GLOBAL EFFECTIVE VERSIONS OF THE BRIANÇON-SKODA-HUNEKE THEOREM

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Abstract. We prove global effective versions of the Briançon-Skoda-Huneke theorem. Our results extend, to singular varieties, a result of Hickel on the membership problem in polynomial ideals in $\mathbb{C}^n$, and a related theorem of Ein and Lazarsfeld for smooth projective varieties. The proofs rely on known geometric estimates and new results on multivariable residue calculus.

1. Introduction

Let V be a reduced algebraic subvariety of $\mathbb{C}^N$ of pure dimension n. If $F_1, \dots, F_m$ are polynomials in $\mathbb{C}^N$ with no common zeros on V, then by the Nullstellensatz there are polynomials Q_j such that $\sum F_j Q_j = 1$ on V. It was proved by Jelonek, [23], that if F_j have degree at most d, then one can find Q_j such that

$$\deg (F_j Q_j) \le c_m d^\mu \deg V$$

on V, where $c_m = 1$ if $m \le n$, $c_m = 2$ if $m > n$, $\deg V$ means the degree of the closure of V in $\mathbb{P}^N$, and, throughout this paper,

$$\mu := \min(m, n).$$

This result generalizes Kollár's theorem[1], [24], for $V = \mathbb{C}^n$ and does not require any smoothness assumptions on V. The bound is optimal[2] when $m \le n$ and almost optimal when $m > n$. However, in view of various known results in the case when $V = \mathbb{C}^n$, one can expect sharper degree estimates if the common zero set of the polynomials F_j behaves nicely at infinity in $\mathbb{P}^N$.

More generally one can take arbitrary polynomials F_j of degree at most d and look for a solution Q_j to

$$F_1 Q_1 + \cdots + F_m Q_m = \Phi \tag{1.1}$$

on V with good degree estimates, provided that the polynomial Φ belongs to the ideal (F_j) generated by the F_j on V. It follows from a result of Hermann, [20], that one can choose Q_j such that $\deg (F_j Q_j) \le \deg \Phi + C(d, N)$, where $C(d, N)$ is like $2(2d)^{2^N - 1}$ for large d, thus doubly exponential. It is shown in [28] that this estimate cannot be substantially improved for $V = \mathbb{C}^n$. However, under additional hypotheses on Φ and the common zero set of the F_j, much sharper estimates are possible. In the extreme case when the polynomials F_j have empty common zero set, even at infinity,

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[1]In Kollár's theorem $c_m = 1$ even for $m > n$ (unless $d = 2$, see also [30]) and this estimate is optimal.

[2]In Kollár's and Jelonek's theorems, as well as in [21], there are more precise results that take into account different degree bounds d_j of F_j, but for simplicity, in this paper we always keep all $d_j = d$.

a classical result of Macaulay, [27], states that when $V = \mathbb{C}^n$, one can solve (1.1) with polynomials Q_j such that $\deg F_j Q_j \leq \max(\deg \Phi, d(n+1)-n)$, cf. Example 1.3 below.

By homogenization, this kind of effective results can be reformulated as geometric statements: Let $z = (z_0, \dots, z_N)$, $z' = (z_1, \dots, z_N)$, let $f_i(z) := z_0^d F_i(z'/z_0)$ be the d-homogenizations of F_i, and let $\varphi(z) := z_0^{\deg \Phi}\Phi(z'/z_0)$. Then there is a representation (1.1) on V with $\deg(F_j Q_j) \leq \rho$ if and only if there are $(\rho - d)$-homogeneous forms q_i on $\mathbb{P}^N$ such that

$$f_1 q_1 + \cdots + f_m q_m = z_0^{\rho - \deg \Phi}\varphi \tag{1.2}$$

on the closure X of V in $\mathbb{P}^N$. As usual, we can consider f_j as holomorphic sections of (the restriction to X of) the line bundle $\mathcal{O}(d) \to \mathbb{P}^N$, $z_0^{\rho-\deg\Phi}\varphi$ as a section of $\mathcal{O}(\rho)$, etc, so that (1.2) becomes a statement about sections of line bundles.

In this paper we present global effective versions of the Briançon-Skoda-Huneke theorem:

Let $\mathcal{V}$ be a germ of a reduced analytic set of pure dimension n at the origin in $\mathbb{C}^N$. There is a number μ_0 such that if $a_1, \dots, a_m, \phi$ are germs of holomorphic functions at 0, $\ell \geq 1$, and $|\phi| \leq C|a|^{\mu+\mu_0+\ell-1}$ in a neighborhood of 0 in $\mathcal{V}$, where C is a positive constant and $|a|^2 = |a_1|^2 + \cdots + |a_m|^2$, then ϕ belongs to the ideal $(a_1, \dots, a_m)^\ell \subset \mathcal{O}_0$.[3]

If $\mathcal{V}$ is smooth, then one can take $\mu_0 = 0$; this is the classical Briançon-Skoda theorem, [13]. The general case was proved by Huneke, [22], by purely algebraic methods. An analytic proof appeared in [7].

Given polynomials $F_1, \dots, F_m$ on V, let f_j denote the corresponding sections of $\mathcal{O}(d)|_X$, and let $\mathcal{J}_f$ be the coherent analytic sheaf on X generated by f_j. Furthermore, let c_∞ be the maximal codimension of the so-called *distinguished varieties* of the sheaf $\mathcal{J}_f$, in the sense of Fulton-MacPherson, that are contained in

$$X_\infty := X \setminus V,$$

see Section 5. If there are no distinguished varieties contained in X_∞, then we interpret c_∞ as $-\infty$. It is well-known that the codimension of a distinguished variety cannot exceed the number m, see, e.g., Proposition 2.6 in [16], and thus

$$c_\infty \leq \mu. \tag{1.3}$$

We let Z^f denote the zero variety of $\mathcal{J}_f$ in X.

Our first result involves the so-called *(Castelnuovo-Mumford) regularity*, $\operatorname{reg} X$, of $X \subset \mathbb{P}^N$, see Section 2.9 for the definition.

Theorem A. *Assume that V is a reduced algebraic subvariety of $\mathbb{C}^N$ of pure dimension n and let X be its closure in $\mathbb{P}^N$.*

(i) There exists a number μ_0 such that if $F_1, \dots, F_m$ are polynomials of degree $\leq d$ and Φ is a polynomial and

$$|\Phi|/|F|^{\mu+\mu_0} \text{ is locally bounded on } V, \tag{1.4}$$

then there are polynomials $Q_1, \dots, Q_m$ such that (1.1) holds on V and

$$\deg(F_j Q_j) \leq \max\big(\deg \Phi + (\mu+\mu_0) d^{c_\infty} \deg X, (d-1)\min(m, n+1) + \operatorname{reg} X\big). \tag{1.5}$$

[3]Often in the literature $\mu+\mu_0$ is replaced by a constant independent of the number of generators m.

(ii) If V is smooth, then there is a number μ' such that if $F_1, \dots, F_m$ are polynomials of degree $\le d$ and Φ is a polynomial and

$$|\Phi|/|F|^{\mu} \text{ is locally bounded on } V, \tag{1.6}$$

then there are polynomials $Q_1, \dots, Q_m$ such that (1.1) *holds on V and*

$$deg\,(F_jQ_j) \le \max\big(deg\,\Phi + \mu d^{c_\infty}\, deg\, X + \mu', (d-1)\min(m, n+1) + \operatorname{reg} X\big). \tag{1.7}$$

If X is smooth, then one can take $\mu' = 0$.

There are analogous results for powers $(F_j)^\ell$ of (F_j), see Theorem 6.6.

Note that if there are no distinguished varieties of $\mathcal{J}_f$ contained in X_∞, then $d^{c_\infty} = 0$.

If $Z^f \cap X_{\text{sing}} = \emptyset$, then the conclusion in (i) holds with $\mu_0 = 0$, see Remark 6.3.

Remark 1.1. The number μ_0 that appears in the proof of Theorem A below only depends on the intrinsic variety X and not on the particular embedding $i : X \to \mathbb{P}^N$, cf. Remark 6.5.

□

Example 1.2. If we apply Theorem A to Nullstellensatz data, i.e., F_j with no common zeros on V and $\Phi = 1$, we get back the optimal result of Jelonek, except for the annoying factor $\mu + \mu_0$ in front of d^{c_∞}. On the other hand, $(\mu + \mu_0)d^{c_\infty} < d^{\mu}$ if $c_\infty < \mu$ and d is large enough. □

Example 1.3. If f_j have no common zeros on X (so that in particular $d^{c_\infty} = 0$), then we can find a solution to $F_1Q_1 + \dots + F_mQ_m = 1$ on V such that

$$\deg F_jQ_j \le \max(\deg \Phi, (d-1)(n+1) + \operatorname{reg} X).$$

If $X = \mathbb{P}^n$, then $\operatorname{reg} X = 1$ and hence we get back the Macaulay theorem, cf., above. □

Remark 1.4. Assume that $X \subset \mathbb{P}^N$ is Cohen-Macaulay; for instance, X is a complete intersection or even $X = \mathbb{P}^N$. Then $\operatorname{reg} X \le \deg X - (N-n)$, see, [18, Corollary 4.15]. If in addition $m \le n$, then the last entries in (1.5) and (1.7) can be omitted, i.e., we get the sharper estimates $\deg\,(F_jQ_j) \le \deg \Phi + (m+\mu_0)d^{c_\infty}\deg X$ and $\deg\,(F_jQ_j) \le \deg \Phi + md^{c_\infty}\deg X + \mu'$ in (i) and (ii), respectively, see the comment right after the proof of Theorem A in Section 6.

□

Remark 1.5. If X is smooth, then

$$\operatorname{reg} X \le (n+1)(\deg X - 1) + 1;$$

this is Mumford's bound, see [26, Example 1.8.48]. □

Example 1.6. For $V = \mathbb{C}^n$, Theorem A gives the estimate

$$\deg\,(F_jQ_j) \le \max\big(\deg \Phi + \mu d^{c_\infty}, d\min(m, n+1) - n\big). \tag{1.8}$$

This estimate was proved by Hickel, [21], but with the term $\min(m, n+1)d^{\mu}$ rather than our μd^{c_∞}. The ideas in [21] are similar to the ones used in [16]. If one applies the geometric estimate in [16], rather than the (closely related) so-called refined Bezout estimate by Fulton-MacPherson that is used in [21], one can replace the exponent μ by c_∞. This refinement was pointed out already in Example 1 in [16]. □

We have the following more abstract variant of Theorem A. It is a generalization to nonsmooth varieties of the geometric effective Nullstellensatz of Ein-Lazarsfeld in [16] (Theorem 7.1 below). Let X be a reduced projective variety. Recall that if $L \to X$ is an *ample* line bundle, then there is a (smallest) number ν_L such that $H^i(X, L^{\otimes s}) = 0$ for $i \geq 1$ and $s \geq \nu_L$, cf., [26, Ch. 1.2]. When X is smooth, by Kodaira's vanishing theorem, ν_L is less than or equal to the least number σ such that $L^\sigma \otimes K_X^{-1}$ is strictly positive, where K_X is the canonical bundle. In particular, if $V = \mathbb{C}^n$, i.e., $X = \mathbb{P}^n$, then $\nu_{\mathcal{O}(1)} = -n$.

Theorem B. *Let X be a reduced projective variety of pure dimension n. There is a number μ_0, only depending on X, such that the following holds: Let $f_1, \dots, f_m$ be global holomorphic sections of an ample Hermitian line bundle $L \to X$, and let ϕ be a section of $L^{\otimes s}$, where*

$$s \geq \nu_L + \min(m, n+1). \tag{1.9}$$

If

$$|\phi| \leq C|f|^{\mu+\mu_0}, \tag{1.10}$$

then there are holomorphic sections q_j of $L^{\otimes(s-1)}$ such that

$$f_1 q_1 + \cdots + f_m q_m = \phi. \tag{1.11}$$

If X is smooth we can choose $\mu_0 = 0$, see Theorem 7.1.

Let $\mathcal{J}_f$ be the ideal sheaf generated by f_j and assume that the associated distinguished varieties Z_k have multiplicities r_k, cf., Section 5. If we assume that ϕ is in $\cap_k \mathcal{J}(Z_k)^{r_k(\mu+\mu_0)}$, where $\mathcal{J}(Z_k)$ is the radical ideal associated with the distinguished variety Z_k, then (1.10) holds, and hence we have a representation (1.11).

Example 1.7. Let X be the cusp $\{z_1^2 z_0^{p-2} - z_2^p = 0\} \subset \mathbb{P}^2$, where $p > 2$ is odd. Then the sections $f = z_2$ of $L := \mathcal{O}(1)|_X$ and $\phi = z_0^{s-1} z_1$ of $L^{\otimes s}$ satisfy $|\phi| \leq C|f|^{\frac{p-1}{2}}$ on X as soon as $s \geq 2$. However, ϕ is not in (f) on X at the singular point $\{z_1 = z_2 = 0\}$ nor at $\{z_0 = z_2 = 0\}$ (unless $p = 3$).

One can check that $\gamma_p = (p-1)/2$ is the smallest integer such that, for any choice of tuples $g_1, \dots, g_m$ of holomorphic germs at $\{z_1 = z_2 = 0\}$, $|\psi| \leq C|g|^{1+\gamma_p}$ implies that the germ ψ is in the local ideal (g_j) at $\{z_1 = z_2 = 0\}$, in other words γ_p is the *Briançon-Skoda number* at $\{z_1 = z_2 = 0\}$. Moreover, one can check that the Briançon-Skoda number at $\{z_0 = z_2 = 0\}$ is $\lceil \frac{(p-3)(p-1)}{p-2} \rceil$, where $\lceil a \rceil$ denotes the smallest integer $\geq a$, see, e.g., [31]. Therefore, μ_0 must be at least

$$a_p := \max\left(\frac{p-1}{2}, \left\lceil \frac{(p-3)(p-1)}{p-2} \right\rceil\right).$$

In fact, in view of [31, Section 2] and the proof below one finds that Theorem B holds with $\mu_0 = a_p$ in this case.

From [18, Proposition 4.16] we know that $\nu_L \leq \operatorname{reg} X - 2 = p - 2$. Since $\mu_0 \geq a_p$, it follows that if our given f and ϕ satisfy (1.10), then $s \geq p - 1 \geq \nu_L + 1$, so the hypothesis (1.9) in Theorem B is vacuous in this case.

□

In particular, Example 1.7 shows that μ_0 can be arbitrarily large.

The starting point for the proofs of Theorems A and B is the framework for solving division problems using residue theory introduced in [2], and further developed in

[5, 32, 33]: Assume that X is a smooth projective variety and that $f_1, \dots, f_m$ are sections of an ample line bundle $L \to X$ with common zero set Z^f. From the Koszul complex generated by the f_j one defines a current R^f with support on Z^f and taking values in a direct sum of negative powers of L. If ϕ is a section of $L^{\otimes s}$ such that the current $R^f\phi$ vanishes, and if in addition $L^{\otimes s}$ is positive enough, so that certain cohomology classes on X vanish, and thus a certain sequence of $\bar\partial$-equations can be solved on X, one ends up with a holomorphic solution $q = (q_1, \dots, q_m)$ to (1.11).

The main novelty in this paper is an extension of this framework to singular X, see Section 4. Given an embedding $i\colon X \to Y$ of X into a smooth manifold Y and a locally free resolution on Y of $\mathcal{O}^Y/\mathcal{J}_X$, where $\mathcal{J}_X$ is the ideal sheaf associated with X, we construct an intrinsic principal value current ω on X, following [6], and a "product current" $R^f\wedge\omega$. If L, f_j, and ϕ admit extensions to Y, which is the situation in Theorem A with $Y = \mathbb{P}^N$, we can proceed basically as before: If $R^f\wedge\omega\phi = 0$, which is indeed an intrinsic condition on X, and certain cohomology classes on Y vanish, we end up with a holomorphic solution to (1.11); this is how we prove Theorem A. By a small variation one can make this procedure more intrinsic and assume vanishing of cohomology classes on X rather than on Y.

For the proof of Theorem B we cannot use this strategy directly, since we have no a priori extensions of L, f_j, and ϕ to a smooth manifold Y. However, for a fixed L it is possible to find an embedding $i\colon X \to Y$ such that L extends. Given such an embedding, without assuming holomorphic extensions of f_j and ϕ, we construct a variant $\widetilde{R} \wedge \omega$ on X of $R^f\wedge\omega$, again with the property that if $\widetilde{R}\wedge\omega\phi = 0$ and the crucial cohomology classes vanish on X we get a holomorphic solution to (1.11).

To verify that the currents $R^f \wedge \omega\phi$ and $\widetilde{R}\wedge\omega\phi$ vanish we need to analyse the singularities of ω. For Theorem A it is enough to consider a fixed ω, coming from the embedding of X in $\mathbb{P}^N$ and a choice of resolution of $\mathcal{O}^Y/\mathcal{J}_X$, whereas for Theorem B we need to control the singularities for ω coming from any possible embedding $i : X \to Y$. To this end we need a new uniform estimate, Proposition 2.5.

In Section 2 we provide necessary background on residue currents. The proofs of our main theorems together with some further results and comments are gathered in Sections 5 to 7.

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2. Some preliminaries on residue theory

Most of the results in this section can be found in the papers [1, 4, 6, 7, 8, 9]. More precisely, the material in Section 2.1 is taken from [9, Sections 2-3]. Sections 2.2-2.4 are mostly based on the first three sections in [8] and Section 2.9 is based on Section 6 in loc. cit. For Section 2.5, see [4, Section 4], and for Sections 2.6 and 2.8, see Sections 1 and 3, respectively, in [6]. Finally Section 2.7 is based on [7, Section 4]. Proposition 2.5 is new; the proof is given in Section 3.

Throughout this paper X is a reduced projective variety of pure dimension n. The sheaf $\mathcal{C}_{\ell,k}$ of currents of bidegree (ℓ, k) on X is by definition the dual of the sheaf $\mathcal{E}_{n-\ell,n-k}$ of smooth $(n-\ell, n-k)$-forms on X. If $i\colon X \to Y$ is an embedding in a smooth manifold Y of dimension N, then $\mathcal{E}_{n-\ell,n-k}$ can be identified with the quotient sheaf $\mathcal{E}^Y_{n-\ell,n-k}/\operatorname{Ker} i^*$, where $\operatorname{Ker} i^*$ is the sheaf of forms ξ on Y such that $i^*\xi$ vanish

on $X_{\rm reg}$. It follows that the currents τ in $\mathcal{C}_{\ell,k}$ can be identified with currents $\tau' = i_*\tau$ on Y of bidegree $(N-n+\ell, N-n+k)$ that vanish on $\operatorname{Ker} i^*$.

Given a holomorphic function f on X, we have the principal value current $[1/f]$, defined for instance as the limit

$$\lim_{\epsilon\to 0}\chi(|f|^2/\epsilon)\frac{1}{f},$$

where $\chi(t)$ is the characteristic function of the interval $[1,\infty)$ or a smooth approximand of it. The existence of this limit for a general f relies on Hironaka's theorem that ensures that there is a modification $\pi\colon \widetilde{X}\to X$ such that π^*f is locally a monomial. It also follows that the function $\lambda\to |f|^{2\lambda}(1/f)$, a priori defined for $\operatorname{Re}\lambda\gg 0$, has a current-valued analytic continuation to $\operatorname{Re}\lambda > -\epsilon$, and that the value at $\lambda=0$ is precisely the current $[1/f]$, see, for instance, [11] or [12]. Although less natural at first sight, it turns out that this latter definition via analytic continuation is often much more convenient. The same idea will be used throughout this paper. For the rest of this paper we skip the brackets and just write $1/f$. It is readily checked that

$$f\frac{1}{f}=1,\qquad f\bar\partial\frac{1}{f}=0. \tag{2.1}$$

2.1. **Pseudomeromorphic currents.** In [9] we introduced the sheaf $\mathcal{PM}$ of *pseudomeromorphic currents* on X in the case X is smooth. The definition when X is singular is identical. In this paper we will use the slightly extended definition introduced in [6]: We say that a current of the form

$$\frac{\xi}{s_1^{\alpha_1}\cdots s_{n-1}^{\alpha_{n-1}}}\wedge\bar\partial\frac{1}{s_n^{\alpha_n}},$$

where s is a local coordinate system and ξ is a smooth form with compact support, is an *elementary pseudomeromorphic current.* The sheaf $\mathcal{PM}$ consists of all possible (locally finite sums of) push-forwards under a sequence of maps $X^m\to\cdots\to X^1\to X$, of elementary pseudomeromorphic currents, where X^m is smooth, and each mapping is either a modification, a simple projection $\widehat{X}\times Y\to\widehat{X}$, or an open inclusion, i.e., X^j is an open subset of X^{j-1}.

The sheaf $\mathcal{PM}$ is closed under $\bar\partial$ (and ∂) and multiplication by smooth forms. If τ is in $\mathcal{PM}$ and has support on a subvariety V and η is a holomorphic form that vanishes on V, then $\overline{\eta}\wedge\tau=0$. We also have the
Dimension principle: If τ is a pseudomeromorphic current on X of bidegree $(*,p)$ that has support on a variety V of codimension $>p$, then $\tau=0$.

If τ is in $\mathcal{PM}$ and V is a subvariety of X, then the natural restriction of τ to the open set $X\setminus V$ has a canonical extension as a principal value to a pseudomeromorphic current $\mathbf{1}_{X\setminus V}\tau$ on X: Let h be a holomorphic tuple with common zero set V. The current-valued function $\lambda\mapsto |h|^{2\lambda}\tau$, a priori defined for $\operatorname{Re}\lambda\gg 0$, has an analytic continuation to $\operatorname{Re}\lambda>-\epsilon$ and its value at $\lambda=0$ is by definition $\mathbf{1}_{X\setminus V}\tau$. One can also take a smooth approximand χ of the characteristic function of the interval $[1,\infty)$ and obtain $\mathbf{1}_{X\setminus V}\tau$ as the limit of $\chi(|h|^2/\epsilon)\tau$ when $\epsilon\to 0$. It follows that $\mathbf{1}_V\tau := \tau-\mathbf{1}_{X\setminus V}\tau$ is pseudomeromorphic and has support on V. Notice that if α is a smooth form, then $\mathbf{1}_V\alpha\wedge\tau=\alpha\wedge\mathbf{1}_V\tau$. Moreover, If $\pi\colon\widetilde{X}\to X$ is a modification, $\tilde\tau$ is in $\mathcal{PM}(\widetilde{X})$, and $\tau=\pi_*\tilde\tau$, then

$$\mathbf{1}_V\tau=\pi_*\big(\mathbf{1}_{\pi^{-1}V}\tilde\tau\big) \tag{2.2}$$

for any subvariety $V \subset X$. There is actually a reasonable definition of $\mathbf{1}_W \tau$ for any constructible set W, and

$$\mathbf{1}_W \mathbf{1}_{W'} \tau = \mathbf{1}_{W \cap W'} \tau. \tag{2.3}$$

Recall that a current is *semi-meromorphic* if it is the quotient of a smooth L-valued form and a holomorphic section of L, for some line bundle L. We say that a current τ is *almost semi-meromorphic* in X if there is a modification $\pi \colon \widetilde{X} \to X$ and a semi-meromorphic current $\tilde{\tau}$ such that $\tau = \pi_* \tilde{\tau}$, see [6, Section 2]. Analogously we say that τ is *almost smooth* if $\tau = \pi_* \tilde{\tau}$ and $\tilde{\tau}$ is smooth. Any almost semi-meromorphic (or smooth) τ is pseudomeromorphic.

2.2. **Residues associated with Hermitian complexes.** Assume that

$$0 \to E_M \xrightarrow{f^M} \dots \xrightarrow{f^3} E_2 \xrightarrow{f^2} E_1 \xrightarrow{f^1} E_0 \to 0 \tag{2.4}$$

is a generically exact complex of Hermitian vector bundles over X and let Z be the subvariety where (2.4) is not pointwise exact. The bundle $E = \oplus E_k$ gets a natural superbundle structure, i.e., a $\mathbb{Z}_2$-grading, $E = E^+ \oplus E^-$, E^+ and E^- being the subspaces of even and odd elements, respectively, by letting $E^+ = \oplus_{2k} E_k$ and $E^- = \oplus_{2k+1} E_k$. This extends to a $\mathbb{Z}_2$-grading of the sheaf $\mathcal{C}_\bullet(E)$ of E-valued currents, so that the degree of $\xi \otimes e$ is the sum of the current degree of ξ and the degree of e, modulo 2. An endomorphism on $\mathcal{C}_\bullet(E)$ is even if it preserves degree and odd if it switches degrees. The mappings $f := \sum f^j$ and $\bar\partial$ are then odd mappings on $\mathcal{C}_\bullet(E)$. We introduce $\nabla = \nabla_f = f - \bar\partial$; it is just (minus) the $(0,1)$-part of Quillen's superconnection $D - \bar\partial$. Since the odd mappings f and $\bar\partial$ anti-commute, $\nabla^2 = 0$. Moreover, ∇ extends to an odd mapping ∇_{End} on $\mathcal{C}_\bullet(\mathrm{End} E)$ so that

$$\nabla(\alpha\xi) = \nabla_{\mathrm{End}}\alpha \cdot \xi + (-1)^{\deg \alpha} \alpha \nabla \xi \tag{2.5}$$

for sections ξ and α of E and $\mathrm{End}E$, respectively, and then $\nabla_{\mathrm{End}}^2 = 0$. In $X \setminus Z$ we define, following [8, Section 2], a smooth $\mathrm{End}E$-valued form u such that

$$\nabla_{\mathrm{End}} u = I,$$

where $I = I_E$ is the identity endomorphism on E. We have that

$$u = \sum_\ell u^\ell = \sum_\ell \sum_{k \geq \ell+1} u_k^\ell,$$

where u_k^ℓ is in $\mathcal{E}_{0,k-\ell-1}(\mathrm{Hom}(E_\ell, E_k))$ over $X \setminus Z$. Following [8][4] we define a pseudomeromorphic current extension U of u across Z, as the value at $\lambda = 0$ of the current-valued analytic function

$$\lambda \mapsto U^\lambda := |F|^{2\lambda} u,$$

a priori defined for $\mathrm{Re}\,\lambda \gg 0$, where F is the tuple f^1. In the same way we define the *residue current* R associated with (2.4) as the value at $\lambda = 0$ of

$$\lambda \mapsto R^\lambda := (1 - |F|^{2\lambda}) I + \bar\partial |F|^{2\lambda} \wedge u.$$

The existence of the analytic continuations follows from a suitable resolution $\widetilde{X} \to X$, see [8], see also Section 5 below. The current R clearly has support on Z, and

$$R = \sum_\ell R^\ell = \sum_\ell \sum_{k \geq \ell+1} R_k^\ell,$$

[4]The definition is the same when X is singular.

where R_k^ℓ is a $\mathrm{Hom}(E_\ell, E_k)$-valued $(0, k-\ell)$-current. The currents U^ℓ and U_k^ℓ are defined analogously. Notice that U has odd degree and that R has even degree. By the dimension principle, R_k^ℓ vanishes if $k-\ell < \operatorname{codim} Z$. In particular, $R_0^0 = (1-|F|^{2\lambda})I_{E_0}|_{\lambda=0}$ is zero, unless some components W of Z has codimension 0, in which case R_0^0 is the characteristic function for W times the identity I_{E_0} on E_0. However, when we define products of currents later on, all components of R^λ may play a role.

Since $\nabla_{\mathrm{End}} U^\lambda = I - R^\lambda$ and $\nabla_{\mathrm{End}} R^\lambda = 0$ when $\operatorname{Re}\lambda \gg 0$, we conclude that

$$\nabla_{\mathrm{End}} U = I - R, \qquad \nabla_{\mathrm{End}} R = 0. \tag{2.6}$$

In particular, if ξ is a section of E, then

$$\nabla(U\xi) = \xi - R\wedge\xi.$$

Also, (2.6) means that, cf. (2.5),

$$f^1 U_1^0 = I_{E_0}, \quad f^{k+1}U_{k+1}^0 - \bar\partial U_k^0 = R_k^0; \quad k \geq 1.$$

Notice that when ϕ is a section of E_0, then $R^0\phi = R\phi$ and $U^0\phi = U\phi$, and we will often skip the upper indices.

Example 2.1 (The Koszul complex). Let $f_1, \dots, f_m$ be holomorphic sections of a Hermitian line bundle $L \to X$. Let E^j be disjoint trivial line bundles with basis elements e_j and define the rank m bundle

$$E = (L^{-1}\otimes E^1) \oplus \cdots \oplus (L^{-1}\otimes E^m)$$

over X. Then $f := \sum f_j e_j^*$, where e_j^* is the dual basis, is a section of the dual bundle $E^* = L\otimes(E^1)^* \oplus \cdots \oplus L\otimes(E^m)^*$. If $S \to X$ is a Hermitian line bundle we can form a complex (2.4) with

$$E_0 = S, \quad E_k = S\otimes\Lambda^k E,$$

where all the mappings f^k in (2.4) are interior multiplication δ_f by the section f. Notice that

$$E_k = S\otimes L^{-k}\otimes\Lambda^k(E^1\oplus\cdots\oplus E^m).$$

The superstructure of $\oplus_k E_k$ in this case coincides with the natural grading of the exterior algebra ΛE of E modulo 2.

Let us recall how the currents U^0 and R^0 are defined in this case. For simplicity we suppress the upper indices throughout this example. We have the natural norm

$$|f|^2 = \sum_j |f_j|_L^2$$

on E^*. Let σ be the section of E over $X\setminus Z$ of pointwise minimal norm such that $f\cdot\sigma = \delta_f\sigma = 1$, i.e.,

$$\sigma = \sum_j \frac{f_j^* e_j}{|f|^2}, \tag{2.7}$$

where f_j^* are the sections of L^{-1} of minimal norm such that $f_j f_j^* = |f_j|_L^2$.

Let us consider the exterior algebra over $E\oplus T^*(X)$ so that $d\bar z_j\wedge e_\ell = -e_\ell\wedge d\bar z_j$ etc. Then, e.g., $\bar\partial\sigma$ is a form of positive degree. We have the smooth form

$$u = \sum u_k, \quad u_k = \sigma\wedge(\bar\partial\sigma)^{k-1} \tag{2.8}$$

in $X \setminus Z$, and it admits a natural current extension U across Z, e.g., defined as the analytic continuation of $U^\lambda = |f|^{2\lambda}u$ to $\lambda = 0$. Furthermore, the associated residue current R is obtained as the evaluation at $\lambda = 0$ of

$$R^\lambda := 1 - |f|^{2\lambda} + \bar\partial |f|^{2\lambda} \wedge u =$$
$$1 - |f|^{2\lambda} + \bar\partial |f|^{2\lambda} \wedge u_1 + \dots + \bar\partial |f|^{2\lambda} \wedge u_{\min(m,n)} =: R_0^\lambda + R_1^\lambda + \dots + R_{\min(m,n)}^\lambda.$$

The current R was introduced in [1] in this form, much inspired by [29] where the coefficients appeared. □

2.3. **The associated sheaf complex.** Given the complex (2.4) we have the associated complex of locally free sheaves

$$0 \to \mathcal{O}(E_M) \xrightarrow{f^M} \dots \xrightarrow{f^3} \mathcal{O}(E_2) \xrightarrow{f^2} \mathcal{O}(E_1) \xrightarrow{f^1} \mathcal{O}(E_0). \tag{2.9}$$

In this paper E_0 is always a line bundle so that $\mathcal{J} := \operatorname{Im} f^1$ is a coherent ideal sheaf over X.

Consider the double sheaf complex $\mathcal{M}_{\ell,k} := \mathcal{C}_{0,k}(E_\ell)$ with mappings f and $\bar\partial$. We have the associated total complex

$$\dots \xrightarrow{\nabla_f} \mathcal{M}_j \xrightarrow{\nabla_f} \mathcal{M}_{j-1} \xrightarrow{\nabla_f} \dots \quad ,$$

where $\mathcal{M}_j = \oplus_{\ell-k=j}\mathcal{M}_{\ell,k}$. If X is smooth, then $\mathcal{M}_{\ell,k}$ is exact in the k-direction except at $k = 0$, and the kernels there are $\mathcal{O}(E_\ell)$. Notice that if ϕ is in $\mathcal{O}(E_\ell)$ and $f^\ell\phi = 0$, then also $\nabla_f\phi = 0$. We therefore have a natural mapping

$$H^j(\mathcal{O}(E_\bullet)) \to H^j(\mathcal{M}_\bullet). \tag{2.10}$$

By standard homological algebra, (2.10) is in fact an isomorphism. We can also consider the corresponding sheaf complexes $\mathcal{M}^{\mathcal{E}}_{\ell,k} := \mathcal{E}_{0,k}(E_\ell)$, $\mathcal{M}^{\mathcal{E}}_j = \oplus_{\ell-k=j}\mathcal{M}^{\mathcal{E}}_{\ell,k}$ of smooth sections, and the analogue of (2.10) is then an isomorphism as well.

Lemma 2.2. *Assume that X is smooth. If ϕ is a holomorphic section of E_0 that annihilates R, i.e., $R\phi = 0$, then ϕ is in $\mathcal{J}$.*

Proof. In fact, by (2.6) we have that

$$\nabla_f(U\phi) = \phi - R\phi = \phi.$$

Since X is smooth, (2.10) is an isomorphism, and thus locally $\phi = f^1\psi$ for some holomorphic ψ, i.e., ϕ is in $\mathcal{J}$. □

The smoothness assumption is crucial, as the following example shows.

Example 2.3. Let f be one single function. Then the residue condition $R\phi = 0$ means that $\bar\partial(\phi/f) = 0$. Thus $\psi = \phi/f$ is in the Barlet-Henkin-Passare class, cf., [19] and [6]; however in general ψ is not (strongly) holomorphic, i.e., in general ϕ is not in $\mathcal{J} = (f)$. □

We shall now see that if X is smooth and there is a global current solution to $\nabla W = \phi$, then there is also a global smooth solution. For further reference however we need a slightly more general statement about the associated complex of global sections. Let $\mathcal{M}_{\ell,k}(X)$ and $\mathcal{M}^{\mathcal{E}}_{\ell,k}(X)$ be the double complexes of global current valued and smooth sections, respectively, and let $\mathcal{M}_\bullet(X)$ and $\mathcal{M}^{\mathcal{E}}_\bullet(X)$ be the associated total complexes. Notice that we have natural mappings

$$H^j(\mathcal{M}^{\mathcal{E}}_\bullet(X)) \to H^j(\mathcal{M}_\bullet(X)), \quad j \in \mathbb{Z}. \tag{2.11}$$

The following result is standard, but we include a proof for the reader's convenience.

Proposition 2.4. *If X is smooth, then the mappings* (2.11) *are isomorphisms.*

Proof. By the de Rham theorem, the natural mappings

$$H^k(\mathcal{E}_{0,\bullet}(X,E_\ell)) \to H^k(\mathcal{C}_{0,\bullet}(X,E_\ell)), \quad k\in\mathbb{Z}, \tag{2.12}$$

are isomorphisms; these spaces are in fact naturally isomorphic to the cohomology groups $H^k(X,\mathcal{O}(E_\ell))$. The short exact sequence

$$0\to \mathcal{M}_\bullet^{\mathcal{E}}(X)\to \mathcal{M}_\bullet(X)\to \mathcal{M}_\bullet(X)/\mathcal{M}_\bullet^{\mathcal{E}}(X)\to 0$$

gives rise to, for each fixed ℓ, the long exact sequence

$$\dots\to H^{k-1}(\mathcal{E}_{0,\bullet}(X,E_\ell))\to H^{k-1}(\mathcal{C}_{0,\bullet}(X,E_\ell))\to \\ H^{k-1}(\mathcal{C}_{0,\bullet}(X,E_\ell)/\mathcal{E}_{0,\bullet}(X,E_\ell))\to H^k(\mathcal{E}_{0,\bullet}(X,E_\ell))\to\dots \quad ,$$

and since (2.12) are isomorphisms the cohomology in the k-direction of $\mathcal{M}_{\ell,k}(X)/\mathcal{M}_{\ell,k}^{\mathcal{E}}(X)$ is zero. By a simple homological algebra argument, using that the double complexes involved are bounded, it follows that

$$H^j(\mathcal{M}_\bullet(X)/\mathcal{M}_\bullet^{\mathcal{E}}(X))=0$$

for each j. The proposition now follows from the long exact sequence

$$\dots\to H^{j-1}(\mathcal{M}_\bullet^{\mathcal{E}}(X))\to H^{j-1}(\mathcal{M}_\bullet(X))\to \\ H^{j-1}(\mathcal{M}_\bullet(X)/\mathcal{M}_\bullet^{\mathcal{E}}(X))\to H^j(\mathcal{M}_\bullet^{\mathcal{E}}(X))\to\dots \quad .$$

□

2.4. **BEF-varieties and duality principle.** We now consider the case when the locally free complex (2.9) is exact, i.e., a resolution of the sheaf $\mathcal{O}(E_0)/\mathcal{J}$. We will refer to a (locally free) resolution $\mathcal{O}(E_0)/\mathcal{J}$ together with a choice of Hermitian metrics on the corresponding vector bundles E_k as a *Hermitian (locally free) resolution.* Let $Z_k^{\rm bef}$ be the set where the mapping f^k does not have optimal rank. Then

$$\cdots Z_{k+1}^{\rm bef}\subset Z_k^{\rm bef}\subset\cdots\subset Z_1^{\rm bef}=Z,$$

and these sets are independent of the choice of resolution; we call them the *BEF varieties*[5]. It follows from the Buchsbaum-Eisenbud theorem that $\operatorname{codim} Z_k^{\rm bef}\geq k$. If moreover $\mathcal{J}$ has pure dimension, for instance $\mathcal{J}$ is the radical ideal sheaf of a pure-dimensional subvariety, then $\operatorname{codim} Z_k^{\rm bef}\geq k+1$ for $k\geq 1+\operatorname{codim}\mathcal{J}$, see [17, Corollary 20.14].

Since (2.9) is exact, by [8, Theorem 3.1], we have that $R^\ell=0$ for each $\ell\geq 1$, i.e., $R=R^0$. Moreover, there are almost semi-meromorphic $\operatorname{Hom}(E_k,E_{k+1})$-valued $(0,1)$-forms α_{k+1}, that are smooth outside $Z_{k+1}^{\rm bef}$, such that

$$R_{k+1}=\alpha_{k+1}R_k$$

there, see [8, Section 3]. From [8, Theorem 1.1] we also have the

Duality principle: If X is smooth and (2.9) *is a resolution of the sheaf $\mathcal{O}(E_0)/\mathcal{J}$, then $\phi\in\mathcal{J}$ if and only if $R\phi=0$.*

[5]The sets $Z_k^{\rm bef}$ are the zero varieties of certain Fitting ideals associated with a free resolution of $\mathcal{O}^X/\mathcal{J}$; the importance of these sets (ideals) was pointed out by Buchsbaum and Eisenbud in the 70's. We have not seen any notion for these sets in the literature, and "Buchsbaum-Eisenbud varieties" is already occupied for another purpose, so we stick to BEF as an acronym for Buchsbaum-Eisenbud-Fitting.

That is, the annihilator ideal sheaf of the residue current R is precisely the ideal sheaf $\mathcal{J}$ generated by f^1.

If for instance $f^1 = (f_1, \dots, f_m)$ defines a complete intersection, i.e, $\operatorname{codim} Z = m$, then the Koszul complex is a resolution of $\mathcal{J}$ and hence the duality principle states that the annihilator of the residue current in Example 2.1 is the ideal itself.

2.5. **Tensor products of complexes.** Assume that $(E^g_\bullet, g)$ and $(E^h_\bullet, h)$ are Hermitian complexes. We can then define a complex $(E^f_\bullet = E^g_\bullet \otimes E^h_\bullet, f)$, where

$$E^f_k = \bigoplus_{i+j=k} E^g_i \otimes E^h_j,$$

and $f = g + h$, or more formally $f = g \otimes I_{E^h} + I_{E^g} \otimes h$, such that

$$f(\xi \otimes \eta) = g\xi \otimes \eta + (-1)^{\deg \xi} \xi \otimes h\eta. \tag{2.13}$$

Notice that E^f_0 is the line bundle $E^g_0 \otimes E^h_0$. If $g^1\mathcal{O}(E^g_1) = \mathcal{J}_g$ and $h^1\mathcal{O}(E^h_1) = \mathcal{J}_h$, then $f^1\mathcal{O}(E^f_1) = \mathcal{J}_g + \mathcal{J}_h$. One extends (2.13) to current-valued sections ξ and η, and $\deg \xi$ then means total degree. We write $\xi \cdot \eta$, or sometimes $\xi \wedge \eta$ to emphasize that the sections may be form- or current-valued, rather than $\xi \otimes \eta$, and define

$$\eta \cdot \xi = (-1)^{\deg \xi \deg \eta} \xi \cdot \eta. \tag{2.14}$$

Notice that

$$\nabla_f(\xi \cdot \eta) = \nabla_g \xi \cdot \eta + (-1)^{deg \xi} \xi \cdot \nabla_h \eta.$$

Let u^g and u^h be the corresponding $\operatorname{End}(E^g)$-valued and $\operatorname{End}(E^h)$-valued forms, cf., Section 2.2. Then $u^h \wedge u^g$ is a $\operatorname{End}(E^f)$-valued form defined outside $Z^g \cup Z^h$. Following the proof of Proposition 2.1 in [9] we can define $\operatorname{End}(E^f)$-valued pseudomeromorphic currents

$$U^h \wedge R^g := U^{h,\lambda} \wedge R^g|_{\lambda=0}, \quad R^h \wedge R^g := R^{h,\lambda} \wedge R^g|_{\lambda=0}.$$

We have that, cf., (2.6) and [4, Section 4],

$$\nabla_{\operatorname{End},f}(U^h \wedge R^g + U^g) = I_{E^f} - R^h \wedge R^g.$$

In general, the current $R^h \wedge R^g$ will change if we interchange the roles of g and h.

In particular we can form the product $E^h_\bullet \otimes E^h_\bullet$ of $E^h_\bullet$ by itself. In this case we consider (2.14) as an identification, so that, for instance,

$$(E^h_\bullet \otimes E^h_\bullet)_1 = E^h_1 \dot{\otimes} E^h_0, \quad (E^h_\bullet \otimes E^h_\bullet)_2 = E^h_2 \dot{\otimes} E^h_0 + \Lambda^2 E^h_1,$$

etc, where $\dot{\otimes}$ denotes symmetric tensor product. In general, $\xi \cdot \xi = 0$ if ξ has odd (total) degree.

We can just as well form a similar product of more than two complexes, and in particular, we can form the product $(E^h)^{\otimes k} = E^h \otimes E^h \otimes \cdots \otimes E^h$ of a given complex by itself.

2.6. **The structure form ω on a singular variety.** Let $i \colon X \to Y$ be an embedding of X in a smooth projective manifold Y of dimension N, let $\mathcal{J}_X$ be the radical ideal sheaf associated with X in Y, and let $S \to Y$ be an ample Hermitian line bundle. Moreover, let E^j_k be disjoint trivial line bundles over Y with basis elements $e_{k,j}$. There is a (possibly infinite) resolution, see, e.g., [26, Ch.1, Example 1.2.21],

$$\dots \xrightarrow{g^3} \mathcal{O}(E_2) \xrightarrow{g^2} \mathcal{O}(E_1) \xrightarrow{g^1} \mathcal{O}(E_0) \tag{2.15}$$

of $\mathcal{O}(E_0)/\mathcal{J}_X = \mathcal{O}^X$, where E_k is of the form

$$E_k = \big(E_k^1 \otimes S^{-d_k^1}\big) \oplus \cdots \oplus \big(E_k^{r_k} \otimes S^{-d_k^{r_k}}\big), \quad E_0 = E_0^0 \simeq \mathbb{C},$$

E_k^i are trivial line bundles, and

$$g^k = \sum_{ij} g_{ij}^k e_{k-1,i} \otimes e_{k,j}^*$$

are matrices of sections

$$g_{ij}^k \in \mathcal{O}(Y, S^{d_k^j - d_{k-1}^i});$$

here $e_{k,j}^*$ are the dual basis elements. There are natural induced norms on E_k. The associated residue current[6] R is annihilated by all smooth forms ξ such that $i^*\xi = 0$. Let Ω be a global nonvanishing $(\dim Y, 0)$-form with values in K_Y^{-1}. By [6, Proposition 16] there is a (unique) almost semi-meromorphic current ω on X, smooth on $X_{\rm reg}$, such that

$$i_*\omega = R\wedge\Omega. \tag{2.16}$$

We say that ω is a *structure form* on X.

As an immediate consequence of the existence of ω, the product $\alpha\wedge R$ is well-defined for all (sufficiently) smooth forms α on X. If $\alpha = i^*a$, we let $\alpha\wedge R := a\wedge R$. This product only depends on α, since if $i^*a = 0$, then $a\wedge R\wedge\Omega = i_*(i^*a\wedge\omega) = 0$ and hence $a\wedge R = 0$ since $\Omega \neq 0$.

Let X_k be the BEF varieties of $\mathcal{J}_X$, and define

$$X^0 = X_{\rm sing}, \quad X^\ell = X_{N-n+\ell},\ \ell \geq 1.$$

Since $\mathcal{J}_X$ has pure dimension it follows that

$$\operatorname{codim} X^k \geq k+1, \tag{2.17}$$

and in particular, $X^n = \emptyset$. These sets X^ℓ are actually independent of the choice of embedding of X, cf., the comment after Lemma 3.1.

Let g_ℓ be the restriction to X of $g^{N-n+\ell}$, and let $\nabla^g = g - \bar\partial$ on X. Let $E^\ell = E_{N-n+\ell}|_X$. Then $\omega = \omega_0+\omega_1+\cdots+\omega_n$, where ω_ℓ is a (n,ℓ)-form on X taking values in E^ℓ, and $\nabla^g\omega = 0$ on X. There are almost semi-meromorphic $\operatorname{Hom}(E^\ell, E^{\ell+1})$-valued $(0,1)$-forms $\alpha^{\ell+1}$ such that

$$\omega_{\ell+1} = \alpha^{\ell+1}\omega_\ell \tag{2.18}$$

there. In fact, α^ℓ is the pullback to X of the form $\alpha_{N-n+\ell}$ associated with a resolution of $\mathcal{O}^Y/\mathcal{J}_X$ in Y, cf., Section 2.4.

Since ω is almost semi-meromorphic, it has the *the standard extension property*, SEP on X, which means that $\mathbf{1}_W\omega = 0$ for all varieties $W \subset X$ of positive codimension.

The singularities of a structure form ω only depend on X, in the following sense:

Proposition 2.5. *Let X be a projective variety. There is a smooth modification $\tau\colon \widetilde{X} \to X$ and a holomorphic section η of a line bundle $S \to \widetilde{X}$ such that the following holds: If $X \to Y$ is an embedding of X in a smooth manifold Y, $\big(\mathcal{O}(E_\bullet^g), g\big)$ is a Hermitian locally free resolution of $\mathcal{O}^Y/\mathcal{J}_X$, and ω is the associated structure form on X, then $\eta\tau^*\omega$ is smooth on $\widetilde{X}$. We can choose η to be nonvanishing in $\widetilde{X} \setminus \tau^{-1}X_{sing}$.*

[6]The fact that (2.15) may be infinite causes no problem, since, for degree reasons, U and R only contain a finite number of terms.

After further resolving we may assume that η is locally a monomial in $\widetilde{X}$.

The proof is postponed to Section 3. Since ω is almost semi-meromorphic, the pullback $\tau^*\omega$ is well-defined; this follows from the proof below, cf., also the remark after Definition 12 in [6].

2.7. **Local division problems on a singular variety.** Still assume that we have the embedding $i\colon X \to Y$, where Y is smooth, and the complex $(E^g_\bullet, g)$ over Y corresponding to a Hermitian locally free resolution of $\mathcal{O}^Y/\mathcal{J}_X$. If $(E^f_\bullet, f)$ is an arbitrary Hermitian complex over Y we have the complex $E^F = E^f \otimes E^g$ with mappings $F = f + g$ as in Section 2.5. Let $F^k = F|_{E_k}$. Since $R^f \wedge R^g = R^{f,\lambda} \wedge R^g|_{\lambda=0}$ and $U^f \wedge R^g = U^{f,\lambda} \wedge R^g|_{\lambda=0}$, cf., Section 2.6, these currents only depend on the values of f on X. From Section 2.5 we also have that

$$\nabla_{\mathrm{End},F} U = I - R^f \wedge R^g \tag{2.19}$$

if $U = U^f \wedge R^g + U^g$. If Φ is a (locally defined) holomorphic function in Y and $R^f \wedge R^g \Phi = 0$, then, following the proof of Lemma 2.2, there is a local holomorphic solution $v = v_f + v_g$ in $E^F_1 = E^f_1 \otimes E^g_0 + E^f_0 \otimes E^g_1$ to $g^1 v_f + f^1 v_g = F^1 v = \Phi$. Notice that in fact $R^f \wedge R^g \Phi$ only depends on the class ϕ of Φ in $\mathcal{O}^Y/\mathcal{J}_X = \mathcal{O}^X$, so $R^f \wedge R^g \phi$ is well-defined for ϕ in $\mathcal{O}^X$. We can define the intrinsic residue current

$$R^f \wedge \omega := R^{f,\lambda} \wedge \omega|_{\lambda=0}$$

on X. Since $i_* R^{f,\lambda} \wedge \omega = R^{f,\lambda} \wedge R^g \wedge \Omega$ when $\operatorname{Re}\lambda \gg 0$, we conclude that

$$i_* R^f \wedge \omega = R^f \wedge R^g \wedge \Omega.$$

Since Ω is nonvanishing, $R^f \wedge \omega\phi = 0$ implies that $R^f \wedge R^g \phi = 0$ and thus we have:

Proposition 2.6. *Assume that $(E^f_\bullet, f)$ is a Hermitian complex on X. If ϕ is a holomorphic section of E^f_0 on X such that $R^f \wedge \omega\phi = 0$, then locally ϕ is in the image of f^1 on X.*

2.8. **A fine resolution of $\mathcal{O}$ on X.** It was proved in [6], see [6, Theorem 2], that there exist sheaves $\mathcal{A}_k$ of $(0,k)$-currents on X with the following properties:
(i) $\mathcal{A}_k$ is equal to $\mathcal{E}_{0,k}$ on X_{reg},
(ii) $\mathcal{A} = \oplus_k \mathcal{A}_k$ is closed under multiplication by smooth $(0,*)$-forms,
(iii) $\bar\partial$ maps $\mathcal{A}_k$ to $\mathcal{A}_{k+1}$ and if E is any vector bundle over X, then the sheaf complex

$$0 \to \mathcal{O}(E) \to \mathcal{A}_0(E) \xrightarrow{\bar\partial} \mathcal{A}_1(E) \xrightarrow{\bar\partial} \mathcal{A}_2(E) \xrightarrow{\bar\partial} \dots$$

is exact.

By standard sheaf theory we have canonical isomorphisms

$$H^k(X, \mathcal{O}(E)) = \frac{\operatorname{Ker}\big(\Gamma(X, \mathcal{A}_k(E)) \xrightarrow{\bar\partial} \Gamma(X, \mathcal{A}_{k+1}(E))\big)}{\operatorname{Im}\big(\Gamma(X, \mathcal{A}_{k-1}(E)) \xrightarrow{\bar\partial} \Gamma(X, \mathcal{A}_k(E))\big)}, \qquad k \geq 1.$$

2.9. **Subvarieties of $\mathbb{P}^N$.** Let X be a subvariety of $Y = \mathbb{P}^N$, $S = \mathcal{O}(1)$, and let $(\mathcal{O}(E_\bullet), g)$ be a resolution of $\mathcal{O}(E_0)/\mathcal{J}_X$ as in (2.15). Then, see [8, Section 6],

$$E_k = \big(E^1_k \otimes \mathcal{O}(-d^1_k)\big) \oplus \dots \oplus \big(E^{r_k}_k \otimes \mathcal{O}(-d^{r_k}_k)\big)$$

and $g^k = (g^k_{ij})$ are matrices of homogeneous forms with $\deg g^k_{ij} = d^j_k - d^i_{k-1}$. We choose the Hermitian metrics so that

$$|\xi(z)|^2_{E_k} = \sum_{j=1}^{r_k} |\xi_j(z)|^2 |z|^{2d^j_k}$$

if $\xi = (\xi_1, \dots, \xi_{r_k})$ is a section of E_k. Moreover,

$$\Omega = const \times \sum (-1)^j z_j dz_0 \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge dz_N$$

in $\mathbb{P}^N$.

Let J_X denote the homogeneous ideal in the graded ring $\mathcal{S} = \mathbb{C}[z_0, \dots, z_N]$ that is associated with X, and let $\mathcal{S}(\ell)$ denote the module $\mathcal{S}$ where all degrees are shifted by ℓ. Then $(\mathcal{O}(E_\bullet), g)$ corresponds to a free resolution

$$\dots \to \oplus_i \mathcal{S}(-d^i_k) \to \dots \to \oplus_i \mathcal{S}(-d^i_2) \to \oplus_i \mathcal{S}(-d^i_1) \to \mathcal{S} \tag{2.20}$$

of the module $\mathcal{S}/J_X$. Conversely, any such free resolution corresponds to a sheaf resolution $(\mathcal{O}(E_\bullet), g)$.

Notice that the ideal J_X has pure dimension in $\mathcal{S}$, so that in particular the ideal associated to the origin is not an associated prime ideal. From Corollary 20.14 in [17], applied to $\mathcal{S}$, it follows that the BEF-variety of dimension zero must vanish; therefore the depth of $\mathcal{S}/J_X$ is at least 1, and hence a minimal free resolution of $\mathcal{S}/J_X$ has length $\leq N$. Recall that the *(Castelnuovo-Mumford) regularity* of a homogeneous module with free graded resolution (2.20) is defined as $\max_{k,i}(d^i_k - k)$, see, e.g., [18, Ch. 4]. The regularity $\operatorname{reg} X$ of $X \subset \mathbb{P}^N$ is defined as the regularity of the ideal J_X, which is, cf., [18, Exercise 4.3], equal to $\operatorname{reg}(\mathcal{S}/J_X) + 1$; note that $\operatorname{reg} X$ depends on the embedding of X in $\mathbb{P}^N$. If the minimal free resolution of $\mathcal{S}/J_X$ has length $M \leq N$ we conclude that

$$\operatorname{reg} X = \max_{k \leq M}(d^i_k - k) + 1. \tag{2.21}$$

The regularity of X is also equal to the *(Castelnuovo-Mumford) regularity* of the sheaf $\mathcal{I}_X$, see again [18, Exercise 4.3].

3. Singularities of the structure form

In this section we provide a proof of Proposition 2.5. Let $i\colon X \to Y$ be an embedding where Y is projective and smooth of dimension N. Recall that the kth *Fitting ideal (sheaf)* of $\mathcal{O}^Y/\mathcal{J}_X$, $\operatorname{Fitt}_0 g^k$, is the ideal generated by the r_k-minors of (the matrix) g^k in a locally free resolution $\big(\mathcal{O}(E^g_\bullet), g\big)$ of $\mathcal{O}^Y/\mathcal{J}_X$, where r_k is the generic rank of g^k, see, e.g., [17]. It is well-known that these ideals are independent of the resolution $\big(\mathcal{O}(E^g_\bullet), g\big)$; the zero variety of $\operatorname{Fitt}_0 g^k$ is just the BEF-variety Z^{bef}_k, cf., Section 2.4. Since X has pure dimension, $\operatorname{Fitt}_0 g^k$ is trivial when $k \geq N$, cf., (2.17). Let $p = N - n$ be the codimension of X in Y. For $\ell = 1, \dots, n-1$, let $\mathfrak{a}_\ell$ be the pullback (restriction) of $\operatorname{Fitt}_0 g^{p+\ell}$ to X. It follows that these ideals only depend on the embedding $i\colon X \to Y$. We call them the *structure ideals* on X associated with the given embedding.

Given a Hermitian resolution $\big(\mathcal{O}(E^g_\bullet), g\big)$ of $\mathcal{O}^Y/\mathcal{J}_X$, let σ_k be the pointwise minimal inverse of g^k. If (after resolution of singularities) $\operatorname{Fitt}_0 g^k$ is principal, generated by the holomorphic section s, Lemma 2.1 in [8] asserts that $s\sigma_k$ is smooth. Thus $i^*\sigma_{p+k} =: \sigma^k$ is well-defined and semi-meromorphic on X.

Lemma 3.1. *Assume that $\mathfrak{a}_\ell$ and $\mathfrak{a}'_\ell$ are the structure ideals associated with the embeddings $i : X \to Y$ and $i' : X \to Y'$, respectively. Then for each $\ell \geq 1$,*

$$\mathfrak{a}_\ell \cdots \mathfrak{a}_{n-1} \subset \mathfrak{a}'_\ell. \tag{3.1}$$

Since the zero set of $\mathfrak{a}_{k+1}$ is contained in the zero set of $\mathfrak{a}_k$ it follows that the zero set of $\mathfrak{a}_\ell$, which is X^ℓ as defined in Section 2.6, coincides with the zero set of $\mathfrak{a}'_\ell$. It follows that X^ℓ is independent of the embedding.

Proof. Given $i\colon X \to Y$ and a point $x \in X$ there is a neighborhood $\mathcal{V} \subset X$ such that the restriction to $\mathcal{V}$ of i factorizes as

$$\mathcal{V} \xrightarrow{j} \widehat{\Omega} \xrightarrow{\iota} \widehat{\Omega} \times \mathbb{B}_M =: \Omega, \tag{3.2}$$

where j is a minimal (and therefore basically unique) embedding at x, $\mathbb{B}_M \subset \mathbb{C}^M_w$ is a ball centered at 0, ι is the trivial embedding $z \mapsto (z, 0)$ if z are coordinates in $\widehat{\Omega}$, and Ω is a neighborhood of x in Y. Let now $\big(\mathcal{O}(E^{\hat{g}}_\bullet), \hat{g}\big)$ be a minimal Hermitian resolution of $\mathcal{O}^{\widehat{\Omega}}/\mathcal{J}_{\mathcal{V}}$ at x in $\widehat{\Omega}$ and assume that $\hat{p}$ is the codimension of $\mathcal{V}$ in $\widehat{\Omega}$. Thus $p = \hat{p} + M$, where as before p is the codimension of X in Y.

Let (E^w, δ_w) be the Koszul complex generated by $w = (w_1, \dots, w_M)$, cf., Example 2.1. The sheaf complex associated with the product complex $E^{\hat{g}} \otimes E^w$ with mappings $g = \hat{g}(z) + \delta_w$, cf., Section 2.5, provides a (minimal) resolution of $\mathcal{O}^\Omega/\mathcal{J}_X$ in Ω, see [4, Remark 8]. Notice that $g^{p+\ell}$ is the mapping

$$(E^{\hat{g}}_{\hat{p}+\ell} \otimes E^w_M) \oplus (E^{\hat{g}}_{\hat{p}+\ell+1} \otimes E^w_{M-1}) \oplus \cdots \oplus (E^{\hat{g}}_{\hat{p}+\ell+M} \otimes E^w_0) \xrightarrow{\hat{g}(z)+\delta_w}$$
$$(E^{\hat{g}}_{\hat{p}+\ell-1} \otimes E^w_M) \oplus (E^{\hat{g}}_{\hat{p}+\ell} \otimes E^w_{M-1}) \oplus \cdots \oplus (E^{\hat{g}}_{\hat{p}+\ell+M-1} \otimes E^w_0).$$

Since $w = 0$ on X, the restriction of $g^{p+\ell}$ to X splits into the direct sum of the separate mappings

$$\hat{g}^{\hat{p}+\ell+j}\colon E^{\hat{g}}_{\hat{p}+\ell+j} \otimes E^w_{M-j} \to E^{\hat{g}}_{\hat{p}+\ell+j-1} \otimes E^w_{M-j}, \quad j = 0, 1, \dots, M.$$

Since the optimal rank $r_{p+\ell}$ of $g^{p+\ell}$ is attained at every point on X_{reg}, it follows that $r_{p+\ell} = \hat{r}_{\hat{p}+\ell} + \hat{r}_{\hat{p}+\ell+1} + \cdots + \hat{r}_{\hat{p}+M}$, where $\hat{r}_k$ is the generic rank of $\hat{g}^k$. Therefore, the restriction to X of $\mathrm{Fitt}_0 g^{p+\ell}$ is equal to (the restriction to X of) the product ideal

$$\mathrm{Fitt}_0 \hat{g}^{\hat{p}+\ell} \cdot \mathrm{Fitt}_0 \hat{g}^{\hat{p}+\ell+1} \cdots \mathrm{Fitt}_0 \hat{g}^{\hat{p}+\ell+M}.$$

Since X has pure dimension, $\mathrm{Fitt}_0 \hat{g}^k$ is trivial for $k \geq \hat{p} + n = \dim \widehat{\Omega}$, and thus if $\hat{\mathfrak{a}}_\ell$ are the structure ideals associated with $j : \mathcal{V} \to \widehat{\Omega}$,

$$\mathfrak{a}_\ell = \hat{\mathfrak{a}}_\ell \cdots \hat{\mathfrak{a}}_{\min(n-1, \ell+M)}. \tag{3.3}$$

Hence

$$\hat{\mathfrak{a}}_\ell \cdots \hat{\mathfrak{a}}_{n-1} \subset \mathfrak{a}_\ell \subset \hat{\mathfrak{a}}_\ell. \tag{3.4}$$

By the same argument, since i' factorizes as $\mathcal{V} \xrightarrow{j} \widehat{\Omega} \xrightarrow{\iota'} \widehat{\Omega} \times \mathbb{B}_{M'}$, at least if $\mathcal{V}$ is small enough, $\mathfrak{a}'_\ell = \hat{\mathfrak{a}}_\ell \cdots \hat{\mathfrak{a}}_{\min(n-1, \ell+M')}$, and so (3.4) holds at x for $\mathfrak{a}'_\ell$ instead of $\mathfrak{a}_\ell$. Combining we see that (3.1) holds in a neighborhood of x. Since $x \in X$ is arbitrary, the inclusion holds globally on X. □

Lemma 3.2. *There is a smooth modification $\tau\colon \widetilde{X} \to X$ and a holomorphic section η_0 of a line bundle $S_0 \to \widetilde{X}$, which is nonvanishing in $\widetilde{X} \setminus \tau^{-1} X_{sing}$, with the following properties: If $i\colon X \to Y$ is an embedding,* $\dim Y = N$, *$p = N - n$, and $\big(\mathcal{O}(E^g_\bullet), g\big)$ is a Hermitian locally free resolution of $\mathcal{O}^Y/\mathcal{J}_X$, then:*

(i) all the ideals $\tau^*\mathfrak{a}_\ell$, $\ell = 1, \dots, n-1$, *are principal,*
(ii) the subbundles $\operatorname{Im}\tau^* i^* g^{p+\ell} \subset \tau^* i^* E_{p+\ell-1}$, $\ell = 1, \dots, n-1$, *a priori defined over* $\widetilde{X} \setminus \tau^{-1} X^\ell$, *all have holomorphic extensions to* $\widetilde{X}$,
(iii) if $\omega = \omega_0 + \cdots + \omega_n$ *is the induced structure form, then* $\eta_0 \tau^* \omega_0$ *is smooth.*

For the proof we will need the following, probably well-known, result.

Lemma 3.3. *Let* E, Q *be holomorphic vector bundles over* X *and let* $g : E \to Q$ *be a holomorphic morphism. Let* $Z \subset X$ *be the analytic set where* g *does not have optimal rank. There is a (smooth) modification* $\pi : \widetilde{X} \to X$ *such that the subbundle* $\pi^* \operatorname{Im} g \subset \pi^* Q$, *a priori defined in* $\widetilde{X} \setminus \pi^{-1} Z$, *has a holomorphic extension to* $\widetilde{X}$.

Proof. Let $G \colon Q \to F$ be a morphism such that F is a direct sum of line bundles, say $S_1, \dots, S_r$, and

$$\mathcal{O}(F^*) \xrightarrow{G^*} \mathcal{O}(Q^*) \xrightarrow{g^*} \mathcal{O}(E^*)$$

is exact, cf., [6, Proposition 3.3]; we write $G = (G_1, \dots, G_r)$, where $G_j : Q \to S_j$. It then follows that

$$E \xrightarrow{g} Q \xrightarrow{G} F \tag{3.5}$$

is pointwise exact in $X \setminus Z$. Therefore,

$$\operatorname{Im} g = \operatorname{Ker} G = \cap_j \operatorname{Ker}(Q \overset{G_j}{\to} S_j)$$

on $X \setminus Z$.

To prove that $\operatorname{Ker} G$ has a holomorphic extension, let us first assume that F has rank 1, so that G defines an ideal sheaf $\mathcal{J}_G \subset \mathcal{O}^X$. Also, let us assume that X is connected; if not we just consider each connected component separately. If G is identically zero we define $K := Q$. Otherwise let $\pi : \widetilde{X} \to X$ be the blow-up of X along $\mathcal{J}_G$, let D be the corresponding divisor, and let $\mathcal{O}(-D)$ be the line bundle defined by D. Then (the pullback to $\widetilde{X}$ of) G is of the form $G^0 G'$, where G' is a nonvanishing mapping $Q \to F \otimes \mathcal{O}(-D)$ and $G^0 : F \otimes \mathcal{O}(-D) \to F$ is generically invertible. Thus $K := \operatorname{Ker} G'$ is a holomorphic subbundle of Q, and it generically coincides with $\pi^* \operatorname{Ker} G$.

For the general case, we proceed by induction: We let K_1 be an extension of $\operatorname{Ker} G_1$ as above. Then we let $K_2 \subset K_1$ be an extension of the kernel of $G_2|_{K_1} \colon K_1 \to S_2$. Proceeding in this way we find subbundles $K_r \subset \cdots \subset K_1 \subset Q$, such that K_j generically coincides with $\operatorname{Ker} G_1 \cap \cdots \cap \operatorname{Ker} G_j$ on X. In particular K_r coincides with $\operatorname{Im} g$ generically on X, and so we have found a holomorphic extension of $\operatorname{Im} g$. □

Proof of Lemma 3.2. Let us first fix an embedding $i \colon X \to Y$ and a Hermitian locally free resolution $\big(\mathcal{O}(E^g_\bullet), g\big)$, and show that there are $\tau \colon \widetilde{X} \to X$ and η_0 such that $(i) - (iii)$ hold. To begin with, by resolution of singularities, we can find a smooth modification $\hat{\tau} \colon \widehat{X} \to X$ such that all $\hat{\tau}^* \mathfrak{a}_\ell$ are principal, so that (i) holds. Next, by repeated use of Lemma 3.3 we can find a modification $\tau : \widetilde{X} \to \widehat{X}$ so that the subbundles $\operatorname{Im} \tau^* \hat{\tau}^* i^* g^{p+\ell}$ have holomorphic extensions.

Let us now consider (iii). According to Proposition 3.3 in [6], ω_0 is of the form

$$\omega_0 = \sigma_G h,$$

where h is holomorphic in the Barlet-Henkin-Passare sense, i.e., $\bar\partial h = 0$ on X, G is a holomorphic map from E_p to a vector bundle F, and $\sigma_G \colon F \to E_p$ is the inverse of G in $X \setminus X^1$ with pointwise minimal norm, vanishing on the orthogonal complement

of $\operatorname{Im} G$. After further resolving we may assume that τ is chosen so that also (the pullback of) the ideal $\mathfrak{a}_G$ is principal in $\widetilde{X}$, say, generated by the section s_G. Then, by [8, Lemma 2.1], $s_G\sigma_G$ is smooth, cf., the text preceding Lemma 3.1. Since h is meromorphic, there is a section η_0 of a line bundle $S_0 \to \widetilde{X}$ such that $\eta_0\tau^*\omega_0$ is smooth. We may also assume that $\tau^{-1}X_{\text{sing}}$ is a divisor, so that τ^*h is meromorphic with poles contained in $\tau^{-1}X_{\text{sing}}$. Since the variety of $\mathfrak{a}_G$ is contained in X_{sing} it follows that we can choose η_0 to be nonvanishing in $\widetilde{X} \setminus \tau^{-1}X_{\text{sing}}$.

We will prove that with the choice of $\tau : \widetilde{X} \to X$ and η_0 above, $(i) - (iii)$, in fact, hold for any choice of embedding and Hermitian resolution. We first keep the embedding $i\colon X \to Y$ and vary the Hermitian resolution. Pick a Hermitian locally free resolution $\big(\mathcal{O}(E_\bullet^{\tilde g}), \tilde g\big)$ of $\mathcal{O}^Y/\mathcal{J}_X$, fix a point $x \in X$, and choose a minimal Hermitian locally free resolution resolution $\big(\mathcal{O}(E_\bullet^{g'}), g'\big)$ at x.

Claim 1: *$(i) - (iii)$ hold for $\big(\mathcal{O}(E_\bullet^{\tilde g}), \tilde g\big)$ at x if and only if they hold for $\big(\mathcal{O}(E_\bullet^{g'}), g'\big)$ at x.*

Since the choices of $\big(\mathcal{O}(E_\bullet^{\tilde g}), \tilde g\big)$ and x are arbitrary, it follows that $(i) - (iii)$ hold for any Hermitian resolution since they hold for $\big(\mathcal{O}(E_\bullet^{g}), g\big)$.

Proof of Claim 1. It is well-known that $(E_\bullet^{g'}, g')$ is a direct summand in $(E_\bullet^{\tilde g}, \tilde g)$, i.e., there is a decomposition $(E_\bullet^{\tilde g} = E_\bullet^{g'} \oplus E_\bullet^{g''}, \tilde g = g' \oplus g'')$, where the complex $(E_\bullet^{g''}, g'')$ is pointwise exact, see, e.g., [17, Theorem 20.2]. Since $\operatorname{Im} \tilde g^{p+\ell} = \operatorname{Im} (g')^{p+\ell} \oplus \operatorname{Im} (g'')^{p+\ell}$ it follows that (the pullback to $\widetilde{X}$ of) $\operatorname{Im} \tilde g^{p+\ell}$ has a holomorphic extension if and only if $\operatorname{Im} (g')^{p+\ell}$ has one, for $\ell \geq 1$, i.e., (ii) holds at x for one of the Hermitian resolutions $\big(\mathcal{O}(E_\bullet^{\tilde g}), \tilde g\big)$ and $\big(\mathcal{O}(E_\bullet^{g'}), g'\big)$ if and only if it holds for the other one. From this decomposition it also follows immediately that $\operatorname{Fitt}_0 \tilde g^{p+\ell} = \operatorname{Fitt}_0 (g')^{p+\ell}$, so that the structure ideals $\mathfrak{a}_\ell$ are independent of the Hermitian resolution, cf. the beginning of this section. In particular, (i) holds for one of the resolutions if and only if it holds for the other one.

Let $\tilde\omega$ and ω' be the structure forms associated with $\big(\mathcal{O}(E_\bullet^{\tilde g}), \tilde g\big)$ and $\big(\mathcal{O}(E_\bullet^{g'}), g'\big)$, respectively. Then ω' can be considered as a structure form associated with $\big(\mathcal{O}(E_\bullet^{\tilde g}), \tilde g\big)$ but with a Hermitian metric that respects the direct sum, cf., [8, Section 4] and (2.16). Moreover $\tilde\omega_0 = \pi\omega_0'$, where π is the orthogonal projection onto the orthogonal complement (with respect to the first metric) of $\operatorname{Im} \tilde g^{p+1}$ in $E_p|_X$ over $X \setminus X^1$, and $\omega_0' = \pi'\tilde\omega_0$, where π' is the orthogonal projection onto the orthogonal complement (with respect to the second metric) of $\operatorname{Im} (g')^{p+1}$ in $E_p|_X$ over $X \setminus X^1$, cf., the proof of Theorem 4.4 in [8]. If (ii) holds for (at least one of) the resolutions, then $\tau^*\pi$ and $\tau^*\pi'$ are smooth and it follows that $\eta_0\tau^*\omega_0$ is smooth if and only if $\eta_0\tau^*\omega_0'$ is, i.e., (iii) holds for one of the resolutions if and only if it holds for the other one.

□

Next, we will vary the embedding of X. Pick an embedding $i^\sharp : X \to Y^\sharp$ and $x \in X$. Then, in a neighborhood $\mathcal{V}$ of x, $i^\sharp$ factorizes as (3.2), where now Ω is a neighborhood of x in $Y^\sharp$.

Claim 2: *$(i) - (iii)$ hold for Hermitian resolutions of $\mathcal{O}^\Omega/\mathcal{J}_X$ at x if and only if they hold for Hermitian resolutions of $\mathcal{O}^{\widehat\Omega}/\mathcal{J}_X$ at x.*

Since $i^\sharp$ and x are arbitrary and all embeddings of X factor through the minimal embedding j in a small neighborhood of x it then follows that $(i) - (iii)$ hold for any embedding of X.

Proof of Claim 2. Let $\big(\mathcal{O}(E^{\hat g}_\bullet), \hat g\big)$ be a Hermitian minimal resolution of $\mathcal{O}^{\widehat\Omega}/\mathcal{J}_X$. Then, using the notation from the proof of Lemma 3.1, $\big(\mathcal{O}(E^{\check g}) := \mathcal{O}(E^{\hat g}) \otimes E^w, \hat g + \delta_w =: \check g\big)$ is a (minimal) resolution of $\mathcal{O}^\Omega/\mathcal{J}_X$. From Claim 1 we know that it suffices to show that $(i)-(iii)$ hold for $\big(\mathcal{O}(E^{\hat g}_\bullet), \hat g\big)$ if and only if they hold for $\big(\mathcal{O}(E^{\check g}_\bullet), \check g\big)$.

The residue current associated to $\big(\mathcal{O}(E^{\check g}_\bullet), \check g\big)$ is equal to $R^{\hat g(z)} \wedge R^w$, see [4, Remark 4.6]. Since a product of local ideals is principal if and only each of its factors is principal it follows from (3.3) that $\tau^*\check{\mathfrak{a}}_\ell$ are principal for $\ell = 1, \dots, n-1$ if and only if $\tau^*\hat{\mathfrak{a}}_\ell$ are principal for $\ell = 1, \dots, n-1$, where $\check{\mathfrak{a}}_\ell$ denotes the structure ideal associated with $i^\sharp$, i.e., (i) holds for $\big(\mathcal{O}(E^{\hat g}_\bullet), \hat g\big)$ if and only if it holds for $\big(\mathcal{O}(E^{\check g}_\bullet), \check g\big)$. Moreover, since the restriction of $\check g^{p+\ell}$ to X is a direct sum of restrictions of $\hat g^{\hat p+\ell+j}$, cf., the proof of Lemma 3.1, it follows that (the pull-back to $\widetilde X$ of) $\operatorname{Im}\check g^{p+\ell}$, $\ell \geq 1$, have holomorphic extensions if and only if $\operatorname{Im}\hat g^{\hat p+\ell}$, $\ell \geq 1$, have, so that (ii) holds for one of the resolutions $\big(\mathcal{O}(E^{\hat g}_\bullet), \hat g\big)$ and $\big(\mathcal{O}(E^{\check g}_\bullet), \check g\big)$ if and only if it holds for the other one.

Since w are just the coordinate functions in $\mathbb{C}^M$, the Poincaré-Lelong formula asserts that

$$R^w_M \wedge dw_1 \wedge \dots \wedge dw_M = (2\pi i)^M [w=0],$$

where $[w=0]$ is the current of integration over the affine set $\{w=0\}$. Let $\widehat N = \dim \widehat\Omega$, and let $\widehat\omega$ denote the structure form in $\mathcal{V}$ associated with $R^{\hat g(z)}$, so that $j_*\widehat\omega = R^{\hat g} \wedge dz_1 \wedge \dots \wedge dz_{\widehat N}$. Then,

$$i_*\widehat\omega = \iota_* R^{\hat g} \wedge dz_1 \wedge \dots \wedge dz_{\widehat N} = R^{\hat g} \wedge dz_1 \wedge \dots \wedge dz_{\widehat N} \wedge [w=0] \sim \\ R^{\hat g} \wedge R^w \wedge dw_1 \wedge \dots \wedge dw_M \wedge dz_1 \wedge \dots \wedge dz_{\widehat N},$$

where $\sim$ denotes "equal to a nonzero constant times". We conclude, cf., (2.16), that $\widehat\omega$ is also a structure form associated with a Hermitian resolution of $\mathcal{O}^\Omega/\mathcal{J}_X$. From the proof of Claim 1 we know that that if we have two Hermitian resolutions of $\mathcal{O}^\Omega/\mathcal{J}_X$, and that (ii) holds (for at least one of the resolutions), then (iii) holds for one of the resolutions if and only it holds for the other resolution. Thus, provided that (ii) holds, $\eta_0\tau^*\check\omega_0$ is smooth if and only if $\eta_0\tau^*\widehat\omega_0$ is, where $\check\omega_0$ denotes the structure form associated with $\big(\mathcal{O}(E^{\check g}_\bullet), \check g\big)$, i.e., (iii) holds for $\big(\mathcal{O}(E^{\hat g}_\bullet), \hat g\big)$ if and only if it holds for $\big(\mathcal{O}(E^{\check g}_\bullet), \check g\big)$.

□

This concludes the proof of Lemma 3.2: With the choice of $\tau : \widetilde X \to X$ and η_0 made above, $(i)-(iii)$ hold for all embeddings $i : X \to Y$ and all Hermitian resolutions of $\mathcal{O}^Y/\mathcal{J}_X$.

□

We can now conclude the proof of Proposition 2.5. Let $\tau\colon \widetilde X \to X$ and η_0 be as in Lemma 3.2. Fix an embedding $i' : X \to Y'$ and let $s'_1, \dots, s'_{n-1}$ be sections on $\widetilde X$ defining (the pull-back to $\widetilde X$ of) the ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_{n-1}$. Let $\eta_\ell = s'_\ell \cdots s'_{n-1}$, $\ell \geq 1$, and $\eta = \eta_0\eta_1 \cdots \eta_{n-1}$. Note that s'_ℓ is nonvanishing outside $\tau^{-1}X_{\text{sing}}$ so that η is nonvanishing in $\widetilde X \setminus \tau^{-1}X_{\text{sing}}$ if η_0 is. We claim that $\eta\tau^*\omega$ is smooth for any structure form ω on X. To see this, let ω be the structure form associated with an embedding $i : X \to Y$ and a Hermitian locally free resolution $\big(\mathcal{O}(E^g_\bullet), g\big)$. Assume that (the pullbacks of) the corresponding structure ideals are defined by sections $s_1, \dots, s_{n-1}$. Outside X^ℓ, $\omega_\ell = \alpha^\ell\omega_{\ell-1}$, where $\alpha^\ell = \mathbf{1}_{X_{\text{reg}}}\bar\partial\sigma^\ell$, cf., (2.18) and (the

notation in) [8, Section 2]. By [8, Lemma 2.1], $s_\ell \tau^* \sigma^\ell$ is smooth in $\widetilde{X}$. Thus, since ω_ℓ has the SEP, $\eta_0 s_1 \cdots s_\ell \omega_\ell$ is smooth, and so $\eta_0 s_1 \cdots s_{n-1}\omega$ is smooth. By Lemma 3.1, s_ℓ divides η_ℓ and hence the claim follows. This concludes the proof of Proposition 2.5.

Remark 3.4. Let ω' be a structure form on X associated with a given embedding $i' : X \to Y'$. From the proof above, using the notation in the proof, it follows that the section $\eta' := \eta_0 s'_1 \cdots s'_n$ satisfies that $\eta' \tau^* \omega'$ is smooth. If $i' : X \to Y'$ is the fixed embedding in the last part of the proof, then

$$\eta = \eta_0 (s'_1) \cdots (s'_\ell)^\ell \cdots (s'_{n-1})^{n-1} = (s'_2) \cdots (s'_\ell)^{\ell-1} \cdots (s'_{n-1})^{n-2} \eta'.$$

In particular, η divides $(\eta')^{n-1}$.

□

4. Global division problems and residues

In this section we will discuss a method for solving division problems on X using residue theory, which originates from [2]. Throughout the section, (2.4) is a generically exact Hermitian complex over X and ϕ is a global holomorphic section of E_0.

Let us first assume that X is smooth and that $R^f \phi = 0$. As we have seen in Section 2, then $\nabla_f (U^f \phi) = \phi$. If the double complex $\mathcal{M}_{\ell,k} = \mathcal{C}_{0,k}(X, E_\ell)$ is exact in the k-direction except at $k = 0$, then it follows, cf., (2.10), that there is a global holomorphic solution to $f^1 q = \phi$. Let us see more precisely what is needed. Notice that $U^f_{\min(M,n+1)} \phi$ is automatically $\bar\partial$-closed. Since X is smooth, by the Dolbeault isomorphism for currents it is possible to successively solve the equations

$$\bar\partial w_{\min(M,n+1)} = U^f_{\min(M,n+1)} \phi, \quad \bar\partial w_k = U^f_k \phi - f^{k+1} w_{k+1}, \ 1 \le k < \min(M, n+1),$$

if

$$H^{k-1}(X, \mathcal{O}(E_k)) = 0, \quad 1 \le k \le \min(M, n+1). \tag{4.1}$$

Then

$$q := U^f_1 \phi - f^2 w_2$$

is a holomorphic solution to $f^1 q = \phi$. To sum up we have

Proposition 4.1. *Assume that X is smooth and ϕ is a holomorphic section of E_0. If $R^f \phi = 0$ and (4.1) holds, then there is a global holomorphic section q of E_1 such that $f^1 q = \phi$.*

Remark 4.2. The essence in Proposition 4.1 is that the vanishing of $R^f \phi$ not only implies that ϕ belongs to the sheaf $\mathcal{J}_f \otimes E_0$ but is in the image of $\Gamma(X, E_1) \to \Gamma(X, \mathcal{J}_f \otimes E_0)$, provided that (4.1) is fulfilled. In general this map is not surjective even if (4.1) is fulfilled. □

We will now look for analogous results when X is nonsmooth. Since we have no access to a $\bar\partial$-theory for currents on X, we need to embed X in a smooth (projective) manifold. We start by considering a special case that is needed for the proof of Theorem A, namely the case when X is embedded in $\mathbb{P}^N$, $(E^f_\bullet, f)$ is the Koszul complex generated by homogeneous forms f_j of degree d, i.e., global sections of $\mathcal{O}(d) \to \mathbb{P}^N$, and ϕ is a section of $\mathcal{O}(\rho) \to \mathbb{P}^N$. Let $(E^g_\bullet, g)$ be an exact Hermitian complex on $\mathbb{P}^N$ associated to X as in Section 2.9 of length $\le N$. If $R^f \wedge R^g \phi = 0$, then $v = (U^g + U^f \wedge R^g)\phi$ is a global current solution to $\nabla v = \phi$ in $\mathbb{P}^N$, see Section 2.7, and, provided that we can solve a sequence of $\bar\partial$-equations on $\mathbb{P}^N$, we get a global

solution to $f\cdot q+g\cdot q'=\phi$ on $\mathbb{P}^N$, and thus a solution q to $f\cdot q=\phi$ on X. However, see, e.g., [15],

$$H^k(\mathbb{P}^N,\mathcal{O}(\ell))=0 \qquad \text{if} \quad \ell\geq -N \quad \text{or} \quad k<N \tag{4.2}$$

so the only possible obstruction is the equation

$$\bar\partial W = U_{N+1}\phi, \tag{4.3}$$

where $U=U^f\wedge R^g+U^g$. Since $(E^g_\bullet,g)$ ends at level N, $U^g_{N+1}=0$. Moreover, $R^g_k=0$ for $k<N-n$ by the dimension principle, so

$$U_{N+1}=\sum_{k=1}^{\min(m,n+1)} U^f_k\wedge R^g_{N+1-k}, \tag{4.4}$$

cf., Section 2.5. The term corresponding to k takes values in a direct sum of line bundles $\mathcal{O}(-dk-d^i_{N+1-k})$. In view of (4.2), one can solve (4.3) if $\rho\geq dk+d^i_{N+1-k}-N$ for all i and $k=1,2,\dots,\min(m,n+1)$. Notice that, cf., (2.21),

$$dk+d^i_{N+1-k}-N=dk+\big(d^i_{N+1-k}-(N+1-k)\big)+1-k\leq (d-1)k+\operatorname{reg}X.$$

It follows that (4.3) is solvable as soon as

$$\rho\geq (d-1)\min(m,n+1)+\operatorname{reg}X. \tag{4.5}$$

Summing up we have:

Lemma 4.3. *If ρ satisfies* (4.5) *and ϕ is a section of $\mathcal{O}(\rho)$ on $\mathbb{P}^N$ such that $R^f\wedge R^g\phi=0$, then there are global sections q_j of $\mathcal{O}(\rho-d)$ such that $f_1q_1+\dots+f_mq_m=\phi$ on X.*

Remark 4.4. To be more precise, only terms where $N+1-k\leq M$ occur in (4.4), where M is the length of $(E^g_\bullet,g)$. If for instance X is Cohen-Macaulay, i.e., the ring $\mathcal{S}/J_X$ is Cohen-Macaulay, and $(E^g_\bullet,g)$ is of minimal length, then $M=N-n$ so that $k\geq n+1$. If in addition $m\leq n$ thus U_{N+1} vanishes, so there is no cohomological obstruction at all. □

In general it is not possible to find an embedding of X into a smooth manifold Y such that $(E^f_\bullet,f)$ and φ extend holomorphically to Y. For our next result (Theorem 4.6), we will still assume that $(E^f_\bullet,f)$ extends. As a substitute for a holomorphic extension of ϕ we will use a ∇_g*-closed extension* Φ of ϕ to Y. If $i:X\to Y$ is an embedding of X into a projective manifold Y, $\big(\mathcal{O}(E^g_\bullet),g\big)$ is a Hermitian resolution of $\mathcal{O}^Y/\mathcal{J}_X$, and ϕ is a global holomorphic section on X of a line bundle $S\to Y$, then we say that a global smooth section $\Phi=\sum_{\ell\geq 0}\Phi_\ell$ of $\oplus_\ell\mathcal{E}_{0,\ell}(E^g_\ell\otimes S)$ on Y is a ∇_g-closed extension of ϕ if $\nabla_g\Phi=0$ on Y and $i^*\Phi_0=\phi$. Recall that $E^g_0\simeq\mathbb{C}$ is a trivial line bundle.

Lemma 4.5. *(i) Any ϕ admits a ∇_g-closed extension.*
(ii) Φ is a ∇_g-closed extension of ϕ if and only if

$$\Phi-R^g\phi=\nabla_g w \tag{4.6}$$

for some current w.

One can obtain a ∇_g-closed extension Φ of ϕ quite elementarily by piecing together local holomorphic extensions, due to the exactness of $\big(\mathcal{O}(E^g_\bullet),g\big)$. However, we prefer an argument that also relates to residue calculus as in (ii), and we also think that Lemma 4.5 (ii) may be of independent interest.

Proof of Lemma 4.5. As noted in Section 2.7, $R^g\phi$ is a well-defined ∇_g-closed current in Y. In view of Proposition 2.4 there is a smooth ∇_g-closed Φ such that (4.6) holds for some current w. Thus (i) follows from (ii).

Assume that Φ is a smooth extension of ϕ as in (i). From (2.6) we have that $\nabla_g(U^g\wedge\Phi) = \Phi - R^g\wedge\Phi$. Since $\big(\mathcal{O}(E^g_\bullet), g\big)$ is exact, $(R^g)^\ell = 0$ for $\ell \geq 1$, cf., Section 2.4, and hence $R^g\wedge\Phi = R^g\Phi_0 = R^g\phi$, since $\Phi_0 = \phi$ on X, i.e., $i^*\Phi_0 = \phi$ on X. Thus

$$\nabla_g(U^g\wedge\Phi) = \Phi - R^g\phi.$$

Conversely, assume that Φ is smooth and (4.6) holds. Then clearly $\nabla_g\Phi = 0$. We have to prove that $\Phi_0 = \phi$ on X. Notice that this is a local statement. Given a point on X there is a neighborhood $\mathcal{U}$ where we have holomorphic extension $\hat\phi$ of ϕ. Then $\nabla_g(U^g\hat\phi) = \hat\phi - R^g\hat\phi = \hat\phi - R^g\phi$ in $\mathcal{U}$. Thus $\nabla_g(w - U^g\hat\phi) = \Phi - \hat\phi$. By Proposition 2.4 there is a smooth ξ such that $\nabla_g\xi = \Phi - \hat\phi$. It follows that $g^1\xi_1 = \Phi_0 - \hat\phi$ and hence $\Phi_0 = \hat\phi = \phi$ in $\mathcal{U}\cap X$. □

We have the following analogue of Proposition 4.1.

Theorem 4.6. *Let $i\colon X \to Y$ be an embedding of X in a projective manifold Y, let $\big(\mathcal{O}(E^g_\bullet), g\big)$ be a locally free Hermitian resolution of $\mathcal{O}^Y/\mathcal{J}_X$ in Y, and let ω be an associated structure form on X.*

Let (2.4) *be a Hermitian complex over (an open neighborhood $\mathcal{U}$ of X in) Y, and let $R^f\wedge\omega$ be the associated residue current. Moreover let ϕ be a global section of E_0 on X.*

(i) If $R^f\wedge\omega\phi = 0$, then there is a global smooth solution W on X to

$$\nabla_f W = \phi. \tag{4.7}$$

(ii) If (4.7) *has a global smooth solution on X and* (4.1) *holds, then there is a global holomorphic section q of $\mathcal{O}(E_1)$ such that $f^1q = \phi$ on X.*

With minor modifications of the proof below we get the following more general version of Theorem 4.6:

With the general hypotheses of Theorem 4.6, assume that ϕ is a global holomorphic section of E_ℓ such that $f^\ell\phi = 0$.

(i) If $R^\ell\wedge\omega\phi = 0$ then there is a smooth global solution to (4.7).

(ii) If (4.7) *has a smooth solution and*

$$H^{0,k-1-\ell}(X, \mathcal{O}(E_k)) = 0, \quad \ell+1 \leq k \leq \min(M, n+1+\ell),$$

then there is a global holomorphic section q of $E_{\ell+1}$ such that $f^{\ell+1}q = \phi$.

Remark 4.7. If we just have a current solution to $\nabla_f T = \phi$ on X it does not follow that there is a holomorphic solution, not even locally. In fact, if X is non-normal, there are holomorphic f and ϕ such that $\bar\partial(\phi/f) = 0$ but $U = \phi/f$ is not holomorphic. Thus $(f - \bar\partial)U = \phi$ but ϕ is not in the ideal (f). If X is normal but nonsmooth, there are similar examples with more generators, see [25]. □

Proof of Theorem 4.6. Recall from Section 2.7 that $R^f\wedge\omega\phi = 0$ implies that $R^f\wedge R^g\phi = 0$. Let Φ be a ∇_g-closed smooth extension of ϕ, as in Lemma 4.5 (i), to Y. As in the proof of Lemma 4.5, $R^g\wedge\Phi = R^g\Phi_0 = R^g\phi$. It follows that $R^f\wedge R^g\wedge\Phi = R^f\wedge R^g\phi = 0$. Hence, from (2.19) we get, cf., Section 2.7,

$$\nabla_F[(U^f\wedge R^g + U^g)\wedge\Phi] = \Phi.$$

By Proposition 2.4 we have a smooth solution Ψ to $\nabla_F\Psi = \Phi$ in Y; i.e.,

$$F^1\Psi_1 = \Phi_0, \quad F^{k+1}\Psi_{k+1} - \bar\partial\Psi_k = \Phi_k,\ k \geq 1.$$

If we let lower indices (i,j) denote values in $E_i^f \otimes E_j^g$, and notice that $\Phi_k = \Phi_{0,k}$, we see that

$$f^1\Psi_{1,0} + g^1\Psi_{0,1} = \Phi_0, \quad f^{k+1}\Psi_{k+1,0} + g^1\Psi_{k,1} - \bar\partial\Psi_{k,0} = 0,\ k \geq 1. \tag{4.8}$$

Since Ψ is smooth we can define the forms $W_k = i^*\Psi_{k,0}$ on X, and (4.8) then implies that

$$f^1W_1 = \phi, \quad f^{k+1}W_{k+1} - \bar\partial W_k = 0,\ k \geq 1.$$

Thus (i) follows.

The proof of (ii) is similar to the case when X is smooth, cf. the beginning of Section 4: Assume that W is a global smooth solution to (4.7). Then $W_{\min(M,n+1)}$ is automatically $\bar\partial$-closed, and thus if (4.1) is satisfied we can successively solve the equations

$$\bar\partial\eta_{\min(M,n+1)} = W_{\min(M,n+1)}, \quad \bar\partial\eta_k = W_k - f^{k+1}\eta_{k+1},\ 1 \leq k < \min(M, n+1),$$

where η_k is in $\mathcal{A}_k$, see Section 2.8. Then $q := W_1 - f^2\eta_2$ is a holomorphic solution to $f^1q = \phi$.

□

Note that the proof of (ii) above only depends on X and not on the embedding $i : X \to Y$.

It should be possible to express the ∇_F-exactness of Φ in Y by means of Čech cohomology, then make the restriction to X, and rely on the vanishing of the relevant Čech cohomology groups on X. In this way one could avoid the reference to the sheaves $\mathcal{A}_k$ over X.

5. Integral closure, distinguished varieties and residues

Let $f_1, \dots, f_m$ be global holomorphic sections of the ample Hermitian line bundle $L \to X$, and let $\mathcal{J}_f$ be the coherent ideal sheaf they generate. Let

$$\nu\colon X_+ \to X$$

be the normalization of the blow-up of X along $\mathcal{J}_f$, and let $W = \sum r_jW_j$ be the exceptional divisor; here W_j are irreducible Cartier divisors. The images $Z_j := \nu(W_j)$ are called the *(Fulton-MacPherson) distinguished varieties* associated with $\mathcal{J}_f$. If $f = (f_1, \dots, f_m)$ is considered as a section of $E^* := \oplus_1^m L$, then $\nu^*f = f^0f'$, where f^0 is a section of the line bundle $\mathcal{O}(-W)$ defined by W, and $f' = (f_1', \dots, f_m')$ is a nonvanishing section of $\nu^*E^* \otimes \mathcal{O}(W)$, where $\mathcal{O}(W) = \mathcal{O}(-W)^{-1}$. Furthermore, $\omega_f := dd^c \log|f'|^2$ is a smooth first Chern form for $\nu^*L \otimes \mathcal{O}(W)$.

Recall that (a germ of) a holomorphic function ϕ belongs to the *integral closure* $\overline{\mathcal{J}_{f,x}}$ of $\mathcal{J}_{f,x}$ at x if $\nu^*\phi$ vanishes to order (at least) r_j on W_j for all j such that $x \in Z_j$. This holds if and only if $|\nu^*\phi| \leq C|f^0|$ (in a neighborhood of the relevant W_j), which in turn holds if and only if $|\phi| \leq C|f|$ in some neighborhood of x. Let $\overline{\mathcal{J}_f}$ denote the integral closure sheaf. It follows that

$$|\phi| \leq C|f|^\ell \quad \text{if and only if} \quad \phi \in \overline{\mathcal{J}_f^\ell}. \tag{5.1}$$

If X is smooth it follows that ϕ is in the integral closure, if for each j, ϕ vanishes to order r_j at a generic point on Z_j. See [26, Section 10.5] for more details (e.g., the proof of Lemma 10.5.2). We will use the geometric estimate

$$\sum r_j \deg_L Z_j \leq \deg_L X \tag{5.2}$$

from [16], see also [26, (5.20)].

Lemma 5.1. *There is a number μ_0, only depending on X, such that if*

$$|\phi| \leq C|f|^{\mu+\mu_0}, \tag{5.3}$$

then $R^f \wedge \omega\phi = 0$ if ω is a structure form of X and R^f is the residue current obtained from the Koszul complex of f. If X is smooth one can take $\mu_0 = 0$.

This proposition (and its proof) is analogous to Proposition 4.1 in [7]; the important novelty here is that μ_0 can be chosen uniform in ω, which is ensured by Proposition 2.5. However, for the readers convenience and future reference we discuss the proof.

Proof. Let us first assume that X is smooth and $\mu_0 = 0$, and that ϕ satisfies (5.3). Then ω is smooth so we have to show that $R^f\phi = 0$. If $f \equiv 0$ on (a component of) X, then $R^f \equiv 1$ and $\phi \equiv 0$, and thus $R^f\phi = 0$. Let us now assume that $\operatorname{codim} Z^f \geq 1$. Then $R^f_0 = 0$ by the dimension principle. Let $\nu\colon X_+ \to X$ be the normalization of the blow-up along $\mathcal{J}_f$ as above, so that $\nu^* f = f^0 f'$. Using the notation in Example 2.1, then $\nu^*\sigma = (1/f^0)\sigma'$, where $1/f^0$ is a meromorphic section of $\mathcal{O}(W)$ and σ' is a smooth section of $\nu^* E \otimes \mathcal{O}(-W)$. It follows that

$$\nu^*(\sigma \wedge (\bar\partial\sigma)^{k-1}) = \frac{1}{(f^0)^k}\sigma' \wedge (\bar\partial\sigma')^{k-1},$$

and hence

$$\nu^* R^{f,\lambda}_k = \bar\partial |f^0 f'|^{2\lambda} \wedge \frac{1}{(f^0)^k}\sigma' \wedge (\bar\partial\sigma')^{k-1},$$

when $k \geq 1$. Since f' is nonvanishing, the value at $\lambda = 0$ is precisely, see, e.g., [1, Lemma 2.1],

$$R^+_k := \bar\partial \frac{1}{(f^0)^k} \wedge \sigma' \wedge (\bar\partial\sigma')^{k-1}. \tag{5.4}$$

Notice that

$$\nu_* R^+_k = R^f_k. \tag{5.5}$$

Since ϕ satisfies (5.3) for $\mu_0 = 0$, $|\nu^*\phi| \leq C|f^0|^\mu$ and, since X_+ is normal it follows that $\nu^*\phi$ contains a factor $(f^0)^\mu$. Therefore,

$$\nu^*\phi\bar\partial\frac{1}{(f^0)^k} = 0, \quad k \leq \mu, \tag{5.6}$$

because of (2.1). Moreover, since $\sigma' \wedge (\bar\partial\sigma')^{k-1}$ is smooth on X_+, it follows from (5.6) and (5.4) that $R^+_k \nu^*\phi = 0$. Therefore, cf., (5.5), $R^f_k\phi = \nu_*(R^+_k\nu^*\phi) = 0$.

Notice that we could have used any normal modification $\pi\colon \widetilde{X} \to X$ such that $\pi^* f$ is of the form $f^0 f'$ in the proof so far.

Now consider a general X. Let us take a smooth modification $\tau\colon \widetilde{X} \to X$ as in Proposition 2.5, so that, for each structure form ω on X, $\tau^*\omega$ is semi-meromorphic with a denominator that divides the section η, and so that η is locally a monomial in suitable coordinates s_j.

Let ω be a structure form on X. In this proof it is convenient to use the regularization

$$R^f = \lim_{\epsilon\to 0} R^{f,\epsilon}, \quad R^{f,\epsilon} := 1 - \chi(|f|^2/\epsilon) + \bar\partial\chi(|f|^2/\epsilon)\wedge u,$$

where u is the form (2.8) and χ is a smooth approximand of the characteristic function of $[1,\infty)$, cf., the beginning of Section 2, so that all the approximands $R^{f,\epsilon}$ are smooth. If $f\equiv 0$ on a component $\widetilde X_j$ of $\widetilde X$, then $R^{f,\epsilon}\equiv 1$ on $\widetilde X_j$ and if ϕ satisfies (5.3) for any μ_0, then $\phi\equiv 0$ on $\widetilde X_j$; here we have suppressed the notation τ^* for simplicity. Hence $\mathbf{1}_{\widetilde X_j} R^{f,\epsilon}\wedge\omega\phi = 0$ and so $\mathbf{1}_{\widetilde X_j} R^{f}\wedge\omega\phi = 0$. We can therefore assume that $f\not\equiv 0$ on $\widetilde X$. Thus the action of $R^{f,\epsilon}\wedge\omega\phi$ on a test form is, via a partition of unity, a sum of integrals like

$$\int_{\widetilde X} \frac{ds_1\wedge\cdots\wedge ds_n}{s_1^{\alpha_1+1}\cdots s_n^{\alpha_n+1}}\wedge R^{f,\epsilon}\phi\wedge\xi,$$

where α_j are nonnegative integers and ξ is a smooth form. Following [7, Section 3] one can integrate by parts $|\alpha| := |\alpha_1| + \cdots + |\alpha_n|$ times, and get a constant times

$$\int_{\widetilde X} \frac{ds_1\wedge\cdots\wedge ds_n}{s_1\cdots s_n}\wedge\partial_s^\alpha\big(R^{f,\epsilon}\phi\wedge\xi\big), \tag{5.7}$$

where $\partial_s^\alpha = \partial^{|\alpha|}/\partial s_1^{\alpha_1}\cdots\partial s_n^{\alpha_n}$.

Let ut consider $\partial_s^\ell(R^{f,\epsilon}\phi)$. Assume that the metric on L is locally given so that $|f_j|^2 = f_j\bar f_j a$, where a is nonvanishing. Then

$$\sigma = \frac{\sum \bar f_j a e_j}{|f|^2},$$

cf. (2.7), and so

$$\frac{\partial}{\partial s_k}\sigma = \frac{\sum \bar f_j \frac{\partial a}{\partial s_k} e_j}{|f|^2} - \frac{\big(\sum \bar f_j a e_j\big)\big(\sum \bar f_j a \frac{\partial f_j}{\partial s_k}\big)}{|f|^4} = \frac{1}{a}\frac{\partial a}{\partial s_k}\sigma - \big(\frac{\partial f}{\partial s_k}\cdot\sigma\big)\sigma,$$

i.e., $\partial\sigma/\partial s_k$ is of the form

$$\frac{\partial}{\partial s_k}\sigma = (\gamma\cdot\sigma)\sigma, \tag{5.8}$$

where γ is smooth. By iterated used of (5.8), since $\sigma\wedge\sigma = 0$, we get that

$$\partial_s^\kappa(\sigma\wedge(\bar\partial\sigma)^{k-1}) = \sigma\wedge(\bar\partial\sigma)^{k-1}\wedge(\gamma_1\cdot\sigma)\wedge\cdots\wedge(\gamma_{|\kappa|}\cdot\sigma), \tag{5.9}$$

where γ_j are smooth. If we take a smooth modification $\pi : \widehat X \to \widetilde X$ such that $\pi^* f = f^0 f'$ as above, then $\pi^*\sigma = smooth/f^0$ and thus $\pi^*(\partial_s^\kappa u)$ is like $1/(f^0)^{\mu+|\kappa|}$. Moreover $\partial_s^\kappa\bar\partial\chi(|f|^2/\epsilon)$ is like $1/|f|^{|\kappa|+1}$ and with support where $|f|^2\sim\epsilon$, see [7]. Thus $\partial_s^\kappa R^{f,\epsilon}$ is like $1/|f|^{\mu+|\kappa|+1}$ and with support where $|f|^2\sim\epsilon$; here we have suppressed π^* for simplicity. Next, assume that $\mu_0\geq\mu+|\alpha|+1$ and that ϕ satisfies (5.3). Then by the smooth Briançon-Skoda theorem, locally in $\widetilde X$, ϕ is in the ideal $(f)^{\mu+|\alpha|+1}$, and therefore,

$$|\partial_s^\kappa\phi| \leq C|f|^{\mu+|\alpha|-|\kappa|+1}.$$

Hence $\partial_s^\ell(R^{f,\epsilon}\phi)$ is bounded and with support where $|f|^2\sim\epsilon$ for $|\ell|\leq|\alpha|$. It follows by dominated convergence that (5.7) tends to zero when $\epsilon\to 0$, cf. [7, Section 4].

We finally choose μ_0 so that $\mu_0\geq n+|\alpha|+1$ for all local representations $\eta = s_1^{\alpha_1+1}\cdots s_n^{\alpha_n+1}$. Then $R^f\wedge\omega\phi = 0$ for all choices of f if ϕ satisfies (5.3).

□

Note that the explicitness of μ_0 in the proof above is directly related to the explicitness of the modification $\tau : \widetilde{X} \to X$. See [31] for a complete description of the optimal μ_0 in the case of plane curves.

6. Proofs of Theorem A and variations

Throughout this section we will use the notation from Theorem A. For the proof of Theorem A, besides the basic Lemma 5.1, we also need

Lemma 6.1. *Assume that $V \subset \mathbb{C}^N$ is smooth, and let ω be a structure form on X. Then there is a number μ' such that $z_0^{\mu'}\omega$ is almost smooth on X.*

Proof. Let $\tau : \widetilde{X} \to X$ be as in Proposition 2.5. Then $\widetilde{\omega} := \tau^*\omega$ is a semi-meromorphic form whose denominator locally is a monomial whose zeros are contained in $\tau^{-1}X_{\text{sing}}$. Since V is smooth, $X_{\text{sing}} \subset X_\infty \subset \{z_0 = 0\}$, and it follows that $\tau^*(z_0^{\mu'})\tau^*\omega$ is smooth for some large enough number μ'. Hence $z_0^{\mu'}\omega$ is almost smooth. □

Proof of Theorem A. Let f_j be the d-homogenizations of F_j, let R^f be the residue current constructed from the Koszul complex $(E^f_\bullet, \delta_f)$ generated by $f_1, \dots, f_m$, and let ϕ be the ρ-homogenization of Φ, with

$$\rho = \max(\deg \Phi + (\mu + \mu_0)d^{c_\infty} \deg X, (d-1)\min(m, n+1) + \operatorname{reg} X), \tag{6.1}$$

where μ_0 is chosen as in Lemma 5.1; in particular, $\mu_0 = 0$ if X is smooth. Note that μ_0 only depends on X and not on the embedding $i : X \to \mathbb{P}^N$. Throughout this proof we will use the notation from Section 5.

The assumption (1.4) implies that $\nu^*\phi$ vanishes to order $(\mu + \mu_0)r_j$ on each W_j such that $\nu(W_j)$ is not contained in X_∞. Now consider W_j such that $\nu(W_j) \subset X_\infty$. If Ω is a first Chern form for $\mathcal{O}(1)|_X$, e.g., $\Omega = dd^c \log |z|^2$, then $d\Omega$ is a first Chern form for $L = \mathcal{O}(d)|_X$ on X (notice that d denotes the degree and not the differential). By (5.2) we therefore have that

$$r_j \int_{Z_j} (d\Omega)^{\dim Z_j} \le \int_X (d\Omega)^n,$$

which implies that

$$r_j \le d^{\operatorname{codim} Z_j} \deg X. \tag{6.2}$$

By the choice (6.1) of ρ, ϕ is of the form $z_0^{(\mu+\mu_0)d^{c_\infty}\deg X}$ times a holomorphic section, and thus $\nu^*\phi$ vanishes to order at least $(\mu + \mu_0)r_j$ on W_j for each j. Hence (5.3) holds, cf., (5.1), and it follows from Lemma 5.1 that $R^f \wedge \omega\phi = 0$.

Since $\rho \ge (d-1)\min(m, n+1) + \operatorname{reg} X$ it follows from Lemma 4.3 that we have a global q such that $f \cdot q = \phi$ on X. After dehomogenization we get a tuple of polynomials Q_j such that (1.1) holds and $\deg F_j Q_j \le \rho$. Thus part (i) of Theorem A is proved.

For the second part choose

$$\rho = \max(\deg \Phi + \mu d^{c_\infty} \deg X + \mu', (d-1)\min(m, n+1) + \operatorname{reg} X),$$

where μ' is chosen as in Lemma 6.1, and let ϕ and ϕ' be the ρ- and $(\deg \Phi + \mu d^{c_\infty} \deg X)$-homogenizations of Φ, respectively. Then, by Lemma 6.1,

$$R^f \wedge \omega\phi = R^f \wedge \beta\phi',$$

where β is almost smooth, and by (1.6) and (6.2),

$$|\phi'| \le C|f|^{\mu}. \tag{6.3}$$

Now take a smooth modification $\pi\colon \widetilde{X} \to X$ such that $\beta = \pi_*\tilde{\beta}$, where $\tilde{\beta}$ is smooth, and $f = f^0 f'$, where f^0 is a section of a line bundle and f' is nonvanishing. Then $R^f \wedge \omega\phi$ is the push-forward under π of a finite sum of currents like

$$(\pi^*\phi')\bar\partial\frac{1}{(f^0)^{\mu}}\wedge smooth,$$

cf., (5.4), (5.5), and in view of (6.3) they must vanish. Thus $R^f \wedge \omega\phi = 0$ and (ii) is proved as (i). If X is smooth even at infinity, then ω is smooth on X so that we can choose $\mu' = 0$ in Lemma 6.1.

□

The statement in Remark 1.4 follows as in the proof above, using Remark 4.4.

Remark 6.2. An alternative way of finding polynomials Q_j such that (1.1) holds would be to first solve the division problem $f \cdot q = \phi$ on X by means of Theorem 4.6 and then extend the solution to $\mathbb{P}^N$. This was indeed done in an earlier version of this paper, see [10, Theorem 1.1]. The degree estimate so obtained coincides with (1.5), except that the last entry in the max is slightly different; in [10, Section 6], however, we show that it is bounded by $d\min(m, n+1) + \operatorname{reg} X - 1$. Thus, expressed in $\operatorname{reg} X$ the estimate in [10] is somewhat less sharp than (1.5). Note that in [10] we used the non-standard convention that $\operatorname{reg} X$ is $\operatorname{reg} \mathcal{S}/J_X$ instead of $\operatorname{reg} J_X$, cf. Section 2.9.

□

Remark 6.3. If

$$\operatorname{codim}(Z^f \cap X^{\ell}) \ge \mu + \ell + 1, \quad \ell \ge 0, \tag{6.4}$$

where X^{ℓ} are as in Section 2.6, thus either $X_{\text{sing}} \cap Z^f = \emptyset$ or $m < n$, then one can find polynomials Q_j such that (1.1) holds and (1.5) holds with $\mu_0 = 0$. To see this, take $\rho \ge \deg \Phi + \mu d^{c_\infty} \deg X$ in the proof of Theorem A. Then $R^f\phi = 0$ on X_{reg}, and thus $R^f \wedge \omega\phi$ has support on $Z^f \cap X^0$. Since $R^f \wedge \omega_0\phi$ has bidegree at most (n, μ) and $\operatorname{codim}(Z^f \cap X^0) \ge \mu + 1$ by (6.4), it follows from the dimension principle that $R^f \wedge \omega_0\phi = 0$. Thus $R^f \wedge \omega_1\phi = R^f \wedge \alpha^1\omega_0\phi$ vanishes outside X^1, so again by (6.4) and the dimension principle we find that $R^f \wedge \omega_1\phi$ vanishes identically. By induction, $R^f \wedge \omega\phi = 0$. □

Example 6.4. In light of the following example due to Masser, Philippon, Brownawell, and Kollár, see [14, page 578] or [24, Example 2.3], one can see that the power c_∞ in Theorem A cannot be improved: Let $X = \mathbb{P}^n$ and let m be an integer with $2 \le m \le n$. Consider the m polynomials

$$z_1^d,\ z_1 z_m^{d-1} - z_2^d, \dots, z_{m-2} z_m^{d-1} - z_{m-1}^d,\ z_{m-1} z_m^{d-1} - 1,$$

in $\mathbb{C}^n$. The associated projective variety $\{z_0 = z_1 = \cdots = z_{m-1} = 0\} \subset X_\infty$ has codimension m, and hence $c_\infty = m$, cf., (1.3). It follows from Theorem A that we have a representation (1.1) with $\Phi = 1$ and $\deg F_j Q_j \le md^m$ (if d is not too small). However, if Q_j are any polynomials so that (1.1) holds with $\Phi = 1$, then by considering the curve

$$t \mapsto (t^{d^{m-1}-1}, t^{d^{m-2}-1}, \dots, t^{d-1}, 1/t, 0, \dots, 0),$$

one can conclude that Q_1 must have degree at least $d^m - d$ so that $\deg F_1 Q_1 \ge d^m$. □

Remark 6.5. In the proof above μ_0 is derived from the section η in Proposition 2.5. Since we have a fixed embedding $X \to \mathbb{P}^N$ we can get a slighly sharper constant μ_0'. In fact, if ω' is an associated structure form we can replace η in the proof by a section η' such that $\eta'\tau^*\omega'$ is smooth, cf., Remark 3.4. If A' is the highest degree of the (in local coordinates) monomial $\tau^*\eta'$ then $\mu_0' = 1 + A'$. If A is the maximal degree of $\tau^*\eta$ then $A \leq (n-1)A'$. It follows that $\mu_0 \leq (n-1)\mu_0'$ so we can however gain at most a factor $n-1$ by considering the special embedding. □

In [3] is used a slight generalization of the Koszul complex to deal with a positive power $\mathcal{J}_f^\ell$ of $\mathcal{J}_f$, cf. [16, p. 439]; this complex is a special case of the Eagon-Northcott complex, see, e.g., [17, Appendix 2.6]. The first mapping in the complex is the natural mapping $E^{\otimes \ell} \to \mathbb{C}$ induced by the f_j. The associated residue current is the push-forward of currents like

$$\bar\partial \frac{1}{(f^0)^k} \wedge smooth$$

for $\ell \leq k \leq \mu + \ell - 1$. By an analogous proof we get the following generalization of Theorem A.

Theorem 6.6. *With the notation in Theorem A, if*

$$|\Phi|/|F|^{\mu+\mu_0+\ell-1} \text{ is locally bounded on } V,$$

then $\Phi \in (F_j)^\ell$ *and there are polynomials* Q_I *such that*

$$\Phi = \sum_{I_1+\cdots+I_m=\ell} F_1^{I_1}\cdots F_m^{I_m} Q_I$$

and

$$deg\,(F_1^{I_1}\cdots F_m^{I_m} Q_I) \leq$$
$$\max\big(deg\,\Phi + (\mu+\mu_0+\ell-1)d^{c_\infty}\,deg\,X, d(\min(m,n+1)+\ell-1) - \min(m,n+1) + \mathrm{reg}\,X\big).$$

There is also an analogous generalization of part (ii) of Theorem A.

7. Proofs of Theorem B and variations

We first look at the case when X is smooth, which is due to Ein-Lazarsfeld [16].

Theorem 7.1. *Let X be a smooth projective variety, let $L \to X$ be an ample Hermitian line bundle, and let $A \to X$ be a line bundle that is either ample or big and nef. Moreover, let $f_1, \dots, f_m$ be global holomorphic sections of L, and let ϕ be a global section of*

$$L^{\otimes s} \otimes K_X \otimes A,$$

where $s \geq \min(m, n+1)$. If

$$|\phi| \leq C|f|^\mu \tag{7.1}$$

on X, then there are holomorphic sections q_j of $L^{\otimes(s-1)} \otimes K_X \otimes A$ such that

$$f_1 q_1 + \cdots + f_m q_m = \phi. \tag{7.2}$$

Let $\mathcal{J}_f$ be the ideal sheaf generated by f_j and assume that the associated distinguished varieties Z_k have multiplicities r_k, cf., Section 5. If ϕ vanishes to (at least) order $r_k\mu$ at a generic point on Z_k for each k, then (7.1) holds, cf., Section 5, and thus we have

Corollary 7.2. *If ϕ vanishes to order $r_k\mu$ at a generic point on Z_k, for each k, then we have a representation* (7.2).

This corollary is precisely part (iii) of the main theorem in [16, p. 430], except for that we have μr_k rather than $(n+1)r_k$, cf., the discussion in Example 1.6. Using (5.2) one gets the estimate $r_k \leq \deg_L X$.

Proof of Theorem 7.1. Let $(E^f_\bullet, \delta_f)$ be the Koszul complex generated by $f_1, \dots, f_m$, as in Example 2.1, tensorized with $L^{\otimes s} \otimes A \otimes K_X$, and let R^f be the associated residue current on X. From the hypothesis (7.1) and Lemma 5.1 we conclude that $R^f\phi = 0$. The bundle E_k is a direct sum of line bundles $L^{\otimes(s-k)} \otimes A \otimes K_X$ and so all the relevant cohomology groups (4.1) vanish by Kodaira's vanishing theorem, or, at the top degree, by the Kawamata-Viehweg vanishing theorem if A is nef and big. Thus Theorem 7.1 follows from Proposition 4.1. □

Proof of Theorem B. Let $(E^f_\bullet, \delta_f)$ be the Koszul complex generated by $f_1, \dots, f_m$ tensorized with $L^{\otimes s}$, see Example 2.1. The choice of s guarantees that (4.1) is satisfied and thus by Theorem 4.6 (ii) we get the desired holomorphic solution to (1.11) as soon as we have a smooth solution to

$$\nabla_f W = \phi \tag{7.3}$$

on X. Indeed, recall that Theorem 4.6 (ii) only depends on X and not on the embedding $i : X \to Y$. Hence to prove the theorem it suffices to show that there is a μ_0 such that we can find a smooth solution to (7.3) for each global section ϕ of $L^{\otimes s}$ that satisfies (1.10). As in the proof of Theorem A the strategy will be to show that ϕ annihilates a certain residue current, which gives a smooth solution to (7.3). Note that we cannot apply Theorem 4.6 (i), since a priori L and the sections f_j are only defined on X.

Let us start by giving an overview of the proof below. First, there is an embedding of X into a smooth manifold Y so that L extends to Y. We cannot assume that f extends holomorphically to Y but in view of Lemma 4.5, if $\big(\mathcal{O}(E^h_\bullet), h\big)$ is a Hermitian resolution of $\mathcal{O}^Y/\mathcal{J}_X$, then there is a ∇_h-closed extension $\tilde f$. Given $\tilde f$ we construct a Koszul complex $(E^H_\bullet \otimes \Lambda^\bullet E, \delta_{\tilde f})$ that extends $(E^f_\bullet, \delta_f)$, and following the ideas in Example 2.1 we construct a residue current $\widetilde R \wedge R^g$, where $\big(\mathcal{O}(E^g_\bullet), g\big)$ is again a resolution of $\mathcal{O}^Y/\mathcal{J}_X$. From the construction it follows that if

$$\widetilde R \wedge R^g \phi = 0, \tag{7.4}$$

then there is a current solution to

$$\nabla W = \Phi, \tag{7.5}$$

where $\nabla = g + \delta_{\tilde f} + h - \bar\partial$. From such a solution we obtain a smooth solution to (7.5), which in turn implies that there is a smooth solution to (7.3). Finally we show that there is a μ_0, only depending on X, such that (7.4), and thus (7.3), holds as soon as ϕ satisfies (1.10).

We first discuss the extension of L. If M is large enough, there are embeddings $i_j \colon X \to \mathbb{P}^{N_j}$, $j = 1, 2$, such that $\mathcal{O}(1)_{\mathbb{P}^{N_1}}|_X = L^M$ and $\mathcal{O}(1)_{\mathbb{P}^{N_2}}|_X = L^{M+1}$. If $\pi_j \colon \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \to \mathbb{P}^{N_j}$, then $\mathcal{L} := \pi_2^*\mathcal{O}(1)_{\mathbb{P}^{N_2}} \otimes \pi_1^*\mathcal{O}(-1)_{\mathbb{P}^{N_1}}$ is a line bundle over $Y := \mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$ and its restriction to $X \simeq \Delta_{X\times X} \subset Y$ is precisely L. This argument was communicated to us by R. Lazarsfeld.

Let $\big(\mathcal{O}(E^h_\bullet), h\big)$ be a Hermitian resolution of $\mathcal{O}^Y/\mathcal{J}_X$ in Y as in Section 2.6. In view of Lemma 4.5, we can choose smooth ∇_h-closed extensions $\tilde f_j \in \oplus_i \mathcal{E}_{0,i}(E^h_i \otimes \mathcal{L})$ of f_j to Y, as defined in Section 4. Let $E^1, \dots, E^m$ be (extensions to Y of) the trivial line bundles used to define $(E^f_\bullet, \delta_f)$ as in Example 2.1, with basis elements $e_1, \dots, e_m$, respectively, and let $\tilde f$ be the section $\tilde f := \tilde f_j e^*_j$ of $E^h_\bullet \otimes E^*$, where $E := \bigoplus_{j=1}^m \mathcal{L}^{-1} \otimes E^j$ and e^*_j are the dual basis elements. Note that each $\tilde f_j$ has even degree so that $\tilde f$ has odd degree.

We next want to construct a Koszul complex of $\tilde f$ as an extension of $(E^f_\bullet, \delta_f)$. To this end we will need to take products of sections of $E^h_\bullet$. We therefore introduce $E^H_\bullet := \bigcup_{k\geq 1} (E^h_\bullet)^{\otimes k}$, where the tensor products $(E^h_\bullet)^{\otimes k}$ are as in Section 2.5. Since E^h_0 is the trivial line bundle, $(E^h_\bullet)^{\otimes k}$ is a natural subcomplex of $(E^h_\bullet)^{\otimes (k+1)}$ and thus the definition makes sense. Next consider the tensor product complexes $E^H_\bullet \otimes \Lambda^k E$, see Section 2.5 and let $\delta_{\tilde f} : E^H_\bullet \otimes \Lambda^k E \to E^H_\bullet \otimes \Lambda^{k-1} E$ be contraction with $\tilde f$, i.e., for a section $\xi = \sum_{I=\{i_1,\dots,i_k\}} \xi_I \cdot e_I$, where ξ_I takes values in $E^H_\bullet$ and $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$, of $E^H_\bullet \otimes \Lambda^k E$, $\delta_{\tilde f}\xi = \sum_I (-1)^{\deg \xi_I} \xi_I \sum_j (-1)^{j-1} \tilde f_{i_j} \cdot e_{I\setminus i_j}$. Note that $\delta_{\tilde f}$ is an anti-derivation. As long as we restrict to X we can write $\tilde f = f - f'$, where $f := \sum f_j e^*_j$ and f' has positive form degree. Let δ_f and $\delta_{f'}$ be defined analogously to $\delta_{\tilde f}$; note that, restricted to X, δ_f is just the differential in the regular Koszul complex $(E^f_\bullet, \delta_f)$.

Inspired by Example 2.1 we now construct the residue current $\widetilde R \wedge R^g$. We start by defining an $(E^H_\bullet \otimes \Lambda^\bullet E)$-valued form $\tilde u$ which will play the role of u; in fact $\tilde u$ will take values in $(E^h_\bullet)^{\otimes n} \otimes \Lambda^k E$. First, let σ be the section of E over $X \setminus Z$ of pointwise minimal norm such that $\delta_f \sigma = 1$ there, cf. Example 2.1. Then

$$\delta_{\tilde f}\sigma = \delta_f \sigma - \delta_{f'}\sigma = 1 - \delta_{f'}\sigma$$

on $X \setminus Z$. Notice that $\delta_{f'}\sigma$ has even degree, and form bidegree at least $(0,1)$, so that

$$\frac{1}{1-\delta_{f'}\sigma} = 1 + \delta_{f'}\sigma + (\delta_{f'}\sigma)^2 + \cdots + (\delta_{f'}\sigma)^n$$

is a form on $X \setminus Z$ with values in $E^H_\bullet \otimes \Lambda^\bullet E$. Let $\tilde\sigma := \sigma/(1-\delta_{f'}\sigma)$ on $X \setminus Z$; then $\delta_{\tilde f}\tilde\sigma = 1$ on $X \setminus Z$. Next, let

$$\tilde u = \frac{\tilde\sigma}{(\delta_{\tilde f} + \nabla_h)\tilde\sigma} = \sum_{k\geq 1} \tilde\sigma \wedge (-\nabla_h \tilde\sigma)^{k-1},$$

cf., Example 2.1 and [1]; now ∇_h plays the role of $-\bar\partial$. Note that $\delta_{\tilde f}$ anti-commutes with (the extension to $E^H_\bullet \otimes \Lambda^\bullet E$ of) ∇_h, i.e., $\delta_{\tilde f} \circ \nabla_h = -\nabla_h \circ \delta_{\tilde f}$. It follows that $(\delta_{\tilde f} + \nabla_h)^2 = 0$ and so

$$(\delta_{\tilde f} + \nabla_h)\tilde u = 1$$

on $X \setminus Z$, cf. Section 2.2.

Let $\big(\mathcal{O}(E^g_\bullet), g\big)$ be a Hermitian resolution of $\mathcal{O}^Y/\mathcal{J}_X$ in Y, let R^g be the residue current associated with the resolution $\big(\mathcal{O}(E^g_\bullet), g\big)$, and let ω be the associated structure form. Recall from Section 2.6 that if α is a (sufficiently) smooth form on X, then $\alpha \wedge R^g$ is a well-defined current in Y; in particular, $\chi(|f|^2/\epsilon)\tilde u \wedge R^g$ is a well-defined current in Y with values in $E^H_\bullet \otimes \Lambda^\bullet E \otimes E^g_\bullet$. Letting

$$\nabla = g + \delta_{\tilde f} + \nabla_h = g + \delta_{\tilde f} + h - \bar\partial, \tag{7.6}$$

note that $\nabla^2 = 0$ and that

$$\nabla\big(\chi(|f|^2/\epsilon)\tilde u\wedge R^g + U^g\big) = I - \tilde R^\epsilon\wedge R^g, \tag{7.7}$$

where $\tilde R^\epsilon = I - \chi(|f|^2/\epsilon)I + \bar\partial\chi(|f|^2/\epsilon)\wedge\tilde u$.

We claim that $\chi(|f|^2/\epsilon)\tilde u\wedge R^g$ has a limit when $\epsilon\to 0$. To see this, recall from Section 2.6, using the notation from that section, that $\chi(|f|^2/\epsilon)\tilde u\wedge R^g\wedge\Omega = i_*(\chi(|f|^2/\epsilon)\tilde u\wedge\omega)$. Next, notice that

$$\tilde\sigma\wedge(-\nabla_h\tilde\sigma)^{k-1} = \sigma\wedge(\bar\partial\sigma)^{k-1}\wedge\sum_{j=0}^n c_j^k(\delta_{f'}\sigma)^j, \tag{7.8}$$

for some numbers c_j^k, since $\sigma\wedge\sigma = 0$ and σ has degree 0 in $E_\bullet^h$. Let $\pi:\widetilde X\to X$ be a smooth modification such that $\pi^*\omega$ is semi-meromorphic and $\pi^*\sigma$ is of the form σ'/f^0, cf. Section 5. Then $\pi^*\tilde u$ is a finite sum of terms $\gamma_k/(f^0)^k$, where γ_k are smooth, and hence $\lim_{\epsilon\to 0}\pi^*(\chi(|f|^2/\epsilon)\wedge\tilde u\wedge\omega)$ exists, see, e.g., [12]. Since Ω is nonvanishing it follows that the limit of $\chi(|f|^2/\epsilon)\tilde u\wedge R^g$ exists.

Let

$$\tilde U\wedge R^g = \lim_{\epsilon\to 0}\chi(|f|^2/\epsilon)\tilde u\wedge R^g,\quad \tilde R\wedge R^g = \lim_{\epsilon\to 0}\tilde R^\epsilon\wedge R^g.$$

Then by (7.7),

$$\nabla(\tilde U\wedge R^g + U^g) = I - \tilde R\wedge R^g,$$

and if Φ is a smooth ∇_g-closed extension of ϕ as in Lemma 4.5 (regarded as a section of $\mathcal{L}^{\otimes s}\otimes E_\bullet^H\otimes\Lambda^\bullet E\otimes E_\bullet^g$), it follows that

$$\nabla\big((\tilde U\wedge R^g + U^g)\wedge\Phi\big) = \Phi \tag{7.9}$$

in Y as soon as (7.4) is satisfied, since, as was noted in the proof of Lemma 4.5, $R^g\wedge\Phi = R^g\phi$.

By a slight modification of Proposition 2.4, if we have a current solution to (7.5) we also have a smooth solution. To see this, let $E_\bullet^F = \Lambda^\bullet E\otimes E_\bullet^g$ and let $\mathcal{M}_\bullet$ and $\mathcal{M}_\bullet^{\mathcal{E}}$ be defined as in Section 2.3, but for the complex $E_\bullet^H$ instead of $E_\bullet^f$. Then we have the double complex

$$\mathcal{B}_{\ell,k} := \oplus_j\mathcal{C}_{0,j}(E_{j+k}^H\otimes E_\ell^F) = \mathcal{M}_k(E_\ell^F)$$

with mappings $\nabla_h\colon\mathcal{B}_{\ell,k}\to\mathcal{B}_{\ell,k-1}$ and $F := g + \delta_{\tilde f}\colon\mathcal{B}_{\ell,k}\to\mathcal{B}_{\ell-1,k}$; indeed note that $\nabla_h\circ F = -F\circ\nabla_h$. If $\mathcal{B}_j := \bigoplus_{\ell+k=j}\mathcal{B}_{\ell,k}$ we get the associated total complex

$$\dots\xrightarrow{\nabla}\mathcal{B}_j\xrightarrow{\nabla}\mathcal{B}_{j-1}\xrightarrow{\nabla}\dots,$$

with ∇ as in (7.6). Analogously let $\mathcal{B}_{\ell,k}^{\mathcal{E}} := \oplus_j\mathcal{E}_{0,j}(E_{j+k}^H\otimes E_\ell^F) = \mathcal{M}_k^{\mathcal{E}}(E_\ell^F)$ with total complex $\mathcal{B}_\bullet^{\mathcal{E}}$. Moreover, let $\mathcal{M}_\bullet(Y,E_\ell^F)$, $\mathcal{M}_\bullet^{\mathcal{E}}(Y,E_\ell^F)$, $\mathcal{B}_\bullet(Y)$, and $\mathcal{B}_\bullet^{\mathcal{E}}(Y)$ be the associated complexes of global sections. Note that we have natural mappings

$$H^j(\mathcal{B}_\bullet^{\mathcal{E}}(Y))\to H^j(\mathcal{B}_\bullet(Y)),\quad j\in\mathbb{Z}. \tag{7.10}$$

Proposition 2.4 implies that the natural mappings $H^k(\mathcal{M}_\bullet^{\mathcal{E}}(Y,E_\ell^F))\to H^k(\mathcal{M}_\bullet(Y,E_\ell^F))$ are isomorphisms. Now, by repeating the proof of Proposition 2.4 with $\mathcal{M}_\bullet$, $\mathcal{M}_\bullet^{\mathcal{E}}$, $\mathcal{C}_{0,\bullet}$, and $\mathcal{E}_{0,\bullet}$ replaced by $\mathcal{B}_\bullet$, $\mathcal{B}_\bullet^{\mathcal{E}}$, $\mathcal{M}_\bullet$, and $\mathcal{M}_\bullet^{\mathcal{E}}$, respectively, using that the double complex $\mathcal{B}_{\ell,k}$ is bounded in the ℓ-direction, we can therefore prove that the mappings (7.10) are in fact isomorphisms. Hence the current solution (7.9) gives a smooth solution to (7.5).

Next we will show that a smooth solution to (7.5) gives a smooth solution to (7.3). Let lower indices (i,j,k) denote components in $\mathcal{L}^{\otimes s}\otimes E_i^H\otimes\Lambda^j E\otimes E_k^g$. Then

$\Phi = \Phi_{0,0,0} + \Phi_{0,0,1} + \cdots + \Phi_{0,0,n}$, where $\Phi_{0,0,k}$ has form bidegree $(0,k)$. Notice that we have the decomposition $\tilde{f} = f_0 - f'$ in Y, where f_0 denotes the 0-component of $\tilde{f}$ and hence is a smooth extension of f to Y. If Ψ is a smooth solution to (7.5) it follows that

$$h\Psi_{1,0,0} + \delta_{f_0}\Psi_{0,1,0} + g\Psi_{0,0,1} = \Phi_{0,0,0}, \tag{7.11}$$

$$h\Psi_{1,j,0} + \delta_{f_0}\Psi_{0,j+1,0} + g\Psi_{0,j,1} - \bar{\partial}\Psi_{0,j,0} = 0,\ j \geq 1. \tag{7.12}$$

Indeed, note that $\delta_{f'}\Psi_{i,j,k}$ has positive degree in $E^H_\bullet$ for all nonvanishing $\Psi_{i,j,k}$. Since Ψ is smooth, we can define the smooth forms $W_j := i^*\Psi_{0,j,0}$ on X. Note that $\mathcal{L}^{\otimes s} \otimes \Lambda^j E|_X = E^f_j$, so that W_j takes values in E^f_j. Since $g\Psi_{0,j,1} = g^1\Psi_{0,j,1}$ and $h\Psi_{1,j,0} = h^1\Psi_{1,j,0}$ are in $\mathcal{E}(\mathcal{J}_X)$ and thus vanish on X, (7.11) and (7.12) implies

$$\delta_f W_1 = \phi, \quad \delta_f W_{j+1} - \bar{\partial} W_j = 0,\ j \geq 1.$$

To sum up so far, we have shown that there is a smooth solution to (7.3) if (7.4) holds.

Claim: *There is a μ_0, only depending on X, such that* (7.4) *holds as soon as ϕ satisfies* (1.10).

Taking the claim for granted we get that there is a μ_0 such that if ϕ satisfies (1.10), then there is a smooth solution to (7.3); this concludes the proof of Theorem B.

The claim is essentially Lemma 5.1, but now R^f is replaced by the current $\widetilde{R}$. Also the proof is analogous to the proof of the lemma and the choice of μ_0 in the proof of the lemma will do here as well; the crucial observation is that the singularities of $\widetilde{R}$ can be controlled in a similar way to the singularities of R^f.

To prove the claim, first note that (7.4) is equivalent to that $\tilde{R}\wedge R^g \wedge \Omega\phi = \lim_{\epsilon\to 0} i_*(\tilde{R}^\epsilon \wedge \omega\phi)$ vanishes, cf. Section 2.7. Let $\tau : \widetilde{X} \to X$ be a smooth modification as in Proposition 2.5, so that locally $\tau^*\omega = smooth/s^{\alpha+1}$, where $s^{\alpha+1}$ is a local representation of the section η, as in the proof of Lemma 5.1. Following that proof, the action of $\tilde{R}^\epsilon \wedge \omega\phi$ on a test form is a sum of integrals like (suppressing τ^* for simplicity)

$$\int_{\widetilde{X}} \frac{ds_1 \wedge \ldots \wedge ds_n}{s_1 \cdots s_n} \wedge \partial_s^\alpha \big(\tilde{R}^\epsilon \phi \wedge \xi\big), \tag{7.13}$$

where ξ is smooth, cf. (5.7). The components of $\widetilde{X}$ where f vanishes identically are taken care of as in the proof of Lemma 5.1. We may therefore assume that $f \not\equiv 0$.

In view of (7.8), $\widetilde{R}^\epsilon$ is a finite sum of terms like

$$\bar{\partial}\chi(|f|^2/\epsilon) \wedge \sigma \wedge (\bar{\partial}\sigma)^{k-1} \wedge (\delta_{f'}\sigma)^j,$$

where $k + j \leq n$ for degree reasons; indeed, recall that f' has form degree at least $(0,1)$. Note that $\delta_{f'}\sigma$ is of the form $(\gamma \cdot \sigma)$, where γ is smooth. Thus by arguments as in the proof of Lemma 5.1, cf. (5.8) and (5.9), we get that $\partial_s^\ell \tilde{R}^\epsilon$ is like $1/|f|^{n+|\ell|+1}$ and with support where $|f|^2 \sim \epsilon$. As in that proof, we choose μ_0 so that $\mu_0 \geq n + |\alpha| + 1$ for all local representations $\eta = s^{\alpha+1}$. If ϕ satisfies (1.10), then $|\partial_s^\ell \phi| \leq C|f|^{n+|\alpha|-|\ell|+1}$, cf. the proof of Lemma 5.1. Now by dominated convergence (7.13) tends to zero when $\epsilon \to 0$, and since the choice of μ_0 only depends on the section η and n and not on the embedding $i : X \to Y$, the resolutions $\big(\mathcal{O}(E^h_\bullet), h\big)$, $\big(\mathcal{O}(E^g_\bullet), g\big)$ or the extension $\tilde{f}$, the claim follows.

□

Remark 7.3. If $(E^h_\bullet, h)$ is a Koszul complex, then we just simply take $E^H_\bullet = E^h_\bullet$, since he desired "product" already exists within $E^h_\bullet$.

□

Remark 7.4. Assume that $i : X \to Y$ is an embedding such that all ample line bundles on X extend to Y. Following the proof above we then obtain Theorem B without relaying on the quite involved Proposition 2.5, since we can then define the μ_0 from the singularities of one fixed structure form, cf. Remark 6.5. It is of course enough that there is a finite number of embeddings of X into smooth manifolds such that each ample line bundle extends to at least one of them.

□

In analogy with Theorem 6.6 we also have the following generalizations of Theorem 7.1 and Theorem B.

Theorem 7.5. *With the notation in Theorem 7.1, if ϕ is a section of $L^{\otimes s} \otimes K_X \otimes A$, where $s \geq \min(m, n+1) + \ell - 1$, and*

$$|\phi| \leq C|f|^{\mu+\ell-1},$$

then there are holomorphic sections q_I of $L^{\otimes(s-\ell)} \otimes K_X \otimes A$, such that

$$\phi = \sum_{I_1+\cdots+I_m=\ell} f_1^{I_1} \cdots f_m^{I_m} q_I. \tag{7.14}$$

With the notation in Theorem B, if ϕ is a section of $L^{\otimes s}$ with $s \geq \nu_L + \min(m, n+1) + \ell - 1$ such that

$$|\phi| \leq C|f|^{\mu_0+\mu+\ell-1},$$

then there are holomorphic sections q_I of $L^{\otimes(s-\ell)}$ such that (7.14) *holds.*

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