a global holomorphic solution q to $hq = \phi$. These $\bar\partial$-equations are solvable if $\rho \geq \beta$ defined by (6.2). Alternatively, one can directly refer to the well-known result that there is a solution to $hq = \phi$ if $\rho \geq \operatorname{reg} J$, where J is the homogeneous ideal corresponding to $\mathcal{J}$, see, e.g., [17, Proposition 4.16].

In the same way Theorems 1.1 and 1.2 follow without any reference to residues. Probably one can also find give an algebraic proof of Theorem 1.4.

□

In the next proof the residue technique plays a more decisive role.

Proof of Theorem 1.6. Let

$$\rho = \max(\deg \Phi + \mu d^{c_\infty} \deg X, (d-1)\min(m, n+1) + \operatorname{reg} X),$$

or if X is Cohen-Macaulay and $m \leq n$ let $\rho = \deg \Phi + m d^{c_\infty} \deg X$, and let ϕ be the ρ-homogenization of Φ considered as a section of $\mathcal{O}(\rho)|_X$. Note that then $\phi = z_0^{\rho - \deg \Phi} \tilde\phi$, where $\tilde\phi$ is the $\deg \Phi$-homogenization of Φ. Moreover, let $R^{\tilde f} \wedge \omega$ be the residue current associated with the (twisted) Koszul complex $E^{\tilde f}_\bullet, \tilde f^\bullet$ of the sections $\tilde f_j$ of $\mathcal{O}(d)|_X$ associated with F_j, and a complex $E^g_\bullet, g^\bullet$ associated with a minimal resolution of $\mathcal{O}/\mathcal{J}_X$ as in Example 5.1 (with $d_j = d$ for all j).

Claim: $R^{\tilde f} \wedge \omega_0 \phi$ has support on $Z^{\tilde f} \cap X^0$.

To prove the claim, since ω is smooth on $X_{\rm reg}$, it is enough to show that $R^{\tilde f}\phi = 0$ on $X_{\rm reg}$. First, since $\operatorname{codim} Z^{\tilde f} \cap V \geq m$, the duality principle for a complete intersection, cf. Example 2.2, implies that $R^{\tilde f}\phi = 0$ on $V_{\rm reg}$.

Next, to prove that $\mathbf{1}_{X_\infty \setminus X^0} R^{\tilde f}\phi = 0$ we consider the normalization of the blow-up $\nu\colon X_+ \to X$, and let $R^+ := \sum R^+_k$ be as in Section 5.1. Let W' be the union of the irreducible components of $W = \nu^{-1} Z^{\tilde f}$ that are contained in $\nu^{-1} X_\infty$. We claim that

$$\mathbf{1}_{X_\infty} R^{\tilde f} = \nu_*\big(\mathbf{1}_{W'} R^+\big). \tag{6.3}$$

In fact, by (2.3),

$$\mathbf{1}_{X_\infty} R^{\tilde f} = \nu_*\big(\mathbf{1}_{\nu^{-1}X_\infty} R^+\big) = \nu_*\big(\mathbf{1}_{\nu^{-1}X_\infty}(\mathbf{1}_{W'} + \mathbf{1}_{W \setminus W'}) R^+\big). \tag{6.4}$$

By, (2.2), $\mathbf{1}_{\nu^{-1}X_\infty}\mathbf{1}_{W'}R^+ = \mathbf{1}_{W'}R^+$. Moreover,

$$\mathbf{1}_{\nu^{-1}X_\infty}\mathbf{1}_{W\setminus W'}\bar\partial\frac{1}{(\tilde f^0)^k} = \mathbf{1}_{\nu^{-1}X_\infty\cap(W\setminus W')}\bar\partial\frac{1}{(\tilde f^0)^k} = 0$$

by (2.2) and the dimension principle, since $\nu^{-1}X_\infty\cap(W\setminus W')$ has codimension at least 2 in X_+. In view of (5.5) we conclude that $\mathbf{1}_{\nu^{-1}X_\infty}\mathbf{1}_{W\setminus W'}R^+ = 0$, and thus (6.3) follows from (6.4).

It follows from (6.3) that $\mathbf{1}_{X_\infty\setminus X^0}R^{\tilde f}\phi = 0$ if $\mathbf{1}_{W'}R^+\nu^*\phi = 0$. To show that $\mathbf{1}_{W'}R^+\nu^*\phi$ vanishes first note that it is sufficient to show that it vanishes in a neighborhood of each point x on W' where W is smooth. Indeed, since $W_{\rm sing}$ has codimension at least 2 in W, $\mathbf{1}_{W_{\rm sing}}\bar\partial(1/(\tilde f^0)^k) = 0$ by the dimension principle. Hence, using (5.5) and (2.2) we get that

$$\mathbf{1}_{W'}R^+ = \mathbf{1}_{W'}(\mathbf{1}_{W_{\rm reg}} + \mathbf{1}_{W_{\rm sing}})R^+ = \mathbf{1}_{W'\cap W_{\rm reg}}R^+.$$

Consider now $x\in 1_{W'\cap W_{\rm reg}}$; say x is contained in the irreducible component W_j of W'. In a neighborhood of x we have that $\tilde f^0 = s^{r_j}v$, where s is a local coordinate function and v is nonvanishing and r_j is as in Section 5.1. Since $\phi = z_0^{\rho-\deg\Phi}\tilde\phi$, by the choice of ρ, $\nu^*\phi$ vanishes to order (at least) $\mu d^{c_\infty}\deg X$ on W'.

If Ω is a first Chern form for $\mathcal{O}(1)|_X$, e.g., $\Omega = dd^c\log|z|^2$, then $d\Omega$ is a first Chern form for $L=\mathcal{O}(d)|_X$ on X (notice that d denotes the degree and not the differential). By (5.4) we therefore have that

$$r_j\int_{Z_j}(d\Omega)^{\dim Z_j}\le\int_X(d\Omega)^n,$$

which implies that

$$r_j\le d^{\operatorname{codim} Z_j}\deg X.$$

It follows that $\nu^*\phi$ vanishes (at least) to order μr_j on W_j and hence it has a factor $s^{\mu r_j}$. In a neighborhood of x,

$$\bar\partial\frac{1}{(\tilde f^0)^k} = \bar\partial\frac{1}{s^{kr_j}}\wedge smooth$$

and thus, in light of (5.5), $R^+_k\nu^*\phi = 0$ for $k\le\mu$ there. Hence $\mathbf{1}_{W'\cap W_{\rm reg}}R^+_k\nu^*\phi = 0$ for $k\le\mu$ and $\mathbf{1}_{X_\infty\setminus X^0}R^{\tilde f}\phi = 0$. We conclude that $\mathbf{1}_{X\setminus X^0}R^{\tilde f}\phi = \mathbf{1}_{V_{\rm reg}}\ R^{\tilde f}\phi + \mathbf{1}_{X_\infty\setminus X^0}R^{\tilde f}\phi = 0$, which proves the claim that $R^{\tilde f}\wedge\omega_0\phi$ has support on $Z^{\tilde f}\cap X^0$.

By (1.8) and the dimension principle we conclude that $R^{\tilde f}\wedge\omega_0\phi$ vanishes identically, since the bidegree of $R^{\tilde f}$ is at most $(0,m)$ and ω_0 has bidegree $(n,0)$. Thus $R^{\tilde f}\wedge\omega_1\phi = R^{\tilde f}\wedge\alpha^1\omega_0\phi$, see (2.9), vanishes outside X^1. By (1.8) and the dimension principle, it vanishes identically since the bidegree of $R^{\tilde f}\wedge\omega_1$ is at most $(n,m+1)$. By induction, it follows that $R^{\tilde f}\wedge\omega_\ell\phi = 0$ for each ℓ. We conclude that $R^{\tilde f}\wedge\omega\phi = 0$.

Since ρ satisfies (5.2) (with $d_j = d$) and $R^{\tilde f}\wedge\omega\phi = 0$, by Example 5.1 there is a global section $q=(q_j)$ of $\sum_1^m\mathcal{O}(\rho-d)$ such that $fq=\phi$ on X. Dehomogenizing gives polynomials Q_j such that (1.1) holds on V and $\deg(F_jQ_j)\le\rho$.

□

Proof of Theorems 1.1 and 1.4. Let

$$\rho = \max(\deg\Phi, d_1+\ldots+d_{\min(m,n+1)} - \min(m,n+1) + \operatorname{reg} X),$$

or if X is Cohen-Macaulay and $m \leq n$ let $\rho = \deg \Phi$. Moreover let ϕ be the ρ-homogenization of Φ and let $R^f \wedge \omega$ be the residue current associated with the twisted Koszul complex $E^f_\bullet, f^\bullet$ of the $\deg F_j$-homogenizations f_j of F_j and a minimal resolution of $\mathcal{O}/\mathcal{J}_X$ as in Example 5.1.

We claim that under the hypotheses of both theorems $R^f \wedge \omega_0 \phi$ has support on $Z^f \cap X^0$. Since ω is smooth outside X^0 it is enough to show that $R^f \phi = 0$ there. First in the case of Theorem 1.1, R^f vanishes for trivial reasons, since Z^f is empty. In the case of Theorem 1.4, first $R^f \phi$ vanishes on $V_{\rm reg}$ by the duality principle. Next, since by assumption (1.2) holds and Z^f has no irreducible components in X_∞, it holds that $\operatorname{codim}(X_\infty \cap Z^f) > m$. Since the components of R^f have bidegree at most $(0, m)$, we conclude that $\mathbf{1}_{X_\infty \setminus X_0} R^f = 0$ by the dimension principle. This proves that $R^f \wedge \omega \phi$ has support on $Z^f \cap X^0$.

Now arguing as in the end of the proof of Theorem 1.6, we get that $R^f \wedge \omega \phi = 0$, and the results follow from Example 5.1.

□

Remark 6.3. If $\deg F_j = d$, then Theorems 1.1 and 1.4 follow directly from Theorem 1.6. First, notice that Theorem 1.1 follows if we apply Theorem 1.6 to F_j with no common zeros on X. Indeed, since Z^f is empty, $\operatorname{codim}(Z^f \cap X) = \infty$ and thus (1.7) and (1.8) are satisfied, and moreover $c_\infty = -\infty$.

Next, assume that F_j satisfy the hypothesis of Theorem 1.4. Since the codimension of a distinguished variety is at most m the condition that Z^f satisfies (1.2) and has no irreducible component contained in X_∞ means that (1.7) is satisfied and no distinguished varieties can be contained in X_∞. Thus $c_\infty = -\infty$ and $d^{c_\infty} = 0$ and Theorem 1.4 follows from Theorem 1.6.

□

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