SOME VARIANTS OF MACAULAY'S AND MAX NÖTHER'S THEOREMS

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Abstract. We use residue currents on toric varieties to obtain bounds on the support of solutions to polynomial ideal membership problems. Our bounds depend on the Newton polytope of the polynomial system and are therefore well adjusted to sparse systems of polynomials. We present variants of classical results due to Macaulay and Max Nöther.

Dedicated to Ralf Fröberg on the occasion of his 65th birthday

1. Introduction

Let $F_1, \dots, F_m$, and Φ be polynomials in $\mathbb{C}^n$. Assume that Φ vanishes on the common zero set of the F_j. Then *Hilbert's Nullstellensatz* asserts that there are polynomials $G_1, \dots, G_m$ such that

$$\sum_{j=1}^{m} F_j G_j = \Phi^\nu \tag{1.1}$$

for some integer ν large enough. The following bound of the degrees of the F_j and ν was obtained by Kollár, [19], for $d \neq 2$, and by Jelonek, [18], for $d = 2$ and $m \leq n$:
Assume that $deg\, F_j \leq d$. *Then one can find* G_j *so that* (1.1) *holds for some* $\nu \leq d^{\min(m,n)}$ *and*

$$deg\,(F_j G_j) \leq (1 + deg \Phi) d^{\min(m,n)}; \tag{1.2}$$

for $d = 2$ and $m \geq n + 1$, the best bound is due to Sombra, [26]: the factor $d^{\min(m,n)}$ in (1.2) should then be replaced by 2^{n+1}. Kollár's and Jelonek's bounds are sharp; the original formulations also take into account different degrees of the F_j. In many cases, however, one can do much better. Classical results due to Max Nöther, [23], and Macaulay, [22], show that the bounds can be substantially improved if (the homogenizations of) the F_j have no zeros at infinity. The aim of this note is to use multidimensional residue on toric varieties to obtain some variants of these results.

Multidimensional residues have been used as a tool to solve polynomial ideal membership problems by several authors, see for example [7].

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In [2] Andersson used residue currents on manifolds to obtain effective solutions; in particular, Macaulay's and Max Nöther's results follow by applying his methods to complex projective space.

Recall that the *support* $\operatorname{supp} F$ of a polynomial $F = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha z^\alpha = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ in $\mathbb{C}^n$ is defined as $\operatorname{supp} F = \{\alpha \in \mathbb{Z}^n \text{ such that } c_\alpha \neq 0\}$ and that the *Newton polytope* $\mathcal{NP}(F_1,\dots,F_m)$ of polynomials $F_1,\dots,F_m$ is the convex hull of $\bigcup_j \operatorname{supp} F_j$ in $\mathbb{R}^n$. In particular, a polynomial of degree d has support in $d\Sigma^n$, where Σ^n is the n-dimensional simplex in $\mathbb{R}^n$ with the origin and the unit lattice points $e_1 = (1,0,\dots,0), e_2 = (0,1,0,\dots,0),\dots,e_n = (0,\dots,0,1)$ as vertices.

Using techniques from toric geometry Sombra [26] obtained a sparse effective Nullstellensatz, which improves Kollár's result when the system of polynomials is sparse, meaning that $\mathcal{NP}(F_1,\dots,F_m)$ is small compared to $d\Sigma^n$. In [28] the author used the residue current techniques developed in [2] applied to toric varieties in order to obtain certain sparse effective versions of polynomial ideal membership problems. This note, in which we focus on the case when F_j have no common zeros at infinity, can be seen as an addendum to [28]. We will specify in Section 4 how no common zeros at infinity should be interpreted.

We work on toric varieties associated with the Newton polytopes or the support of the F_j. Given a lattice polytope $\mathcal{P}$, i.e., a polytope in $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$, one can construct a toric variety $X_\mathcal{P}$ and a line bundle $\mathcal{O}(D_\mathcal{P})$ on $X_\mathcal{P}$ whose global sections correspond to polynomials with support in $\mathcal{P}$, see Section 3. The toric variety $X_\mathcal{P}$ is smooth if for each vertex v of $\mathcal{P}$ the smallest integer normal directions of the facets of $\mathcal{P}$ containing v form a base for the $\mathbb{Z}^n$, see [16, p. 29]. We then say that the lattice polytope $\mathcal{P}$ is *smooth* (or *Delzant*) with respect to the lattice $\mathbb{Z}^n$.

The following sparse version of Macaulay's Theorem is due to Castryck-Denef-Vercauteren [10].

Theorem 1.1. *Let $F_1,\dots,F_m$, and Φ be polynomials in $\mathbb{C}^n$. Assume that the F_j have no common zeros even at infinity, and that supp $\Phi \subseteq e\mathcal{NP}(F_1,\dots,F_m)$, where $e\mathcal{NP}(F_1,\dots,F_m)$ is a lattice polytope. Then there are polynomials G_j that satisfy*

$$\sum_{j=1}^m F_j G_j = \Phi \tag{1.3}$$

and

$$\textit{supp}\,(F_jG_j) \subseteq \max(n+1,e)\mathcal{NP}(F_1,\dots,F_m).$$

In particular, one can find polynomials G_j that satisfy

$$\sum_{j=1}^m F_j G_j = 1 \tag{1.4}$$

and

$$\text{supp}\,(F_jG_j) \subseteq (n+1)\mathcal{NP}(F_1,\dots,F_m). \tag{1.5}$$

Macaulay's Theorem, [22], corresponds to the case when $\mathcal{P} = d\Sigma^n$, i.e., $\deg F_j \leq d$. Then (1.5) reads $\deg\,(F_jG_j) \leq (n+1)d$, which is slightly worse that Macaulay's original result:
Assume that F_j have no common zeros even at infinity (in $\mathbb{P}^n$). Then one can find G_j that satisfy (1.4) *and* $\deg\,(F_jG_j) \leq (n+1)d - n$.

Theorem 1.1 can be seen as a special case of the following sparse version of Max Nöther's Theorem, [23]. Let (F) denote the ideal generated by $F_1,\dots,F_m$.

Theorem 1.2. *Let $F_1,\dots,F_m$ be polynomials in $\mathbb{C}^n$ and let $\mathcal{P}$ be a smooth lattice polytope that contains the origin and the support of the F_j and the coordinate functions $z_1,\dots,z_n$. Assume that the F_j have no common zeros at infinity. Then there is a number ν_F, such that if $\Phi \in (F)$ satisfies that $\text{supp}\,\Phi \subseteq e\mathcal{P}$, where $e\mathcal{P}$ is a lattice polytope, then there are polynomials G_j that satisfy* (1.3) *and*

$$\text{supp}\,(F_jG_j) \subseteq \max(\nu_F, e)\mathcal{P}. \tag{1.6}$$

In fact, Theorem 1.2 is a sparse version of a result in the forthcoming paper [6]. As Theorem 1.2 is stated above the common zero set of the F_j has to be discrete. It is, however, possible to replace the assumption that the F_j lack common zeros at infinity by a less restrictive assumption, see Remark 4.2.

The reason that we require $\mathcal{P}$ to be smooth in Theorem 1.2 is that we need a certain line bundle to be ample, see Section 4. For example, $\mathcal{P} = d\Sigma^n$ is smooth; with this choice (1.6) reads $\deg\,(F_jG_j) \leq \max(\nu_F d, \deg\Phi)$.

Theorem 1.2 is a variant of Max Nöther's Theorem, [23], in the sense that Φ is assumed to be in (F) and the F_j are assumed to have no zeros at infinity. In the original formulation, $F_1,\dots,F_m$ are moreover assumed to form a *complete intersection*, i.e., the codimension of $\{F_1 = \ldots = F_m = 0\}$ is m:
Assume that the zero-set of $F_1,\dots,F_n$ is discrete and contained in $\mathbb{C}^n$ and that $\Phi \in (F)$. Then there are G_j that satisfy (1.3) *and* $\deg\,(F_jG_j) \leq \deg\Phi$.
Note that if $\text{supp}\,\Phi$ (or $\deg\Phi$) is large enough, then the bound (1.6) coincides with Max Nöther's bound; indeed ν_F only depends on the F_j. In [28, Theorem 1.2] was presented a sparse versions of Nöther's Theorem, which essentially says, that if the F_j are a complete intersection, then Theorem 1.2 holds with $\nu_F = 0$. To be precise, the polytope $e\mathcal{P}$ has to satisfy an additional condition.

If the F_j lack common zeros, then Theorem 1.1 says that we can choose $\nu_F = n + 1$. In general, we do not have an explicit description of ν_F; see the discussion after the proof of Theorem 1.2.

Recall that the polynomial Φ lies in the *integral closure* of (F) if Φ satisfies a monic equation $\Phi^r + H_1\Phi^{r-1} + \cdots + H_r = 0$, where $H_j \in (F)^j$ for $1 \le j \le r$ or, equivalently, if Φ locally satisfies $|\Phi| \le C|F|$, where $|F|^2 = |F_1|^2 + \cdots + |F_m|^2$. If Φ is in the integral closure of (F), then the Briançon-Skoda Theorem , [9], asserts that one can solve (1.1) with $\nu = \min(m, n)$. Our next result is a sparse effective Briançon-Skoda Theorem, which also can be seen as a generalization of Macaulay's Theorem. Indeed, when the F_j have no common zeros, the assumption below that $\mathcal{P}$ contains the origin is automatically satisfied and then any polynomial Φ is in the integral closure of (F).

Theorem 1.3. *Let $F_1, \dots, F_m$, and Φ be polynomials in $\mathbb{C}^n$ and let $\mathcal{P}$ be a lattice polytope that contains the origin and the support of the F_j. Assume that the F_j have no common zeros at infinity. Moreover assume that Φ is in the integral closure of (F) and that $\text{supp}\,\Phi \subseteq e\mathcal{P}$, where $e\mathcal{P}$ is a lattice polytope. Then there are polynomials G_j that satisfy*

$$\sum_{j=1}^{m} F_j G_j = \Phi^n \tag{1.7}$$

and

$$\text{supp}\,(F_j G_j) \subseteq \max(n+1, ne)\mathcal{P}. \tag{1.8}$$

The assumption that the F_j have no common zeros at infinity could be replaced by a less restrictive assumption, see Remark 4.2. If $\mathcal{P} = d\Sigma^n$, then (1.8) reads $\deg(F_j G_j) \le \max((n+1)d, n\deg\Phi)$.

Morally, Theorems 1.2 and 1.3 say that when the F_j have no zeros at infinity and $\text{supp}\,\Phi$ is large enough compared to $\text{supp}\,F_j$, then the bounds on $\text{supp}\,(F_j G_j)$ in (1.3) and (1.7) are as good as possible; in fact, $\text{supp}\,(F_j G_j)$ is then bounded by $\text{supp}\,\Phi$ and $\text{supp}\,\Phi^n$, respectively. Andersson-Götmark, [3, Thm 1.3], and Hickel [17, Thm 1.1] proved effective Max Nöther's and Briançon-Skoda Theorem's, respectively, in which they allow common zeros at infinity. Then typically terms of size d^n appear, cf. (1.2).

Let us sketch the idea of the proofs of our results. A standard way of reformulating the kind of division problems we consider is the following. There are polynomials G_j that satisfy (1.1) and $\text{supp}\,(F_j G_j) \subseteq c\mathcal{P}$ if and only if there are sections g_j of line bundles $\mathcal{O}(D_{(c-1)\mathcal{P}})$ over $X_{\mathcal{P}}$ such that

$$\sum_{j=1}^{m} f_j g_j = \psi, \tag{1.9}$$

where f_j and ψ are sections of line bundles $\mathcal{O}(D_\mathcal{P})$ and $\mathcal{O}(D_{c\mathcal{P}})$ over $X_\mathcal{P}$ corresponding to F_j and Φ^ν, respectively. Now there is a local solution to (1.9) on $X_\mathcal{P}$ if ψ annihilates a certain residue current, see Section 2. To obtain a global solution to (1.9) the constant c has to be large enough so that certain Dolbeault cohomology on $X_\mathcal{P}$ vanishes. By analyzing when these conditions are satisfied we obtain our results.

The proofs of Theorem 1.1- 1.3 occupy Section 4. In sections 2 and 3 we provide some necessary background on residue currents and toric varieties, respectively.

2. Residue currents

Let $f_1, \ldots, f_m$ be holomorphic functions whose common zero set $V_f = \{f_1 = \ldots = f_m = 0\}$ has codimension m. Then the *Coleff-Herrera product*, introduced in [11],

$$R^f_{CH} = \bar\partial\left[\frac{1}{f_1}\right] \wedge \cdots \wedge \bar\partial\left[\frac{1}{f_m}\right],$$

represents the ideal (f) generated by the f_j in the sense that it has support on V_f and moreover a holomorphic function ψ is in (f) if and only if the current ψR^f_{CH} vanishes, see [13, 24].

When $\operatorname{codim} V_f < m$, there is no such canonical residue current associated with $f_1, \ldots, f_m$. Passare-Tsikh-Yger, [25], constructed residue currents by means of the Bochner-Martinelli kernel that generalize the Coleff-Herrera product to when the codimension of V_f is arbitrary. Their construction was later developed by Andersson, [1], and by Andersson and the author, [4].

Theorem 2.1. *Assume that $E_0, E_1, \ldots, E_N$ are Hermitian holomorphic vector bundles over a complex manifold X of dimension n and assume that E_0 has rank 1. Moreover assume that the complex*

$$0 \longrightarrow E_N \xrightarrow{f^N} \ldots \xrightarrow{f^3} E_2 \xrightarrow{f^2} E_1 \xrightarrow{f^1} E_0, \tag{2.1}$$

is exact outside an analytic set Z of positive codimension. Then one can construct an $\operatorname{End}(\bigoplus_k E_k)$-valued residue current R on X, which has support on Z and satisfies the following:

(a) *If ψ is a holomorphic section of E_0 that annihilates R, i.e., the current $R\psi$ vanishes, then ψ is in the ideal sheaf $\operatorname{Im} f^1$ generated by the image of f^1.*
(b) *If the associated complex of locally free sheaves of $\mathcal{O}$-modules of sections of E_k*

$$0 \longrightarrow \mathcal{O}(E_N) \xrightarrow{f^N} \ldots \xrightarrow{f^2} \mathcal{O}(E_1) \xrightarrow{f^1} \mathcal{O}(E_0) \tag{2.2}$$

is exact, then $\psi \in \operatorname{Im} f^1$ if and only if $R\psi = 0$.

(c) *Assume that f is a holomorphic section of a Hermitian vector bundle E of rank m over X and that* (2.1) *is the Koszul complex of f, i.e., $E_k = \Lambda^k E^*$ and f^k is contraction (interior multiplication) with f. Moreover assume that ψ locally satisfies that*

$$|\psi| \leq C|f|^{\min(m,n)}$$

for some constant C. Then $R\psi = 0$.

The idea of the proof of Theorem 2.1 is that outside Z one can obtain a local holomorphic solution to the division problem

$$f^1 g = \psi \tag{2.3}$$

by means of (2.1); here ψ is a section of E_0 and g a section of E_1. The residue current $R\psi$ appears as an obstruction when one tries to extend the solution from $X \setminus Z$ to X; we refer to [1] and [4] for details.

The explicitness of the current R of course directly depends on the explicitness of (2.1). If (2.1) is the Koszul complex of f, then R has support on the zero locus V_f of f and locally the coefficients of R are the residue currents introduced by Passare-Tsikh-Yger, [25]. In particular, if $\operatorname{codim} V_f = m$, then R is locally a Coleff-Herrera product. Note that in this case $\operatorname{Im} f^1$ is the ideal sheaf $\mathcal{J}(f)$ generated by f.

Morally, the residue current R is the obstruction to solve (2.3) locally. To obtain a global solution one also needs certain $\bar\partial$-cohomology on X to vanish. The construction of the currents in [4] implies the following, cf. [4, Prop. 6.1]:

Theorem 2.2. *Let L be a line bundle over X. Assume that*

$$H^{0,q}(X, L \otimes E_{q+1}) = 0 \tag{2.4}$$

for $1 \leq q \leq \min(N-1, n)$. Let ψ be a holomorphic section of $L \otimes E_0$. If $R\psi = 0$, then there is a global section g of $L \otimes E_1$ that satisfies (2.3).

The current R allows for multiplication with characteristic functions of varieties and more generally constructible sets in such a way that ordinary calculus rules hold, see [5]. In particular, if $V \subseteq X$ is a variety, then $R\psi = 0$ if and only if $\mathbf{1}_V R\psi = 0$ and $\mathbf{1}_{X\setminus V} R\psi = 0$. Moreover R is said to have the *Standard Extension Property (SEP)* in the sense of Björk, [8], if $\mathbf{1}_W R = 0$ for all subvarieties $W \subset V_f$ of positive codimension.

3. Toric varieties from polytopes

For a general reference on toric varieties, see [16]. A toric variety can be constructed from a *fan* Δ, which is a certain collection of $\mathbb{Z}^n$ cones, by gluing together copies of $\mathbb{C}^n$ corresponding to the n-dimensional cones of Δ; we denote the resulting toric variety by X_Δ. Let $\mathcal{P}$ be a lattice polytope in $\mathbb{R}^n$. Then $\mathcal{P}$ determines a fan $\Delta_\mathcal{P}$, the so-called *normal fan* of $\mathcal{P}$, whose rays correspond to the normal directions of the faces of maximal dimension of $\mathcal{P}$. The corresponding toric variety $X_{\mathcal{P}} = X_{\Delta_{\mathcal{P}}}$ is projective, see [15, Section VII.3].

A toric variety X_Δ is smooth if and only if each cone in Δ is generated by a part of a basis for the lattice $\mathbb{Z}^n$. Such a fan is said to be *regular*. The fan $\Delta_{\mathcal{P}}$ is regular precisely when $\mathcal{P}$ is smooth, cf. the introduction. For each fan Δ there exists a refinement $\widetilde{\Delta}$ of Δ such that $X_{\widetilde{\Delta}} \to X_\Delta$ is a resolution of singularities. Also if Δ_1 and Δ_2 are two different fans, there exists a regular fan $\widetilde{\Delta}$ that refines both Δ_1 and Δ_2. If Δ is a refinement of $\Delta_{\mathcal{P}}$ we say that Δ and $\mathcal{P}$ are *compatible*.

Assume that $\mathcal{P}$ is compatible with Δ. Then $\mathcal{P}$ defines a divisor $D_{\mathcal{P}}$ on X_Δ, such that the global holomorphic sections of the line bundle $\mathcal{O}(D_{\mathcal{P}})$ correspond precisely to the polynomials with support in $\mathcal{P}$, see [16, p. 66]. Moreover $\mathcal{O}(D_{\mathcal{P}})$ is generated by its sections, and if $\Delta = \Delta_{\mathcal{P}}$, then $\mathcal{O}(D_{\mathcal{P}})$ is ample, see [16, p. 73]. Also, $\mathcal{O}(D_{\mathcal{P}}) \otimes \mathcal{O}(D_{\mathcal{Q}}) = \mathcal{O}(D_{\mathcal{P}+\mathcal{Q}})$.

If Δ is compatible with a polytope and L is a line bundle over X_Δ that is generated by its sections, then $H^{0,q}(X_\Delta, L) = 0$ for all $q \geq 1$.

In the situation of Theorems 1.2 and 1.3 we want to consider toric varieties that are compactifications of $\mathbb{C}^n$. Assume that Δ contains the first orthant σ_0 as an n-dimensional cone; observe that if $\mathcal{P} \subseteq \mathbb{R}^n_+$ contains the origin, then one can find such a Δ, which is regular and compatible with $\mathcal{P}$. Then we can identify the corresponding affine chart $\mathcal{U}_{\sigma_0}$ with $\mathbb{C}^n$; we refer to the complement $X_\Delta \setminus \mathcal{U}_{\sigma_0}$ as the *variety at infinity* and denote it by V_∞. If $\mathcal{P}$ is compatible with Δ and moreover contains the origin, then in local coordinates in $\mathcal{U}_{\sigma_0} = \mathbb{C}^n$, a section ψ of $\mathcal{O}(D_{\mathcal{P}})$ coincides with the corresponding polynomial Ψ in $\mathbb{C}^n$, so that ψ can really be seen as a homogenization of Ψ, see [12] and also [28, Section 3.4].

4. Proofs

In Theorem 1.1 the F_j are assumed to have no common zeros even at infinity. This should be interpreted as that the corresponding sections f_j of $\mathcal{O}(D_{\mathcal{P}})$ lack common zeros in X_Δ, where Δ is compatible with $\mathcal{P} = \mathcal{NP}(F_1, \ldots, F_m)$. Observe that whether the f_j have common zeros in X_Δ in fact only depends on $\mathcal{P}$ and not on the particular choice of Δ, as long as it is compatible with $\mathcal{P}$. In Theorems 1.2 and 1.3, $\mathcal{P}$ is assumed to contain the origin. It follows that Δ can be chosen compatible with $\mathcal{P}$ so that it contains the first orthant as a cone. The assumption that the f_j lack common zeros at infinity should be interpreted as that, given such a Δ, the corresponding sections of $\mathcal{O}(D_{\mathcal{P}})$ lack common zeros at V_∞ in X_Δ.

Consider polynomials F_j with support in polytopes $\mathcal{P}_j$. Whether or not the F_j, or rather the corresponding sections f_j of line bundles $\mathcal{O}(D_{\mathcal{P}_j})$, have common zeros (at infinity) clearly depends on the polytopes $\mathcal{P}_j$. Assume that f_j are sections of a line bundle $\mathcal{O}(D_{\mathcal{P}})$ over

X_Δ, where Δ is compatible with $\mathcal{P}$. Then the f_j do have common zeros unless $\mathcal{P} = \mathcal{NP}(F_1, \dots, F_m)$ and they have common zeros at infinity unless $\mathcal{P}$ is the convex hull of the Newton polytope and the origin. On the other hand, any generic choice of $n+1$ sections of $\mathcal{O}(D_\mathcal{P})$ will lack common zeros and any choice of n polynomials with support in $\mathcal{P}$ will lack common zeros at V_∞, see for example [28, Section 6.2] or [27, Lma 4.1]. Thus the sparse versions of Macaulay's and Max Nöther's results generalize their classical counterparts in the sense that they apply to more general situations.

Theorem 1.1 is a consequence of the following more general result, which is due to Tuitman [27]; we include a proof for completeness. Recall that the polytope $\mathcal{Q}$ is a *summand* of the polytope $\mathcal{P}$ if there exist another a polytope $\mathcal{S}$ such that $\mathcal{P} = \mathcal{Q} + \mathcal{S}$.

Theorem 4.1. *[Tuitman [27]] Let $F_1, \dots, F_m$, and Φ be polynomials in $\mathbb{C}^n$. Let $\mathcal{P}_j$ and $\mathcal{P}$ be polytopes that contain the support of the F_j and Φ, respectively. Assume that the F_j have no common zeros even at infinity, meaning that the corresponding sections of line bundles $\mathcal{O}(D_{\mathcal{P}_j})$ over a toric variety lack common zeros. Assume that $\mathcal{P}_{j_1} + \dots + \mathcal{P}_{j_q}$ is a summand of $\mathcal{P}$ for all $1 \le q \le \min(m, n+1)$ and $\mathcal{J} = \{j_1, \dots, j_q\} \subseteq \{1, \dots, m\}$. Then there are polynomials that satisfy (1.3) and*

$$\text{supp}\,(F_j G_j) \subseteq \mathcal{P}. \tag{4.1}$$

In particular, we can let $\mathcal{P} = \sum_{j=1}^m \mathcal{P}_j$. Also, if we choose $\mathcal{P}$ as $\max(n+1, e)\mathcal{NP}(F_1, \dots, F_m)$ we get back Theorem 1.1.

Proof. Let Δ be a regular fan that is compatible with $\mathcal{P}_1, \dots, \mathcal{P}_m$, and $\mathcal{P}$, let E be the bundle $\mathcal{O}(D_{\mathcal{P}_1}) \oplus \dots \oplus \mathcal{O}(D_{\mathcal{P}_m})$ over X_Δ, and let L be the line bundle $\mathcal{O}(D_\mathcal{P})$. We identify polynomials with support in $\mathcal{P}_j$ and $\mathcal{P}$ with sections of $\mathcal{O}(D_{\mathcal{P}_j})$ and L, respectively. Accordingly, let f_j, f, and ψ be the sections of $\mathcal{O}(D_{\mathcal{P}_j})$, E, and $\mathcal{O}(D_\mathcal{P})$ corresponding to F_j, the tuple $F_1, \dots, F_m$, and Φ, respectively.

Let (2.1) be the Koszul complex of f and let R be the associated residue current. By assumption, the f_j have no common zeros, and hence $R = 0$.

Now

$$L \otimes E_Q = L \otimes \Lambda^q E^* = \bigoplus_{|\mathcal{J}|=q} \mathcal{O}(D_\mathcal{P} - (D_{\mathcal{P}_{j_1}} + \dots + D_{\mathcal{P}_{j_q}})), \tag{4.2}$$

Since for each term in the right hand side of (4.2), $\mathcal{P}_{j_1} + \dots + \mathcal{P}_{j_q}$ is a summand of $\mathcal{P}$, $\mathcal{O}(D_{\mathcal{P}-(\mathcal{P}_{j_1}+\dots+\mathcal{P}_{j_q})})$ is generated by its sections, see Section 3. Hence (2.4) holds for $1 \le q \le n$, cf. (the proof of) Theorem 4.1 in [28].

Now Theorem 2.2 asserts that we can find a section $g = (g_1, \dots, g_m)$ of $L \otimes E^*$ that satisfies (2.3), and thus polynomials G_j that satisfy (1.3) and (4.1). □

The original proof by Tuitman is very similar to our proof. In fact, the residue current does not really play a role in our proof, since it trivially vanishes.

Theorem 1.3 is proved along the same lines as Theorem 1.1, using residue currents constructed from the Koszul complex. It would be possible to give a more general formulation of Theorem 1.3, that would take into account that the F_j might have different supports, as was done in Theorem 4.1.

Proof of Theorem 1.3. Let Δ be a regular fan that is compatible with $\mathcal{P}$ and that contains the first orthant as cone. Moreover, let E be the vector bundle $\mathcal{O}(D_{\mathcal{P}})^{\oplus m}$ over X_Δ, and let L be the line bundle $\mathcal{O}(D_{\max(n+1,ne)\mathcal{P}})$. Let f_j, f, and ψ be the sections of $\mathcal{O}(D_{\mathcal{P}})$, E, and L corresponding to F_j, the tuple $F_1, \dots, F_m$, and Φ^n, respectively.

Let (2.1) be the Koszul complex of f and let R be the associated residue current. By assumption, the f_j have no common zeros at infinity, and hence $\mathbf{1}_{V_\infty} R = 0$. Moreover, since Φ is in the integral closure of (F) in $\mathbb{C}^n$, $\mathbf{1}_{\mathbb{C}^n} R\psi = 0$ by Theorem 2.1 (c) and the end of Section 3.

By Section 3, $L \otimes E_q = L \otimes \Lambda^q E^*$ is a direct sum of line bundles $\mathcal{O}(D_{(\max(n+1,ne)-q)\mathcal{P}})$, and since $\mathcal{O}(D_{c\mathcal{P}})$ is generated by its sections if $c \geq 0$, by Section 3, (2.4) holds for $1 \leq q \leq n$.

Now Theorem 2.2 asserts that we can find a section $g = (g_1, \dots, g_m)$ of $L \otimes E^*$ that satisfies (2.3), and thus polynomials $G_1, \dots, G_m$ in $\mathbb{C}^n$ that satisfy (1.7) and (1.8). □

Proof of Theorem 1.2. Let E be the vector bundle $\mathcal{O}(D_{\mathcal{P}})^{\oplus m}$ over $X_{\mathcal{P}}$, and let f be the section of E corresponding to $F_1, \dots, F_m$. Let E_0 be the trivial bundle of rank 1 over $X_{\mathcal{P}}$, let $E_1 = E^*$, and let f^1 be multiplication with f. Since $X_{\mathcal{P}}$ is projective, $E_1 \xrightarrow{f^1} E_0$ can be continued to a complex (2.1), such that the associated complex (2.2) is exact, see for example [20, Ex. 1.2.21]. Since, by assumption, $\mathcal{P}$ is smooth, the line bundle $\mathcal{O}(D_{\mathcal{P}})$ over $X_{\mathcal{P}}$ is ample and thus for some large enough number ν_F, $H^{0,q}(X_{\mathcal{P}}, \mathcal{O}(D_{\mathcal{P}})^{\otimes\nu} \otimes E_{q+1}) = 0$ for $1 \leq q \leq \min(N-1, n)$ and $\nu \geq \nu_F$. In particular, $L := \mathcal{O}(D_{\max(\nu_F,e)\mathcal{P}})$ satisfies (2.4) for $1 \leq q \leq \min(N-1, n)$.

The assumption that $\mathcal{P}$ contains the origin and the support of the coordinate functions $z_1, \dots, z_n$ implies that the first orthant in $\mathbb{R}^n$ is a cone of $\Delta_{\mathcal{P}}$. Let R be the residue current associated with (2.1) and let ψ be the section of L corresponding to Φ. By assumption, the f_j have no common zeros at infinity, and hence $\mathbf{1}_{V_\infty} R = 0$. Moreover, since (2.2) is exact and $\Phi \in (F)$ in $\mathbb{C}^n$, $\mathbf{1}_{\mathbb{C}^n} R\psi = 0$ by Theorem 2.1 (b) and the end of Section 3.

Now Theorem 2.2 asserts that we can find a section $g = (g_1, \dots, g_m)$ of $L \otimes E_1 = L \otimes E^*$ that satisfies (2.3), and thus polynomials $G_1, \dots, G_m$ in $\mathbb{C}^n$ that satisfy (1.3) and (1.6). □

The constant ν_F in Theorem 1.2 depends on the degrees of the mappings in the resolution (2.1), which are closely related to the Castelnuovo-Mumford regularity of (F), see [14, Chapter 20.5].

Remark 4.2. Observe that the proofs of Theorems 1.2 and 1.3 only use that R vanishes along V_∞, i.e., $\mathbf{1}_{V_\infty} R = 0$. In fact, this allows us to replace the assumptions that the F_j lack common zeros at infinity by less restrictive assumptions.

Let Z_k be the set where the mapping f^k in (2.1) does not have optimal rank. When (2.2) is exact R admits a decomposition $R = \sum_k \mathbf{1}_{Z_k \setminus Z_{k-1}} R$, where $\mathbf{1}_{Z_k \setminus Z_{k-1}} R$ has support on and the SEP with respect to Z_k, see [5, Ex. 7]. Thus in Theorem 1.2 we could replace the assumption that the F_j lack common zeros at infinity by the assumption that the Z_k have no irreducible components contained in V_∞.

Let $\{V_j\}$ be the set of so-called *distinguished subvarieties* of $\mathcal{J}(f)$, see [21, p. 263], and let R be the residue current constructed from the Koszul complex of f. It follows from the construction that R admits a decomposition $R = \sum \mathbf{1}_{V_j} R$, where $\mathbf{1}_{V_j} R$ has support on and the SEP with respect V_j, see for example [3]. Hence in Theorem 1.3 we could replace the assumption that F_j lack common zeros at infinity by the assumption that $\mathcal{J}(f)$ has no distinguished subvarieties contained in V_∞. □

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