RESIDUE CURRENTS OF MONOMIAL IDEALS

ELIZABETH WULCAN

ABSTRACT. We compute residue currents of Bochner-Martinelli type associated with a monomial ideal I, by methods involving certain toric varieties. In case the variety of I is the origin, we give a complete description of the annihilator of the currents in terms of the associated Newton diagram. In particular, we show that the annihilator is strictly included in I, unless I is defined by a complete intersection. We also provide partial results for general monomial ideals.

1. INTRODUCTION

Let f be a tuple of holomorphic functions $f_1, \ldots, f_m$ in $\mathbb{C}^n$ and let $Y = \{f_1 = \ldots = f_m = 0\}$. If f is a complete intersection, that is, the codimension of Y is m, the duality theorem, due to Dickenstein-Sessa, [6], and Passare, [10], asserts that a holomorphic function h locally belongs to the ideal $(f) = (f_1, \ldots, f_m)$ if and only if $hR_{CH}^f = 0$, where R_{CH}^f is the Coleff-Herrera residue current of f. In [11], Passare, Tsikh and Yger introduced residue currents for arbitrary f by means of the Bochner-Martinelli kernel. For each ordered index set $\mathcal{I} \subseteq \{1, \ldots, m\}$ of cardinality k, let $R_{\mathcal{I}}^f$ be the analytic continuation to $\lambda = 0$ of

$$\bar{\partial}|f|^{2\lambda} \wedge \sum_{\ell=1}^{k} (-1)^{\ell-1} \frac{\overline{f_{i_\ell}} \bigwedge_{q \neq \ell} \overline{df_{i_q}}}{|f|^{2k}},$$

where $|f|^2 = |f_1|^2 + \ldots + |f_m|^2$. Then $R_{\mathcal{I}}^f$ is a well-defined $(0, k)$-current with support on Y, that vanishes whenever $k < \operatorname{codim} Y$ or $k > \min(m, n)$. In case f defines a complete intersection, the only nonvanishing current, $R_{\{1,\ldots,m\}}^f$, is shown to coincide with the Coleff-Herrera current.

The concept of Bochner-Martinelli residue currents was further developed by Andersson in [1]. From his construction, based on the Koszul complex, follows that $hR_{\mathcal{I}}^f = 0$ for all $\mathcal{I}$ implies that the holomorphic function h belongs to the ideal (f) locally. Thus, letting

Ann R^f denote the annihilator ideal, $\{h \text{ holomorphic}, hR_{\mathcal{I}}^f = 0, \forall \mathcal{I}\}$, we have that

$$(1.1) \qquad\qquad\qquad \text{Ann } R^f \subseteq (f).$$

The inclusion is strict in general, and thus the currents $R_{\mathcal{I}}^f$ do not fully characterize (f) as in the complete intersection case. Still the ideal Ann R^f is big enough to catch in some sense the "size" of (f). Recall that a holomorphic function h belongs locally to the *integral closure* of (f), denoted by $\overline{(f)}$, if $|h| \leq C|f|$ for some constant C, or equivalently if h fulfills a monic equation $h^r + g_1 h^{r-1} + \ldots + g_r = 0$ with $g_i \in (f)^i$ for $1 \leq i \leq r$. In [11] it was proved that $hR_{\mathcal{I}}^f = 0$ for any h that is locally in the integral closure of $(f)^k$, where $k = |\mathcal{I}|$, and thus we get

$$(1.2) \qquad\qquad\qquad \overline{(f)^\mu} \subseteq \text{Ann } R^f,$$

where $\mu = \min(m, n)$. Now, combining (1.1) and (1.2) yields a proof of the Briançon-Skoda theorem [5]: $\overline{(f)^\mu} \subseteq (f)$. This motivates us to study the ideal Ann R^f.

In this paper we compute the Bochner-Martinelli currents $R_{\mathcal{I}}^f$ in case the generators f_i are all monomials. Because of their simplicity and nice combinatorial description monomial ideals serve as a good toy model for illustrating general ideas and results in commutative algebra and algebraic geometry, such as resolution of singularities and Briançon-Skoda type theorems, see [13]. On the other hand many results for general ideals can be proved by specializing to monomial ideals. Recall from [11] and [1] that the existence of (Bochner-Martinelli) residue currents indeed is proved by reducing to a monomial situation by resolving singularities. Monomial ideals are therefore a natural starting point for investigating the inclusions (1.1) and (1.2) and for producing examples of $R_{\mathcal{I}}^f$. A first result in this direction was obtained in [11] (Proposition 3.1) where $R_{\{1,\ldots,n\}}^f$ was computed explicitly for monomial ideals generated by exactly n monomials.

Our main result, Theorem 3.1, gives a complete description of Ann R^f in terms of the Newton diagram associated with the generators, when (f) is a monomial ideal of dimension 0. In particular it turns out that Ann R^f depends only on (f), not on the particular choice of generators. Also, it follows that we have equality in (1.1) if and only if (f) is a complete intersection and moreover that the inclusion (1.2) is always strict. The proof of Theorem (3.1), given in Section 4, amounts to computing residue currents in a certain toric variety constructed from the generators, using ideas originally from Varchenko, [14], and Khovanskii, [8]. In Section 5 we provide partial results for the case of general monomial ideals.

2. PRELIMINARIES AND NOTATION

Let A be a set in $\mathbb{Z}_+^n$ and let z^A denote the tuple of monomials $\{z^a\}_{a \in A}$, where $z^a = z_1^{a_1} \cdots z_n^{a_n}$ if $a = (a_1, \ldots, a_n)$. The ideal (z^A) admits a nice geometric interpretation as the set $\cup_{a \in A}(a + \mathbb{R}_+^n) \subset \mathbb{R}^n$. Indeed, a holomorphic function is in the ideal precisely when its support $(\mathrm{supp} \sum \varphi_a z^a = \{a \in \mathbb{Z}_+^n, \varphi_a \neq 0\})$ is in $\cup_{a \in A}(a + \mathbb{R}_+^n)$. The *Newton polyhedron* $\Gamma^+(A)$ of A is defined as the convex hull of $\cup_{a \in A}(a + \mathbb{R}_+^n) \subset \mathbb{R}^n$ and the *Newton diagram* $\Gamma(A)$ of A is the union of all compact faces of the Newton polyhedron. Recall that a face of maximal dimension is called a *facet*. For further reference we remark that the set of vertices of the Newton polyhedron is a subset of A, see for example [15].

We will work in the framework from [1] and use the fact that the currents $R_{\mathcal{I}}^f$ appear as the coefficients of full Bochner-Martinelli current introduced there. We identify z^A with a section of the dual bundle E^* of a trivial vector bundle E over $\mathbb{C}^n$ of rank $m = |A|$ (the number of generators), endowed with the trivial metric. If $\{e_a\}_{a \in A}$ is a global holomorphic frame for E and $\{e_a^*\}_{a \in A}$ is the dual frame, we can write $z^A = \sum_{a \in A} z^a e_a^*$. We let s be the dual section $\sum_{a \in A} \bar{z}^a e_a$ of z^A. Also, we fix an ordering of A.

Next, we let

$$u = \sum_\ell \frac{s \wedge (\bar{\partial} s)^{\ell - 1}}{|z^A|^{2\ell}},$$

where $|z^A|^2 = \sum_{a \in A} |z^a|^2$, be the full Bochner-Martinelli form, introduced in [2] in order to construct integral formulas with weight factors in a convenient way. Then u is a smooth section of $\Lambda(E \oplus T_{0,1}^*(\mathbb{C}^n))$ (where $e_a \wedge d\bar{z}_i = -d\bar{z}_i \wedge e_a$), that is clearly well defined outside $Y = f^{-1}(0)$, and moreover

$$\bar{\partial}|z^A|^{2\lambda} \wedge u$$

has an analytic continuation as a current to $\mathrm{Re}\,\lambda > -\epsilon$. The *(full) Bochner-Martinelli residue current* R^{z^A} is defined as the value at $\lambda = 0$. Then R^{z^A} has support on Y and $R^{z^A} = R_p + \ldots + R_\mu$, where $p = \mathrm{codim}\,Y$ and $\mu = \min(m, n)$, and where $R_k \in \mathcal{D}_{0,k}'(\mathbb{C}^n, \Lambda^k E)$, by analogy with the fact that the current $R_{\mathcal{I}}^f$ vanishes if $|\mathcal{I}|$ is smaller than p or greater than μ. We should remark that Andersson's construction of residue currents, using kernels of Cauchy-Fantappiè-Leray type, works for sections of any holomorphic vector bundle equipped with some Hermitian metric. Observe that in our case (trivial bundle and trivial metric), though, the coefficients of R^{z^A} are just currents of the type $R_{\mathcal{I}}^f$. Indeed, letting s_B be the section $\sum_{a \in B} \bar{z}^a e_a$, we can write u as a sum, taken over subsets B of A, of terms

$$u_B = \frac{s_B \wedge (\bar{\partial} s_B)^{k-1}}{|z^A|^{2n}},$$

where k is the cardinality of B. The corresponding current,

$$(2.1) \qquad\qquad \bar{\partial}|z^A|^{2\lambda} \wedge u_B$$

evaluated at $\lambda = 0$, denoted by $R_B^{z^A}$ or R_B for short, is then merely the current $R_{\mathcal{I}}^f$ with $\mathcal{I}$ corresponding to the subset B, times the basis element $e_B = \bigwedge_{a \in B} e_a$, where the wedge product is taken with respect to the ordering. Henceforth we will deal with the Bochner-Martinelli currents rather then currents $R_{\mathcal{I}}^f$.

Let us make an observation that will be of further use. If the section s can be written as $\mu s'$ for some smooth function μ we have the following homogeneity:

$$(2.2) \qquad\qquad s \wedge (\bar{\partial}s)^{k-1} = \mu^k s' \wedge (\bar{\partial}s')^{k-1},$$

that holds since s is of odd degree.

We will use the notation $\bar{\partial}[1/f]$ for the value at $\lambda = 0$ of $\bar{\partial}|f|^{2\lambda}/f$ and analogously by $[1/f]$ we will mean $|f|^{2\lambda}/f|_{\lambda=0}$, that is just the principal value of $1/f$. By iterated integration by parts we have that

$$(2.3) \qquad\qquad \int_z \bar{\partial}\Big[\frac{1}{z^p}\Big] \wedge \varphi dz = \frac{2\pi i}{(p-1)!} \frac{\partial^{p-1}}{\partial z^{p-1}} \varphi(0).$$

In particular, the annihilator of $\bar{\partial}[1/z^p]$ is (z^p). The currents $R_B^{z^A}$ will typically be tensor products of currents of this type.

3. Main results

Our main result is an explicit computation of the Bochner-Martinelli residue current R^{z^A} in case Y is the origin. Before stating it let us introduce some notation. We say that a subset $B = \{a_1, \ldots, a_n\} \subseteq A$ is *essential* if there exists a facet F of $\Gamma^+(A)$ such that B lies in F and if in addition B spans $\mathbb{R}^n$, that is $\det(a_1, \ldots, a_n) \neq 0$. It follows, when $Y = \{0\}$, that the essential sets are contained in the Newton diagram $\Gamma(A)$. Indeed, $Y = \{0\}$ precisely when A intersects all axes in $\mathbb{Z}^n$ and thus the only non-compact faces of Γ^+ are contained in the coordinate planes in $\mathbb{Z}^n$. But if B is contained in a coordinate plane, B can not span $\mathbb{R}^n$. Also, when $Y = \{0\}$, all points in $A \cap \Gamma(A)$ are in fact contained in some essential set. Next, if B is a subset of A, let $\alpha^B = \sum_{a \in B} a$. Notice that if B is essential, then α^B lies on $n\Gamma$. In fact, α^B/n is the barycenter of the simplex spanned by B. We are now ready to formulate our main theorem.

Theorem 3.1. *Let z^A, $A \subseteq \mathbb{Z}_+^n$ be a tuple of monomials in $\mathbb{C}^n$ such that $\{z^A = 0\} = \{0\}$, and let R^{z^A} be the corresponding Bochner-Martinelli residue current . Then*

$$R^{z^A} = \sum_{B \subseteq A} R_B,$$

where

$$(3.1) \qquad R_B = C_B \ \bar{\partial}\Big[\frac{1}{z_1^{\alpha_1^B}}\Big] \wedge \ldots \wedge \bar{\partial}\Big[\frac{1}{z_n^{\alpha_n^B}}\Big] \wedge e_B,$$

and where C_B is a constant that is nonzero if B is an essential set and zero otherwise.

An immediate consequence is that if B is essential then

$$\text{Ann } R_B = (z_1^{\alpha_1^B}, \ldots, z_n^{\alpha_n^B}),$$

where $\text{Ann } R_B$ just denotes the ideal of holomorphic functions annihilating R_B. Note in particular that $\text{Ann } R_B$ depends only on the set B and not on the remaining A. Furthermore, since the basis elements e_B are all different it follows that

$$\text{Ann } R^{z^A} = \bigcap_{B \text{ essential}} \text{Ann } R_B.$$

Thus, $\text{Ann } R^{z^A}$ is fully determined by the Newton diagram $\Gamma(A)$ and the points in A lying on it. In particular $\text{Ann } R^{z^A}$ depends only on the ideal, not on the particular choice of generators. We also see that different monomial ideals (z^A) and $(z^{A'})$ give rise to the same annihilator ideal if and only if $A \cap \Gamma(A) = A' \cap \Gamma(A')$.

Furthermore, Theorem 3.1 implies that the inclusion (1.1) is strict unless we have a complete intersection.

Theorem 3.2. *Let $z^A, A \subseteq \mathbb{Z}_+^n$, be a tuple of monomials such that $\{z^A = 0\} = \{0\}$, and let R^{z^A} be the corresponding Bochner-Martinelli residue current. Then*

$$Ann \, R^{z^A} = (z^A)$$

if and only if (z^A) can be generated by a complete intersection.

For the proof we need a simple lemma.

Lemma 3.3. *Let B be an essential subset of A such that $(z^B) \subseteq Ann \, R_B$. Then (z^B) is a complete intersection.*

Proof. Denote the elements in B by $a_i, i = 1, \ldots, n$ and let $\geq$ be the natural partial order on $\mathbb{Z}^n$. Suppose that $(z^B) \subseteq \text{Ann } R_B$. We have that $z^{a_i} \in \text{Ann } R_B$ precisely when one of the generators of $\text{Ann } R_B$ divides z^{a_i}, that is, when

$$(a_{1i}, \ldots, a_{ni}) \geq (\sum_j a_{1j}, 0, \ldots, 0) \text{ or}$$

$$(a_{1i}, \ldots, a_{ni}) \geq (0, \sum_j a_{2j}, 0, \ldots, 0) \text{ or}$$

$$\vdots$$

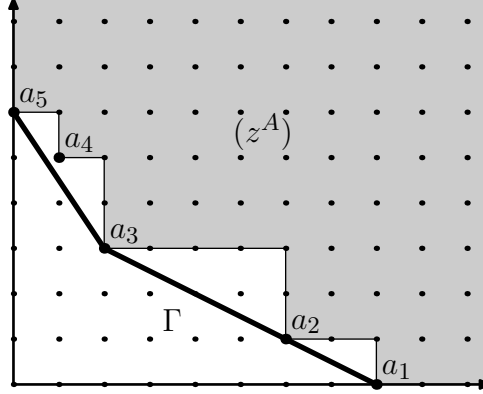$$(a_{1i}, \ldots, a_{ni}) \geq (0, \ldots, 0, \sum_j a_{nj}).$$

FIGURE 1. The ideal (z^A) and the Newton diagram $\Gamma(A)$ in Example 1

This set of inequalities holds for all $1 \leq i \leq n$, and it is easy to see that this implies first that $a_{k\ell} \neq 0$ for at most one k, which means that a_ℓ lies in one of the coordinate axes, and second that there is at least one a_ℓ intersecting each coordinate axis. Thus, B intersects all coordinate axes in $\mathbb{Z}^n$, which in turn implies that (z^B) is a complete intersection. $\square$

Proof of Theorem 3.2. We need to show the "only if" direction. Suppose that $(z^A) = \operatorname{Ann} R^{z^A}$ and let B be an essential subset. Clearly essential subsets always exist, since otherwise $R^{z^A} = 0$ and $\operatorname{Ann} R^{z^A} = (z^A)$ is the whole ring of holomorphic functions, which contradicts that $Y = \{0\}$. Now, in particular $(z^B) \subseteq \operatorname{Ann} R_B$, and by Lemma 3.3, (z^B) is a complete intersection. Thus

$$(z^B) = \operatorname{Ann} R^{z^B} = \operatorname{Ann} R_B^{z^A} \supseteq \operatorname{Ann} R^{z^A} = (z^A) \supseteq (z^B),$$

where the second equality follows since $\operatorname{Ann} R_B$ only depends on B and not on A. Hence $(z^A) = (z^B)$ and the result follows. $\square$

We give some examples to illustrate Theorems 3.1 and 3.2.

Example 1. Let

$$A = \{a^1 = (8,0), a^2 = (6,1), a^3 = (2,3), a^4 = (1,5), a^5 = (0,6)\} \subseteq \mathbb{Z}^2.$$

We identify the ideal (z^A) with the set $\bigcup_{a \in A}(a + \mathbb{R}_+^n)$ as in Figure 1, where we have also depicted the Newton diagram Γ. Such pictures of monomial ideals are usually referred to as *staircase diagrams*, see [9]. The points in A should be recognized as the "inner corners" of the staircase. The Newton diagram $\Gamma(A)$ consists of two facets, one with vertices a^1 and a^3 and the other one with vertices a^3 and a^5, and thus we have the essential sets

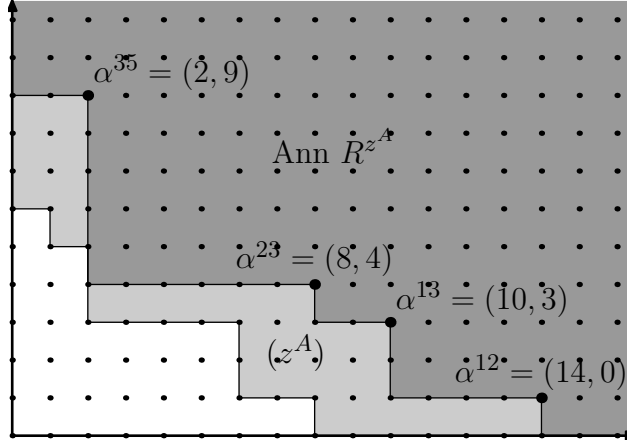$$\{a^1, a^2\}, \{a^1, a^3\}, \{a^2, a^3\}, \{a^3, a^5\},$$

FIGURE 2. The ideals $\operatorname{Ann} R^{z^A}$ (dark gray) and (z^A) (light gray) in Example 1

with
$$\alpha^{12} = (14, 1), \alpha^{13} = (10, 3), \alpha^{23} = (8, 4), \alpha^{35} = (2, 9),$$
respectively. It follows from Theorem 3.1 that
$$\operatorname{Ann} R^{z^A} = (z_1^{14}, z_2) \cap (z_1^{10}, z_2^3) \cap (z_1^8, z_2^4) \cap (z_1^2, z_2^9),$$
which is equal to the ideal $(z_1^{14}, z_1^{10}z_2, z_1^8 z_2^3, z_1^2 z_2^4, z_2^9)$, see Figure 2. Observe that $\operatorname{Ann} R^{z^A}$ is given by the staircase diagram with α^{ij} as "outer corners". Note also that $\operatorname{Ann} R^{z^A}$ does not depend on a^4, which lies in the interior of $\Gamma^+(A)$. □

Example 2. Consider the complete intersection $\{z_1^{a^1}, \ldots, z_n^{a^n}\}$. The associated Newton diagram is the n-simplex spanned by
$$A = \{(a^1, 0, \ldots, 0), \ldots, (0, \ldots, 0, a^n)\}$$
and there exists only one essential set, namely A itself, with $\alpha^A = (a^1, \ldots, a^n)$. Thus according to Theorem 3.1,
$$\operatorname{Ann} R^{z^A} = (z_1^{a^1}, \ldots, z_n^{a^n}),$$
so the annihilator ideal is equal to (z^A), which we already knew. Figure 3 illustrates the two ways of thinking of the ideal when $n = 2$; either as a staircase with $(a^1, 0)$ and $(0, a^2)$ as inner corners or as a staircase with $\alpha^A = (a^1, a^2)$ as the (only) outer corner. □

Example 3. We should remark that not all monomial ideals arise as annihilator ideals associated with monomial ideals. The idea is that the outer corners of the staircase of an annihilator ideal must lie on a hypothetical Newton diagram. Indeed, from the discussion just before Theorem 3.1 we know that each α^B corresponding to an essential set B lies on $n\Gamma$. In other words, the lines joining adjacent outer corners must lie on the boundary of a convex domain above the staircase, and
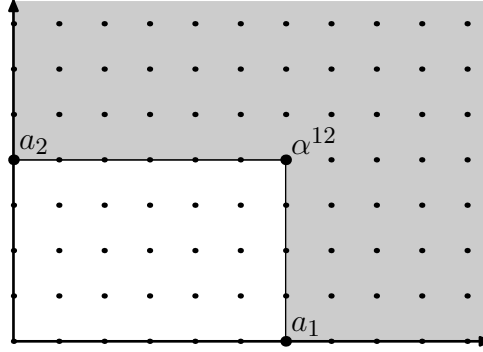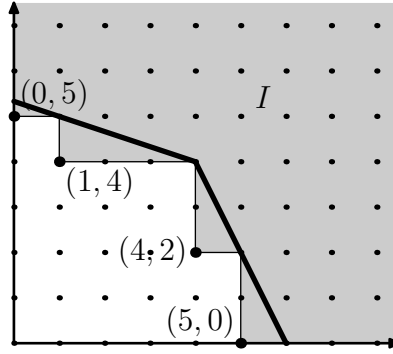
FIGURE 3. A complete intersection

FIGURE 4. The ideal in Example 3. The thick lines illustrate the "slope" of the staircase.

thus a necessary condition is that the "slope" of the staircase decreases while we are descending it.

For example, consider the ideal

$$I = (z_1^5, z_1^4 z_2^2, z_1 z_2^4, z_2^5)$$

with staircase diagram as in Figure 4, where we have also marked the slope. Clearly, the outer corners cannot lie on the boundary of a convex Newton polyhedron, and thus I is not an annihilator ideal.   □

Remark 1. Observe that adding an extra generator to an ideal (z^A) does not necessarily make the corresponding annihilator ideal smaller or larger. However, with a fixed Newton diagram an extra generator can only make the annihilator ideal smaller. In fact, given Γ, Ann R^{z^A} is maximal if A is chosen as the vertex set of Γ and minimal if A is all integer points on Γ, as we will see in Example 4.   □

Let us now consider the inclusion (1.2). We start by interpreting the left hand side in case f is monomial.

Lemma 3.4. *The integral closure of the monomial ideal (z^A) is the monomial ideal generated by $z^a, a \in \Gamma^+(A)$.*

The result is well known from algebraic contexts, see for example [12].

Next, we claim that the ideal $\overline{(z^A)^r}$ is generated by $z^a, a \in r\Gamma^+(A)$. The ideal $(z^A)^r$ is generated by $z^a, a \in A + \ldots + A$ (r times), so we need to show that the Newton polytope of $A + \ldots + A$ is equal to $r\Gamma^+(A)$. But $A + \ldots + A \supseteq rA$ and thus $\Gamma^+(A + \ldots + A) \supseteq \Gamma^+(rA) = r\Gamma^+(A)$. On the other hand $A + \ldots + A \subseteq \Gamma^+(A) + \ldots + \Gamma^+(A) = r\Gamma^+(A)$, where the equality holds since $\Gamma^+(A)$ is a convex set, and so it follows that $\Gamma^+(A + \ldots + A) \subseteq r\Gamma^+(A)$.

Corollary 3.5. *Suppose $n \geq 2$. Let z^A be as in Theorem 3.1. Then the integral closure of the ideal $(z^A)^n$ is strictly included in $\operatorname{Ann} R^{z^A}$.*

Observe that Corollary 3.5 fails when $n = 1$. Then, in fact, $(z^A) = \operatorname{Ann} R^{z^A} = \overline{(z^A)}$.

Proof. Let $(b_1, 0, \ldots, 0)$ be the intersection between $\Gamma(A)$ and the x_1-axis and let $f = z_1^{nb_1 - 1}$. Then $(nb_1 - 1, 0, \ldots, 0) \notin n\Gamma^+(A)$ and thus $f \notin \overline{(z^A)^n}$. However, $f \in \operatorname{Ann} R_B$ for all essential B. To see this, observe that the simplex spanned by the intersection points between Γ and the axes separates Γ from $\{x_1 = b_1\}$, and so Γ intersects the hyperplane $\{x_1 = b_1\}$ only at the point $(b_1, 0, \ldots, 0)$. This implies in particular that $\alpha_1^B \leq nb_1 - (n-1)$ for all essential B and thus $f \in (z_1^{\alpha_1^B}) \subseteq \operatorname{Ann} R_B$. Hence we have found a function f in $\operatorname{Ann} R^{z^A} \setminus \overline{(z^A)^n}$. $\qquad \square$

Another, probably more illuminating, way of thinking of the ideals is in terms of staircase diagrams as in the examples above. The fact that the ideal $\overline{(z^A)^n}$ is generated by $\{z^a\}, a \in n\Gamma^+$ means that its staircase lies just above $n\Gamma$. On the other hand we know that the outer corners of the staircase of $\operatorname{Ann} R^{z^A}$, the α^B, lie on $n\Gamma$ and therefore the staircase must lie under $n\Gamma$. Thus the staircase of $\operatorname{Ann} R^{z^A}$ is "strictly lower" than the staircase of $\overline{(z^A)^n}$ and so the corresponding inclusion of ideals is strict. For an illustration, see Figure 5, where we have drawn the staircases of the three ideals (z^A), $\operatorname{Ann} R^{z^A}$ and $\overline{(z^A)^n}$ for A from Example 1.

Example 4. Let Γ be the simplex with vertices $(3, 0)$ and $(0, 3)$. In Figure 6 we have drawn the staircases of the ideals (z^A) (light gray), $\operatorname{Ann} R^{z^A}$ (medium gray) and $\overline{(z^A)^2}$ (dark gray) for different $A = A_i$ with Γ as Newton diagram; more precisely for $A_1 = \{(3, 0), (0, 3)\}$, $A_2 = \{(3, 0), (2, 1), (0, 3)\}$, and finally for $A_3 = \{(3, 0), (2, 1), (1, 2), (0, 3)\}$. We see that $\operatorname{Ann} R^{z^A}$ decreases when we add points to A. In particular integrally closed ideals, that is ideals I such that $\overline{I} = I$, have the smallest annihilator ideals. $\qquad \square$

4. Proof of Theorem 3.1

The proof of Theorem 3.1 is very much inspired by the the proof of Lemma 2.2 in [11] and the proof of Theorem 1.1 in [1]. We will compute
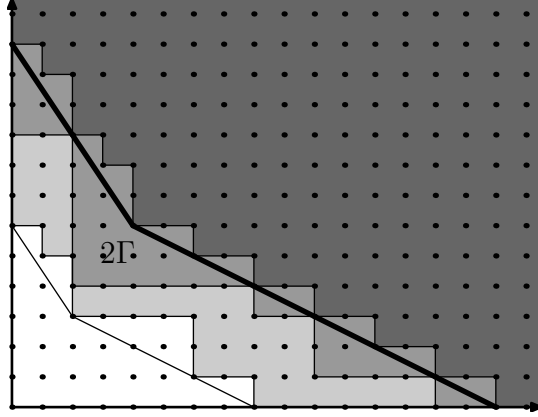
FIGURE 5. The ideals (z^A) (light gray) $\operatorname{Ann} R^{z^A}$ (medium gray) and $\overline{(z^A)^2}$ (dark gray) in Example 1
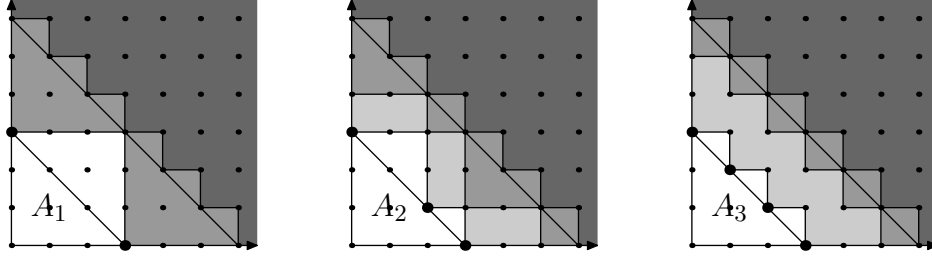

FIGURE 6. The various ideals in Example 4

R^{z^A} as the push-forward of a corresponding current on a certain toric variety $\mathcal{X}$ constructed from the Newton polyhedron $\Gamma^+(A)$. To do this we will have use for the following simple lemma which is proved essentially by integration by parts.

Lemma 4.1. *Let v be a strictly positive smooth function in $\mathbb{C}$, φ a test function in $\mathbb{C}$, and p a positive integer. Then*

$$\lambda \mapsto \int v^\lambda |s|^{2\lambda} \varphi(s) \frac{ds \wedge d\bar{s}}{s^p}$$

and

$$\lambda \mapsto \int \bar{\partial}(v^\lambda |s|^{2\lambda}) \wedge \varphi(s) \frac{ds}{s^p}$$

both have meromorphic continuations to the entire plane with poles at rational points on the negative real axis. At $\lambda = 0$ they are both independent of v and the second one only depends on the germ of φ at the origin. Moreover, if $\varphi(s) = \bar{s}\psi(s)$ or $\varphi = d\bar{s} \wedge \psi$, then the value of the second integral at $\lambda = 0$ is zero.

Throughout this section we will write $\mathbf{1}$ for the unit vector $(1, 1, \ldots, 1)$. We will regard the elements in A as column vectors and denote by B

the matrix with the vectors in the set B as columns. Also we will use the notation $\widehat{\alpha_i}$ for $\alpha_1 \wedge \ldots \wedge \alpha_{i-1} \wedge \alpha_{i+1} \wedge \ldots \wedge \alpha_n$.

Let us start by describing $\mathcal{X}$, following [4]. Let $\mathcal{S}$ be the set of normal directions to the facets of Γ^+ represented by vectors ρ with minimal integer non-negative coefficients. Then $\mathcal{S}$ provides a partition of the first orthant of $\mathbb{R}^n$ into a finite number of distinct n-dimensional cones. Such a system of cones with the same apex together with their faces is called a *fan*. We say that the fan is generated by $\mathcal{S}$ and we denote it by $\Delta(\mathcal{S})$. By techniques due to Mumford et al., [7], $\mathcal{S}$ can be completed into a system $\widetilde{\mathcal{S}}$ of vectors ρ such that if $\rho_1, \ldots, \rho_n$ generate one of the n-dimensional cones of $\Delta(\widetilde{\mathcal{S}})$, then $\det(\rho_1, \ldots, \rho_n) = \pm 1$. Such a fan is called *regular*. We will construct $\mathcal{X}$ by gluing together different copies of $\mathbb{C}^n$, one for each n-dimensional cone of $\Delta(\widetilde{\mathcal{S}})$. Let τ be such a cone and denote its generators by $\rho_1, \ldots, \rho_n$. Let $\mathcal{U}$ be the corresponding copy of $\mathbb{C}^n$ with local coordinates $t = (t_1, \ldots, t_n)$. Let P be the matrix with $\rho_i = (\rho_{1i}, \ldots, \rho_{ni})$ as rows and let Π be the mapping

$$\begin{aligned} \Pi : \mathcal{U} &\rightarrow \mathbb{C}^n \\ t &\mapsto t^P, \end{aligned}$$

where t^P is a shorthand notation for $(t_1^{\rho_{11}} \cdots t_n^{\rho_{n1}}, \ldots, t_1^{\rho_{1n}} \cdots t_n^{\rho_{nn}})$.

Two points $t \in \mathcal{U}$ and $t' \in \mathcal{U}'$ are identified if the monoidal map $\Pi'^{-1} \circ \Pi : \mathcal{U} \to \mathcal{U}'$ is defined at t and maps t to t'. Gluing the charts $\mathcal{U}$ together induces a proper map $\widetilde{\Pi} : \mathcal{X} \to \mathbb{C}^n$ that is biholomorphic from $\mathcal{X} \setminus \widetilde{\Pi}^{-1}(\{z_1 \cdots z_n = 0\})$ to $\mathbb{C}^n \setminus \{z_1 \cdots z_n = 0\}$, that is, outside the coordinate planes. It holds that $\widetilde{\Pi}^{-1}(\{z_1 \cdots z_n = 0\})$ is a set of measure zero in $\mathcal{X}$, and moreover $\widetilde{\Pi}^{-1}(0)$ consists of a system of various $\mathbb{CP}^{n-i}$, corresponding to i-dimensional cones of the fan $\Delta(\widetilde{\mathcal{S}})$. In particular, each vector ρ, that generates a 1-dimensional cone, corresponds to a $\mathbb{CP}^{n-1}$, denoted by S_ρ and obtained by gluing together parts of the charts from the cones determined by n-dimensional cones that have ρ as one of its generators. In fact, if the vector ρ determines the coordinate t_1 in $\mathcal{U}$, then S_ρ is covered by the $\{t_1 = 0\}$-part of $\mathcal{U}$.

Observe that $R^{z^A} = R_n$ since $Y = \{0\}$. Therefore, we only need to compute the currents R_B when B is a subset of cardinality n. For $\operatorname{Re} \lambda$ large enough, (2.1) is integrable and since $\widetilde{\Pi}$ is biholomorphic outside a set of measure zero it holds that

$$\int_{\mathbb{C}^n} \bar{\partial} |z^A|^{2\lambda} \wedge u_B \wedge \phi = \int_{\mathcal{X}} \widetilde{\Pi}^*(\bar{\partial} |z^A|^{2\lambda} \wedge u_B) \wedge \widetilde{\Pi}^* \phi,$$

if ϕ is a test form of bidegree $(n, 0)$. It is easy to see that the analytic continuation to $\operatorname{Re} \lambda > -\epsilon$ of $\widetilde{\Pi}^*(\bar{\partial} |z^A|^{2\lambda} \wedge u_B)$ exists in each chart $\mathcal{U}_\tau$; we will actually compute it below. Thus, because of the uniqueness of analytic continuations,

$$\widetilde{R}_B := \widetilde{\Pi}^*(\bar{\partial} |z^A|^{2\lambda} \wedge u_B)|_{\lambda=0}$$

defines a (globally defined) current on $\mathcal{X}$ such that $\widetilde{\Pi}_* \widetilde{R}_B = R_B$. We will start by computing $\widetilde{R}_B$ in a fixed chart $\mathcal{U}_0$ parametrized by Π corresponding to the cone τ_0.

Claim 1. *The current $\widetilde{R}_B$ vanishes in $\mathcal{U}_0$ whenever B is not contained in a facet whose normal direction is one of the generators of τ_0. Moreover $\widetilde{R}_B$ vanishes if $\det B = 0$.*

In particular, a necessary condition for $\widetilde{R}_B$ not to vanish is that B is essential.

Proof. First, note that the pullback Π^* transforms the exponents of monomials by the linear mapping P;

$$(4.1) \quad \Pi^* z^a = \Pi^* z_1^{a_1} \cdots z_n^{a_n} = t_1^{\rho_{11} a_1 + \ldots + \rho_{1n} a_n} \cdots t_n^{\rho_{n1} a_1 + \ldots + \rho_{nn} a_n} = t^{Pa}.$$

It is well known that for some $a_0 \in A$, $\Pi^* z^{a_0}$ divides $\Pi^* z^a$ for all $a \in A$, and moreover, in view of (4.1) one easily checks that a_0 has to be a vertex of $\Gamma^+(A)$. Using this we can write

$$\Pi^* s = \bar{t}^{Pa_0} s',$$

where s' is the nonvanishing section

$$s' = \sum_{a \in A} \bar{t}^{P(a-a_0)} e_a,$$

and furthermore

$$\Pi^* |z^A|^2 = |t|^{2Pa_0} \nu(t),$$

where

$$\nu(t) = \sum_{a \in A} |t|^{2P(a-a_0)n}$$

is nonvanishing. By homogeneity, see (2.2),

$$\Pi^* (s \wedge (\bar{\partial} s)^{n-1}) = \bar{t}^{nPa_0} s' \wedge (\bar{\partial} s')^{n-1},$$

and thus

$$(4.2) \qquad \widetilde{R}_B = \bar{\partial}\big(|t|^{2\lambda Pa_0} \nu^\lambda\big) \frac{s'_B \wedge (\bar{\partial} s'_B)^{n-1}}{t^{nPa_0} \nu(t)^n}\bigg|_{\lambda=0}.$$

By Leibniz' rule and Lemma 4.1, (4.2) is equal to a sum of currents

$$(4.3) \qquad \bar{\partial}\left[\frac{1}{t_i^{n\rho_i \cdot a_0}}\right] \otimes \left[\frac{1}{\prod_{j \neq i} t_j^{n\rho_j \cdot a_0}}\right] \wedge \frac{s'_B \wedge (\bar{\partial} s'_B)^{n-1}}{\nu(t)^n}.$$

We need to compute $s'_B \wedge (\bar{\partial} s'_B)^{n-1}$. Denote the elements in B by $b_1, \ldots, b_n$ in such a way that $e_B = e_{b_n} \wedge \ldots \wedge e_{b_1}$. Furthermore, let C be the matrix with columns $Pb_i - Pa_0$ so that

$$s'_B = \sum_i \bar{t}_1^{c_{1i}} \cdots \bar{t}_n^{c_{ni}} e_{b_i},$$

and let D_i be the determinant of C with row i replaced with the unit vector $\mathbf{1}$. Then we have the following lemma.

Lemma 4.2. *We have that*

$$(4.4) \qquad s'_B \wedge (\bar\partial s'_B)^{n-1} = (n-1)! \bar{t}^{C\mathbf{1}} \sum_i (-1)^{i-1} \, D_i \frac{\widehat{d\bar{t}_i}}{\bar{t}_i} \wedge e_B,$$

where

$$\frac{\widehat{d\bar{t}_i}}{\bar{t}_i} = \frac{d\bar{t}_1}{\bar{t}_1} \wedge \ldots \wedge \frac{d\bar{t}_{i-1}}{\bar{t}_{i-1}} \wedge \frac{d\bar{t}_{i+1}}{\bar{t}_{i+1}} \wedge \ldots \wedge \frac{d\bar{t}_n}{\bar{t}_n}.$$

Observe that all $\bar{t}_i$ in the denominator are cancelled since (4.4) is in fact smooth.

Proof. Let $\alpha_j = \bar{t}_1^{c_{1i}} \cdots \bar{t}_n^{c_{ni}} e_{b_j}$ and $\beta_i = \frac{d\bar{t}_i}{\bar{t}_i}$. Then $s'_B = \sum_{j=1}^n \alpha_j$ and

$$\bar\partial s'_B = \sum_{j=1}^n \sum_{i=1}^n c_{ij} \frac{d\bar{t}_i}{\bar{t}_i} \wedge \bar{t}_1^{c_{1i}} \cdots \bar{t}_n^{c_{ni}} e_{b_j} = \sum_{j=1}^n \sum_{i=1}^n c_{ij} \beta_i \wedge \alpha_j.$$

Thus we get

$$s'_B \wedge (\bar\partial s'_B)^{n-1} = \sum_{j=1}^n \alpha_j \wedge (\sum_{j=1}^n \sum_{i=1}^n c_{ij} \beta_i \wedge \alpha_j)^{n-1} =$$

$$\sum_{\sigma \in S^n} \sum_{\tau \in S^n} c_{\sigma(2)\tau(2)} \cdots c_{\sigma(n)\tau(n)} \alpha_{\tau(1)} \wedge \beta_{\sigma(2)} \wedge \alpha_{\tau(2)} \wedge \ldots \wedge \beta_{\sigma(n)} \wedge \alpha_{\tau(n)} =$$

$$\sum_{\sigma \in S^n} \sum_{\tau \in S^n} c_{\sigma(2)\tau(2)} \cdots c_{\sigma(n)\tau(n)} \beta_{\sigma(2)} \wedge \ldots \wedge \beta_{\sigma(n)} \wedge \alpha_{\tau(n)} \wedge \ldots \wedge \alpha_{\tau(1)} =$$

$$\sum_{\sigma \in S^n} \sum_{\tau \in S^n} (-1)^{\mathrm{sgn}\ \tau} c_{\sigma(2)\tau(2)} \cdots c_{\sigma(n)\tau(n)} \beta_{\sigma(2)} \wedge \ldots \wedge \beta_{\sigma(n)} \wedge \alpha_n \wedge \ldots \wedge \alpha_1 =$$

$$\sum_{i=1}^n \sum_{\sigma \in S^n; \sigma(1)=i} D_i (-1)^{\mathrm{sgn}\ \sigma} \beta_{\sigma(2)} \wedge \ldots \wedge \beta_{\sigma(n)} \wedge \alpha_n \wedge \ldots \wedge \alpha_1 =$$

$$\sum_{i=1}^n (n-1)! D_i (-1)^{i-1} \beta_1 \wedge \ldots \widehat{\beta_i} \ldots \wedge \beta_n \wedge \alpha_n \wedge \ldots \wedge \alpha_1 =$$

$$(n-1)! \bar{t}^{C\mathbf{1}} \sum_{i=1}^n (-1)^{i-1} \, D_i \frac{\widehat{d\bar{t}_i}}{\bar{t}_i} \wedge e_B.$$

Here S^n just denotes the set of permutations of $\{1, \ldots, n\}$. $\qquad \square$

Now (4.3) is equal to

$$(4.5) \qquad \bar\partial \left[\frac{1}{t_i^{n\rho_i \cdot a_0}} \right] \otimes \left[\frac{1}{\prod_{j \neq i} t_j^{n\rho_j \cdot a_0}} \right] \wedge \frac{(n-1)! \, D_i \, \bar{t}^{C\mathbf{1}}}{\nu(t)^n} \frac{\widehat{d\bar{t}_i}}{\bar{t}_i} \wedge e_B,$$

that can vanish for two reasons. First, by Lemma 4.1, (4.5) vanishes whenever the numerator contains a factor $\bar{t}_i$, that happens if $c_{ij} > 0$ for some j, which means that Pb_j has a greater t_i-coordinate than Pa_0. Thus, a necessary condition for (4.5) not to vanish is that $P(B)$ is contained in the facet of $P(\Gamma^+)$ parallel to the coordinate plane

$\{t_i = 0\}$; in other words, since P is invertible, that B is contained in the facet F_i of Γ^+ with normal direction ρ_i. Hence the first part of Claim 1 follows.

Second, (4.5) vanishes if $D_i = 0$. Assume for simplicity that $i = 1$. Then $\rho_1 \cdot a$ is constant and equal to $\rho_1 \cdot a_0$ on F_1, that is, $(PB)_{1j} = (Pa_0)_1$ for all j, and we get

$$
D_1 = \begin{vmatrix} 1 & \cdots & 1 \\ c_{21} & \cdots & c_{2n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{vmatrix} = \begin{vmatrix} 1 & \cdots & 1 \\ (PB)_{21} - (Pa_0)_2 & \cdots & (PB)_{2n} - (Pa_0)_2 \\ \vdots & & \vdots \\ (PB)_{n1} - (Pa_0)_n & \cdots & (PB)_{nn} - (Pa_0)_n \end{vmatrix} =
$$

$$
\begin{vmatrix} 1 & \cdots & 1 \\ (PB)_{21} & \cdots & (PB)_{2n} \\ \vdots & & \vdots \\ (PB)_{n1} & \cdots & (PB)_{nn} \end{vmatrix} = \frac{1}{(Pa_0)_1} \begin{vmatrix} (PB)_{11} & \cdots & (PB)_{1n} \\ (PB)_{21} & \cdots & (PB)_{2n} \\ \vdots & & \vdots \\ (PB)_{n1} & \cdots & (PB)_{nn} \end{vmatrix} = \frac{\det(PB)}{(Pa_0)_1}.
$$

But since P is invertible $(Pa_0)_1 \neq 0$ and $\det P \neq 0$. Thus $D_i = 0$ if and only if $\det B = 0$. $\qquad\square$

Note that it follows from the proof of Claim 1 that $\widetilde{R}_B$ has support on S_ρ if B is contained in the facet with normal direction ρ. Indeed $\widetilde{R}_B$ survives precisely in the charts corresponding to cones τ with ρ as one of its generators and in each such chart it has support on the part covering S_ρ.

Now let us fix a set B contained in the facet with normal direction ρ_i, so that (4.5) is nonvanishing, and compute the action of $\widetilde{R}_B$ on the pullback of a test form $\phi = \varphi(z)\, dz$ of bidegree $(n, 0)$. Here dz is just a shorthand notation for $dz_1 \wedge \ldots \wedge dz_n$. Let $\{\chi_\tau\}$ be a partition of unity on $\mathcal{X}$ subordinate the cover $\{\mathcal{U}_\tau\}$. It is not hard to see that we can choose the partition in such a way that the χ_τ are circled, that is they only depend on $|t_1|, \ldots, |t_n|$. Now $\widetilde{R}_B = \sum_\tau \chi_\tau \widetilde{R}_B$. We will start by computing the contribution from our fixed chart $\mathcal{U}_0$ where $\widetilde{R}_B$ is realized by (4.5).

Since R has support at the origin it does only depend on finitely many derivatives of φ and therefore to determine R_B it is enough to consider the case when φ is a polynomial. We can write φ as a finite Taylor expansion,

$$
\varphi = \sum_{\alpha, \beta} \frac{\varphi_{\alpha, \beta}(0)}{\alpha! \beta!} z^\alpha \bar{z}^\beta,
$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$

$$
\varphi_{\alpha, \beta} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}} \frac{\partial^{\beta_1}}{\partial \bar{z}_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial \bar{z}_n^{\beta_n}} \varphi,
$$

and $\alpha! = \alpha_1! \cdots \alpha_n!$, $\beta! = \beta_1! \cdots \beta_n!$ with pullback to $\mathcal{U}_0$ given by

$$\Pi^* \varphi = \sum_{\alpha,\beta} \frac{\varphi_{\alpha,\beta}(0)}{\alpha! \beta!} t^{P\alpha} \bar{t}^{P\beta}.$$

A computation similar to the one in the proof of Lemma 4.2 yields

$$\Pi^* dz = \det P \, t^{(P-I)\mathbf{1}} \, dt.$$

Hence $\chi_\tau \widetilde{R}_B . \Pi^* \phi$ is equal to

$$K \int \bar{\partial} \left[\frac{1}{t_i^{n\rho_i \cdot a_0}} \right] \otimes \left[\frac{1}{\prod_{j \neq i} t_j^{n\rho_j \cdot a_0}} \right] \wedge \frac{\bar{t}_i \bar{t}^{(C-I)\mathbf{1}} \, \widehat{d\bar{t}_i} \wedge e_B}{\nu(t)^n} \wedge$$

$$\chi_\tau(t) \sum_{\alpha,\beta} \frac{\varphi_{\alpha,\beta}(0)}{\alpha! \beta!} t^{P\alpha} \bar{t}^{P\beta} t^{(P-I)\mathbf{1}} dt = K \sum_{\alpha,\beta} I_{\alpha,\beta} \wedge \frac{\varphi_{\alpha,\beta}(0)}{\alpha! \beta!} e_B,$$

where $K = (n-1)! D_1 \det P$ and

$$(4.6) \qquad I_{\alpha,\beta} = \int \bar{\partial} \left[\frac{1}{t_i^{\rho_i \cdot (na_0 - \alpha - 1) + 1}} \right] \otimes [\mu_{\alpha,\beta}] \frac{\chi_\tau(t) \, \bar{t}_i^{\rho_1 \cdot \beta}}{\nu(t)^n} \wedge \widehat{d\bar{t}_i} \wedge dt,$$

and where $\mu_{\alpha,\beta}$ is the Laurent monomial in t_j and $\bar{t}_j$ for $j \neq i$:

$$\mu_{\alpha,\beta} = \prod_{j \neq i} t_j^{\rho_j \cdot (\alpha + \mathbf{1} - na_0) - 1} \, \bar{t}_j^{\rho_j \cdot (\beta + B\mathbf{1} - na_0) - 1}.$$

Observe that $\rho_j \cdot (\beta + B\mathbf{1} - na_0) - 1 \geq 0$ so there are no $\bar{t}_j$ in the denominator. Recalling (2.3), we evaluate the t_i-integral. Since ν and χ_τ depend on $|t_1|, \ldots, |t_n|$ it follows that $\frac{\partial^\ell}{\partial t_i^\ell} \frac{\chi_\tau}{\nu} \big|_{t_i = 0} = 0$ for $\ell \geq 1$ and thus (4.6) is equal to

$$(4.7) \qquad\qquad 2\pi i \int_{\widehat{t_i}} \frac{\chi_\tau(t)|_{t_i=0}[\mu_{\alpha,\beta}]}{\nu(t)^n|_{t_i=0}} \, \widehat{d\bar{t}_i} \wedge \widehat{dt_i},$$

if

$$(4.8) \qquad\qquad \rho_i \cdot (na_0 - \alpha - \mathbf{1}) + 1 = 1$$

and

$$(4.9) \qquad\qquad \rho_i \cdot \beta = 0,$$

and zero otherwise. Moreover, for symmetry reasons (4.7) vanishes unless

$$(4.10) \qquad \rho_j \cdot (\alpha + \mathbf{1} - na_0) - 1 = \rho_j \cdot (\beta + B\mathbf{1} - na_0) - 1$$

for $j \neq i$. From the discussion just before Theorem 3.1 we know that the facet containing B is compact, which means that its normal vector has nonzero entries. Thus (4.9) implies that $\beta = (0, \ldots, 0)$. Using the fact that $\rho_i \cdot a = \rho_i \cdot a_0$ for all $a \in B$ we can rewrite the left hand side of (4.8) as $\rho_i \cdot (B\mathbf{1} - 1 - \alpha - \mathbf{1}) + 1$ and thus summarize the conditions (4.8) and (4.10) on α as

$$(4.11) \qquad\qquad P(\alpha + \mathbf{1}) = PB\mathbf{1}.$$

But, since P is invertible there exists exactly one α that fulfills (4.11), namely $\alpha = (B-I)\mathbf{1}$, which is precisely $\alpha^B - \mathbf{1}$. With these values of α and β the Laurent monomial $\mu_{\alpha,\beta}$ is nonsingular and so the integrand of (4.7),

$$(4.12) \qquad 2\pi i \int_{\widehat{t_i}} \frac{\chi_\tau(t)|_{t_i=0} \ \prod_{j\neq i} |t_j|^{2(\rho_j \cdot (B\mathbf{1}-na_0)-1)}}{\nu(t)^n|_{t_i=0}} \widehat{dt_i} \wedge \widehat{d\bar{t}_i}$$

becomes integrable.

To compute $\widetilde{R}_B.\widetilde{\Pi}^*\phi$ we want to add contributions from all charts. However, $\mathcal{U}_0$ covers the support of $\widetilde{R}_B$ except for a set of measure zero, since $\widetilde{R}_B$ has support on S_{ρ_i}, and moreover all integrands that appear are of the form (4.12) and therefore integrable. Thus $\widetilde{R}_B.\widetilde{\Pi}^*\phi$ is equal to

$$\int_{\mathcal{X}} \sum_\tau \widetilde{\Pi}^*(\bar\partial|z^A|^{2\lambda} \wedge u_B) \wedge \chi_\tau \widetilde{\Pi}^*\phi \Big|_{\lambda=0} =$$

$$\int_{\mathcal{U}_0} \widetilde{\Pi}^*(\bar\partial|z^A|^{2\lambda} \wedge u_B) \wedge \widetilde{\Pi}^*\phi \Big|_{\lambda=0} = C_B \ \frac{\varphi_{\alpha^B-\mathbf{1},0}(0)}{\alpha!\beta!} e_B,$$

where

$$C_B = 2\pi i K \int_{\widehat{t_i}} \frac{\prod_{j\neq i} |t_j|^{2(\rho_j \cdot (B\mathbf{1}-na_0)-1)}}{(\sum_{a\in A} \prod_{j\neq i} |t_j|^{2\rho_j \cdot (a-a_0)})^n} \widehat{dt_i} \wedge \widehat{d\bar{t}_i}.$$

Hence R_B is of the form (3.1) and the result follows.

5. General monomial ideals

If the zero variety of z^A is of positive dimension the computations of R^{z^A} get more involved. Recall that in general $R^{z^A} = R_p + \ldots + R_\mu$, where $p = \operatorname{codim} Y$, $\mu = \min(m,n)$ and $R_k \in \mathcal{D}'_{0,k}(\mathbb{C}^n, \Lambda^k E)$. Parts of the top degree term R_n can be computed by the techniques from the proof of Theorem 3.1. Our method for dealing with the terms of lower degree, though, is to perform the computations outside certain varieties, where some of the coordinates are zero. This amounts to projecting A and brings us back to the more familiar top degree case in a lower dimension. The price we have to pay is that we miss parts of $\mathbb{C}^n$. More precisely, we will compute the current R_k outside the $(k+1)$-dimensional variety

$$V_k := \bigcup_{\mathcal{I}, |\mathcal{I}|=k+1} \bigcap_{i\in\mathcal{I}} H_i,$$

where H_i denotes the hyperplane $\{z_i = 0\}$. However, it turns out that R_k will not carry any essential information on such "small" varieties. To be precise, we have the following lemma, which can be proved analogously to the proof of Lemma 2.2 in [3].

Lemma 5.1. *Let $h_1, \ldots, h_s$ be a tuple of holomorphic functions and let $Y_h = \{h_1 = \ldots h_s = 0\}$. Suppose that $\mathrm{codim}\, Y_h \cap Y > k$. Then the current $|h|^{2\lambda_0} R_k$, where $|h|^2 = |h_1|^2 + \ldots + |h_s|^2$, has an analytic continuation to $\mathrm{Re}\, \lambda_0 \geq -\epsilon$ and*

$$|h|^{2\lambda_0} R_k|_{\lambda_0=0} = R_k.$$

It follows, in particular, that to annihilate R_k it suffices to do it outside V_k (or any variety of codimension $k + 1$). Indeed $hR_k = 0$ outside V_k implies that $hR_k = 0$.

Before stating our result, a word of notation: For $\mathcal{I} = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, let $T_{\mathcal{I}}$ be the projection

$$T_{\mathcal{I}} : \mathbb{Z}^n \;\; \to \;\; \mathbb{Z}^k$$
$$(a_1, \ldots, a_n) \;\; \mapsto \;\; (a_{i_1}, \ldots, a_{i_k}).$$

We say that $T_{\mathcal{I}}(B)$ is essential if $T_{\mathcal{I}}(B)$ is contained in a facet of $\Gamma^+(T_{\mathcal{I}}(A))$ and if $T_{\mathcal{I}}(B)$ spans $\mathbb{R}^{|\mathcal{I}|}$.

Theorem 5.2. *Let z^A, $A \subseteq \mathbb{Z}^n_+$, be a tuple of monomials in $\mathbb{C}^n$, and let*

$$R^{z^A} = \sum_{B \subset A} R_B$$

be the corresponding Bochner-Martinelli residue current. Then outside $V_{|B|}$,

$$R_B = \sum_{\mathcal{I} \subset \{1, \ldots, n\}, |\mathcal{I}| = |B|} R_{B,\mathcal{I}},$$

where the current $R_{B,\mathcal{I}}$ vanishes unless $T_{\mathcal{I}}(B)$ is essential. Moreover if $T_{\mathcal{I}}(B)$ is essential and contained in a compact facet of $\Gamma^+(T_{\mathcal{I}}(A))$, then

$$(5.1) \qquad R_{B,\mathcal{I}} = C_{B,\mathcal{I}}(\eta) \otimes \bigwedge_{i \in \mathcal{I}} \bar{\partial} \left[\frac{1}{z_i^{\alpha_i^B}} \right] \wedge e_B,$$

where η denotes the $z_i, i \notin \mathcal{I}$, and $C_{B,\mathcal{I}}(\eta)$ is a smooth function not identically equal to zero.

Several remarks are in order. First, an immediate consequence is that

$$\mathrm{Ann}\, R_{B,\mathcal{I}} = (z_i^{\alpha_i^B})_{i \in \mathcal{I}}$$

if $R_{B,\mathcal{I}}$ is of the form (5.1), since annihilating such a current clearly is equivalent to annihilating the $\bigwedge_{i \in \mathcal{I}} \bar{\partial} \left[\frac{1}{z_i^{\alpha_i^B}} \right]$ part. Moreover the support of (5.1) is the set $\cap_{i \in \mathcal{I}} \{z_i = 0\}$. Note that all the computable $R_{B,\mathcal{I}}$ have different supports.

Remark 2. Observe that adding elements to A that lie in any of the non-compact facets of $\Gamma^+(A)$, not contained in any coordinate plane, gives rise to new essential sets. For example we can add redundant generators to (z^A) and thus in general $\mathrm{Ann}\, R^{z^A}$ is not independent of the choice of generators as in the case of a discrete zero variety. $\qquad \square$

Remark 3. Theorem 3.1 is just a special case of Theorem 5.2. Let us say a word about how to see that the currents of lower degree vanish when Y is the origin. This hypothesis means precisely that A intersects all axes, which in turn implies that the image of A under any projection $T_{\mathcal{I}}$, $|\mathcal{I}| < n$, contains the origin. However, if $0 \in A$, the Newton polyhedron $\Gamma^+(A)$ equals the first orthant and there are no essential sets; note that this corresponds to the case when f contains a nonvanishing function. Thus $R^{z^A} = \sum_{|B|=n} R_B$, where $R_B = R_{B,\{1,\dots,n\}}$ and Theorem 3.1 follows. Of course, by slightly refined arguments one can see how the currents $R_k, k < \operatorname{codim} Y$ vanish in general.

We should also mention that Theorem 5.2 includes Proposition 3.1 in [11] except that the smooth contributions $C_{B,\mathcal{I}}$ are not made explicit. $\qquad\square$

Remark 4. By Theorem 5.2 we can extend Theorem 3.2 to hold for a much larger class of ideals. Recall that a crucial point of the proof of Theorem 3.2 was the existence of essential sets. If the Newton diagram of A is of dimension $n - 1$, though, we can always find essential sets, for example take the vertices of one of the facets, and the proof applies immediately. In fact, one can show that Theorem 3.2 holds unless $\Gamma(A)$ is not parallel to any of the coordinate planes. Yet, there are ideals for which Theorem 5.2 does not give enough information to decide whether the inclusion (1.1) is strict or not, as we will see in Example 6. Still, in this particular case, one can show by explicit computations that the annihilator ideal is strictly included in the ideal and we believe that Theorem 3.2 holds for monomial ideals in general, although we do not know enough to prove it. $\qquad\square$

Let us illustrate Theorem 5.2 with some simple examples.

Example 5. Let $A = \{a^1 = (6,1), a^2 = (3,2), a^3 = (2,4)\}$. There are two essential subsets of A, $\{a^1, a^2\}$ and $\{a^2, a^3\}$, with $\alpha^{12} = (9,3)$ and $\alpha^{23} = (5,6)$, respectively. Moreover, $\Gamma^+(T_{\{1\}}(A))$ is the interval $[2, \infty)$ and consequently $\Gamma(T_{\{1\}}(A)) = \{2\}$. Thus the only set such that its image under $T_{\{1\}}$ is essential is $\{a^3\}$, with $\alpha^3 = a^3$, and according to Theorem 5.2 $\operatorname{Ann} R_{\{a^3\},\{1\}} = (z_1^2)$. Similarly, projecting A on the second axis yields one current, $R_{\{a^1\},\{2\}}$, with annihilator (z_2). Altogether we get
$$\operatorname{Ann} R^{z^A} = (z_1^9, z_2^3) \cap (z_1^5, z_2^6) \cap (z_1^2) \cap (z_2),$$
that is equal to $(z_1^9 z_2, z_1^5 z_2^3, z_1^2 z_2^6)$, see Figure 7. Observe, apropos of Remark 2, that adding a point to A in any of the noncompact facets gives a new essential set and thereby essentially changes R^{z^A}. $\qquad\square$

In view of Example 5 it should be clear that Theorem 5.2 actually gives a complete description of $\operatorname{Ann} R^{z^A}$ in case $n = 2$, provided we choose a minimal set of generators (or at least avoid to pick redundant generators from the unbounded facets of $\Gamma^+(A)$).

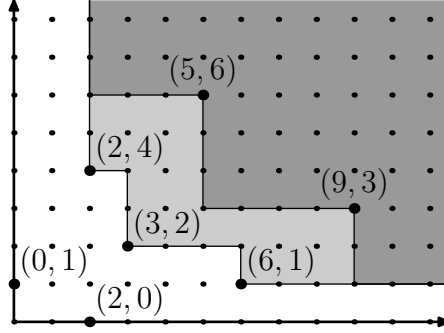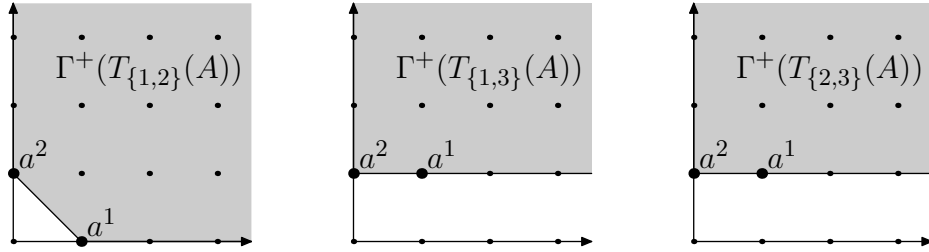FIGURE 7. The ideals $\operatorname{Ann} R^{z^A}$ (dark gray) and (z^A) (light gray) in Example 5

FIGURE 8. The image of A under the various projections to $\mathbb{Z}^2$ in Example 6

Example 6. Let I be the ideal (z^A), where $A = \{a^1 = (1,0,1), a^2 = (0,1,1)\} \subset \mathbb{Z}^3$. The codimension of $\{z^a = 0\}$ is 1 and thus I is not a complete intersection (nor can be defined by one). Note that the set A is to small to be essential, whereas the image of A under any projection to $\mathbb{Z}^2$ is, as shown in Figure 8. Still, Theorem 5.2 gives the annihilator ideal only for one of the corresponding currents, namely $\operatorname{Ann} R_{A,\{1,2\}} = (z_1, z_2)$. In both of the other cases the projection of A lies in a noncompact facet of the Newton polyhedron. Furthermore, projecting A to $\mathbb{Z}$ yields the currents $R_{\{a^1\},\{3\}}$ and $R_{\{a^2\},\{3\}}$, both with annihilator (z_3). Observe that the intersection of the computable currents is precisely I. Thus we have found an example of an non-complete intersection where Theorem 5.2 does not give enough information to decide whether the inclusion (1.1) is strict or not. In this simple example, however, it is easy to compute the remaining parts of R^{z^A} and see that the inclusion is indeed strict. $\qquad\square$

Proof of Theorem 5.2. We start by considering the term of top degree,

$$R_n = \sum_{B \subset A, |B|=n} R_B,$$

for which the result follows easily from the proof of Theorem 3.1. To see this, observe first that the proof of Claim 1 does not depend on the codimension of Y. Thus we conclude that $R_B = 0$ unless B is essential.

Next, suppose that B is contained in a compact facet F_B of Γ^+ with normal direction ρ_i. As in the proof of Theorem 3.1 let $\mathcal{U}_0$ be a chart parametrized by Π, determined by the cone τ_0 that has ρ_i as its ith generator. Recall from the proof that the support of $\widetilde{R}_B$ in $\mathcal{U}_0$ is given by $\{t_i = 0\}$. That F_B is compact means precisely that all entries of ρ_i are strictly positive, which implies that $\Pi(\{\rho_i = 0\}) = \{0\}$. Consequently, when computing $\widetilde{R}_B$ in $\mathcal{U}_0$, we only need to consider it acting on test forms $\phi = \varphi\, dz$, where φ is a polynomial. Hence the rest of the proof of Theorem 3.1 applies, and we get that $R_B = R_{B,\{1,\dots,n\}}$ is of the form (3.1) that is equivalent to (5.1) in case $k = n$.

We will compute the terms of lower degree by looking outside certain coordinate planes, which will correspond to projections of A. More precisely, to determine R_k we will look where $n-k$ of the z_i are nonzero. To do this let us fix $\mathcal{I} = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ and let $M_{\mathcal{I}}$ be the set where z_i is nonvanishing if $i \notin \mathcal{I}$, that is

$$M_{\mathcal{I}} = (\bigcup_{i \notin \mathcal{I}} H_i)^C.$$

Denote the z_i, $i \in \mathcal{I}$, by ζ and the z_i, $i \notin \mathcal{I}$, by η and write $z^a = \zeta^{a_\zeta} \eta^{a_\eta}$, where a_ζ and a_η are the images of a under $T_{\mathcal{I}}$ and $T_{\mathcal{I}^C}$, respectively. Let A_ζ and A_η denote the corresponding images of A, and let ϕ be a test form of bidegree (n, k) with (compact) support in $M_{\mathcal{I}}$. Now R_k acting on ϕ is the analytic continuation to $\lambda = 0$ of

$$\int \bar{\partial}|z^A|^{2\lambda} \wedge \frac{s \wedge (\bar{\partial} s)^{k-1}}{|z^A|^{2k}} \wedge \phi(z),$$

that is equal to a sum, taken over B such that $|B| = k$, of terms

$$(5.2) \qquad \int_\eta \int_\zeta \bar{\partial}|z^A|^{2\lambda} \wedge \frac{s_B \wedge (\bar{\partial} s_B)^{k-1}}{|z^A|^{2k}} \wedge \varphi(\zeta, \eta)\, d\zeta \wedge d\bar{\eta} \wedge d\eta.$$

It is easily checked that R_k vanishes unless ϕ is of the form $\varphi(\zeta, \eta)\, d\bar{\eta} \wedge d\eta \wedge d\zeta$. We can now compute the inner integral of (5.2) as in the top degree case (with A^ζ in $\mathbb{C}^k_\zeta$). Indeed, since η is nonvanishing, we can regard z^A as the monomials ζ^{A_ζ} times the parameters η^{A_η}. It follows that, at $\lambda = 0$, (5.2) vanishes unless $T_{\mathcal{I}}(B)$ is essential, and moreover, if $T_{\mathcal{I}}(B)$ is contained in a compact facet of $\Gamma(T_{\mathcal{I}}(A))$, then the inner integral is equal to

$$C_{B,\mathcal{I}}(\eta) \otimes \ \bar{\partial}\left[\frac{1}{\zeta_1^{\alpha_1^B}}\right] \wedge \dots \wedge \bar{\partial}\left[\frac{1}{\zeta_k^{\alpha_k^B}}\right] \wedge e_B \wedge \varphi(\zeta, \eta)\, d\zeta,$$

where $C_{B,\mathcal{I}}$ depends smoothly on η.

In other words, if we let $R_{B,\mathcal{I}}$ be defined by (5.2) (meaning that its action on a test form ϕ is the value of (5.2) at $\lambda = 0$), then $R_{B,\mathcal{I}}$ is of the form (5.1).

When looking in $M_{\mathcal{J}}$ for each index set $\mathcal{J}$ of cardinality k we miss

$$(\bigcup_{\mathcal{J},|\mathcal{J}|=k} M_{\mathcal{J}})^C = (\bigcup_{\mathcal{J},|\mathcal{J}|=k} (\bigcup_{i \notin \mathcal{J}} H_i)^C)^C =$$

$$\bigcap_{\mathcal{J},|\mathcal{J}|=k} \bigcup_{i \notin \mathcal{J}} H_i = \bigcup_{\mathcal{J},|\mathcal{J}|=k+1} \bigcap_{i \in \mathcal{J}} H_i,$$

that is precisely V_k. Clearly each current $R_{B,\mathcal{I}}$ extends to $\bigcup_{\mathcal{J},|\mathcal{J}|=k} M_{\mathcal{J}}$. In fact $R_{B,\mathcal{I}}$ has support only in $M_{\mathcal{I}}$. Thus outside V_k we have $R_k = \sum R_{B,\mathcal{I}}$, where the $R_{B,\mathcal{I}}$ are of the desired form and we are done. $\qquad\square$

References

[1] Mats Andersson, *Residue currents and ideals of holomorphic functions*, Bull. Sci. Math. **128** (2004), 481–512. MR **2074610 (2005e:**32006)

[2] _____, *Integral representation with weights. I*, Math. Ann. **326** (2003), 1–18. MR **1981609 (2004f:**32003)

[3] _____, *The membership problem for polynomial ideals in terms of residue currents*, Ann. Inst. Fourier (Grenoble) **56** (2006), 101–119. MR 2228682 (English, with English and French summaries)

[4] Carlos A. Berenstein, Roger Gay, Alekos Vidras, and Alain Yger, *Residue currents and Bezout identities*, Progress in Mathematics, vol. 114, Birkhäuser Verlag, Basel, 1993, ISBN 3-7643-2945-9. MR **1249478 (94m:**32006)

[5] Henri Skoda and Joël Briançon, *Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de $\mathbf{C}^n$*, C. R. Acad. Sci. Paris Sér. A **278** (1974), 949–951. MR 0340642 (49 #5394) (French)

[6] A. Dickenstein and C. Sessa, *Canonical representatives in moderate cohomology*, Invent. Math. **80** (1985), 417–434. MR **791667 (87a:**32013)

[7] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Springer-Verlag, Berlin, 1973, Lecture Notes in Mathematics, Vol. 339. MR 0335518 (49 #299)

[8] A. G. Khovanskii, *Newton polyhedra, and toroidal varieties*, Funct. Anal. Appl. **11** (1977), 289–296.

[9] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005, ISBN 0-387-22356-8. MR **2110098 (2006d:**13001)

[10] Mikael Passare, *Residues, currents, and their relation to ideals of holomorphic functions*, Math. Scand. **62** (1988), 75–152. MR **961584 (90d:**32019)

[11] Mikael Passare, August Tsikh, and Alain Yger, *Residue currents of the Bochner-Martinelli type*, Publ. Mat. **44** (2000), 85–117. MR **1775747 (2001i:**32006)

[12] Bernard Teissier, *Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney* (1982), 314–491. MR **708342 (85i:**32019) (French)

[13] _____, *Monomial ideals, binomial ideals, polynomial ideals* (2004), 211–246. MR **2132653 (2006c:**13032)

[14] A. N. Varčenko, *Newton polyhedra and estimation of oscillating integrals*, Funct. Anal. Appl. **18** (1976), 175–196.

[15] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995, ISBN 0-387-94365-X. MR **1311028 (96a:**52011)

MATHEMATICAL SCIENCES, CHALMERS UNIVERSITY OF TECHNOLOGY AND MATHEMATICAL SCIENCES, GÖTEBORG UNIVERSITY, SE-412 96 GÖTEBORG, SWEDEN

E-mail address: wulcan@math.chalmers.se