REGULARITY OF PSEUDOMEROMORPHIC CURRENTS

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Abstract. Let X be a complex manifold. We prove that direct images of principal value and residue currents on X are smooth outside sets that are small in a certain sense. We also prove that the sheaf of such currents is a stalkwise injective $\mathcal{O}_X$-module.

1. Introduction

Let f be a generically nonvanishing holomorphic function on a reduced analytic space X of pure dimension n. Herrera and Lieberman, [13], proved that the principal value

$$\lim_{\epsilon\to 0}\int_{|f|^2>\epsilon}\frac{\xi}{f}$$

exists for test forms ξ and defines a current $[1/f]$. It follows that $\bar\partial[1/f]$ is a current with support on the zero set $Z(f)$ of f; such a current is called a residue current. Coleff and Herrera, [10], introduced (non-commutative) products of principal value and residue currents, like

$$[1/f_1]\cdots[1/f_r]\bar\partial[1/f_{r+1}]\wedge\cdots\wedge\bar\partial[1/f_m]. \tag{1.1}$$

The theory of (products of) residue and principal value currents has been further developed by a number of authors since then, see, e.g., the references given in [7].

In order to obtain a coherent approach to questions about residue and principal value currents were introduced in [6, 4] the sheaf $\mathcal{PM}_X$ of *pseudomeromorphic currents* on X, consisting of direct images under holomorphic mappings of products of test forms and currents like (1.1). Pseudomeromorphic currents play a decisive role in several recent papers concerning, e.g., effective division problems and the $\bar\partial$-equation on singular spaces; see [7] for various references.

The objective of this paper is to study regularity properties of pseudomeromorphic currents in the case when X is smooth. To understand the singular support of a pseudomeromorphic current one is lead to study non-proper images of analytic sets. Our first main result Theorem 3.14 states that a pseudomeromorphic current is smooth outside a set that is small in a certain sense.

Our second main result Theorem 5.1 asserts that $\mathcal{PM}_X$ is "ample" in the sense that it is a stalkwise injective $\mathcal{O}_X$-module. The simplest instance of this result is that the equation $f\nu=\mu$ has a pseudomeromorphic solution for any pseudomeromorphic current μ and nontrivial holomorphic function f. In particular this means that, although smooth outside small sets, pseudomeromorphic currents can be quite singular. The analogue of Theorem 5.1 for general currents is a classical result by Malgrange [15].

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Combining Theorem 5.1 with the fact that $\mathcal{PM}_X^{0,\bullet}$ is a fine resolution of $\mathcal{O}_X$, which was noticed already in [4], we obtain a generalization of the classical Dickenstein-Sessa decomposition, [11], in Section 5.3.

The proof of Theorem 5.1 is based on an integral formula and relies heavily on the regularity result Theorem 3.14. Another important ingredient is the fact from [7] that one can "multiply" arbitrary pseudomeromorphic currents by proper direct images of principal value currents.

In Section 2 we recall some basic facts about pseudomeromorphic currents on reduced analytic spaces and in Sections 3 and 4 we prove Theorem 3.14 and some variants. The last two sections are devoted to a discussion and the proof of Theorem 5.1.

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2. Pseudomeromorphic currents

In one complex variable s one can define the principal value current $[1/s^m]$ for instance as the limit

$$\Big[\frac{1}{s^m}\Big] = \lim_{\epsilon\to 0} \chi(|s|/\epsilon)\frac{1}{s^m},$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth function that is equal to 0 in a neighborhood of 0 and 1 in a neighborhood of ∞; we write $\chi \sim \chi_{[1,\infty)}$ to denote such a χ. We have the relations

$$\frac{\partial}{\partial s}\Big[\frac{1}{s^m}\Big] = -m\Big[\frac{1}{s^{m+1}}\Big], \quad s\Big[\frac{1}{s^{m+1}}\Big] = \Big[\frac{1}{s^m}\Big]. \tag{2.1}$$

It is also well-known that

$$\bar\partial\Big[\frac{1}{s^{m+1}}\Big].\xi ds = \frac{2\pi i}{m!}\frac{\partial^m}{\partial s^m}\xi(0). \tag{2.2}$$

for test functions ξ; in particular, $\bar\partial[1/s^{m+1}]$ has support at $\{s=0\}$. It follows from (2.2) that

$$\bar s\bar\partial\Big[\frac{1}{s^{m+1}}\Big] = 0, \quad d\bar s\wedge\bar\partial\Big[\frac{1}{s^{m+1}}\Big] = 0. \tag{2.3}$$

Let t_j be coordinates in an open set $\Omega \subset \mathbb{C}^N$ and let α be a smooth form with compact support in Ω. Then

$$\tau = \alpha\wedge\Big[\frac{1}{t_1^{m_1}}\Big]\cdots\Big[\frac{1}{t_k^{m_k}}\Big]\bar\partial\Big[\frac{1}{t_{k+1}^{m_{k+1}}}\Big]\wedge\ldots\wedge\bar\partial\Big[\frac{1}{t_r^{m_r}}\Big] \tag{2.4}$$

is a well-defined current, since it is the tensor product of one-variable currents (times α). We say that τ is an *elementary pseudomeromorphic current*, and we refer to $[1/t_j^{m_j}]$ and $\bar\partial[1/t_\ell^{m_\ell}]$ as its *principal value factors* and *residue factors*, respectively. It is clear that (2.4) is commuting in the principal value factors and anti-commuting in the residue factors. If $\alpha \not\equiv 0$ we say the the intersection of Ω and the coordinate plane $\{t_{k+1} = \cdots = t_r = 0\}$ is the *elementary support* of τ. Clearly the support of τ is contained in the intersection of the elementary support and the support of α.

Remark 2.1. In view of (2.1), notice that $\partial\tau$ is an elementary current, whose elementary support either equals the elementary support H of τ or is empty. Also $\bar\partial\tau$ is a finite sum of elementary currents, whose elementary supports are either equal to H or coordinate planes of codimension 1 in H, cf. (2.1) and (2.2). $\square$

Throughout this section X is a reduced analytic space of pure dimension n.

2.1. **Definition and basic properties.** If X is smooth we say that a germ μ of a current at $x \in X$ is *pseudomeromorphic* at x, $\mu \in \mathcal{PM}_x$, if it is a finite sum of currents of the form

$$\mu = \varphi_* \tau,$$

where $\varphi\colon \Omega \to \mathcal{U}$ is a holomorphic mapping, $\mathcal{U} \subset X$ is a neighborhood of x, and τ is elementary in $\Omega \subset \mathbb{C}^N$. By definition the union $\mathcal{PM} = \mathcal{PM}_X = \cup_x \mathcal{PM}_x$ is an open subset (of the étalé space) of the sheaf $\mathcal{C} = \mathcal{C}_X$ of currents, and hence it is a subsheaf which we call the sheaf of *pseudomeromorphic* currents. It follows from [7, Theorem 2.15] that this definition is equivalent to the definition given in [4, 7][1]. Thus a section μ of $\mathcal{PM}$ in an open set $\mathcal{V} \subset X$, $\mu \in \mathcal{PM}(\mathcal{V})$, can be written as a locally finite sum

$$\mu = \sum (\varphi_\ell)_* \tau_\ell, \tag{2.5}$$

where each φ_ℓ is holomorphic and each τ_ℓ is elementary. For simplicity we will always suppress the subscript ℓ in φ_ℓ.

If X is a general analytic pure-dimensional space and $\pi\colon Y \to X$ is a modification where Y is smooth, then $\mathcal{PM}_X$ consists of all direct images of currents in $\mathcal{PM}_Y$. It follows from [7, Theorem 2.15] that the sheaf so obtained is independent of the choice of Y. Thus we again have a representation (2.5), where in this case each φ_ℓ is a holomorphic mapping into a complex manifold composed by a modification.

Remark 2.2. Note that each elementary current τ is a finite sum of currents τ_ℓ such that the support of τ_ℓ is contained in an irreducible component of the elementary support of τ. We may therefore assume that each τ_ℓ in (2.5) has irreducible elementary support. □

From [7, Corollary 2.16] we have

Lemma 2.3. *Assume that $\varphi\colon W \to X$ is of the form $\pi \circ \psi$, where $\psi : W \to Y$ is a holomorphic mapping, Y is a complex manifold, and $\pi : Y \to X$ is a modification. If μ is pseudomeromorphic in W with compact support, then $\varphi_*\mu$ is pseudomeromorphic in X.*

In particular if X is smooth, and $\varphi : W \to X$ is any holomorphic mapping, then $\varphi_*\mu$ is pseudomeromorphic in X.

Notice that if ξ is a smooth form, then

$$\xi \wedge \varphi_* \mu = \varphi_*(\varphi^* \xi \wedge \mu). \tag{2.6}$$

Applying (2.6) to the representation (2.5) we see that $\mathcal{PM}_X$ is closed under exterior multiplication by smooth forms, since this is true for elementary currents. For the same reason $\mathcal{PM}_X$ is closed under $\bar\partial$ and ∂, cf. Remark 2.1.

Another important property that is inherited from elementary currents, cf. (2.3), is the fact that

$$\bar h \mu = 0, \quad d\bar h \wedge \mu = 0 \tag{2.7}$$

if h is a holomorphic function that vanishes on the support of the pseudomeromorphic current μ. This means in particular that the action of the current μ only involves holomorphic derivatives of test forms. From (2.7) we get the *dimension principle:*

[1]The definition of pseudomeromorphic currents in [6] was slightly more restrictive.

If μ is pseudomeromorphic of bidegree $(*,p)$ and has support on the analytic variety V, where $\operatorname{codim} V > p$, then $\mu = 0$.

Given an analytic subvariety V of an open subset $\mathcal{U} \subset X$, the natural restriction of a pseudomeromorphic current μ to $\mathcal{U} \setminus V$ has a canonical extension to a pseudomeromorphic current $\mathbf{1}_{X\setminus V}\mu$ in $\mathcal{U}$. The following lemma is Lemma 2.6 in [7]:

Lemma 2.4. *Let V be a subvariety of $\mathcal{U} \subset X$, let h be a holomorphic tuple in $\mathcal{U}$ whose common zero set is precisely V, let v be a smooth and nonvanishing function, and let $\chi \sim \chi_{[1,\infty)}$. For each pseudomeromorphic current μ in $\mathcal{U}$ we have*

$$\mathbf{1}_{\mathcal{U}\setminus V}\mu = \lim_{\epsilon\to 0} \chi(|h|^2 v/\epsilon)\mu.$$

Because of the factor v, the lemma holds just as well for a holomorphic section h of a Hermitian vector bundle.

It follows that

$$\mathbf{1}_V\mu := \mu - \mathbf{1}_{\mathcal{U}\setminus V}\mu$$

has support on V. It is proved in [6] that this operation extends to all constructible sets and that

$$\mathbf{1}_V\mathbf{1}_W\mu = \mathbf{1}_{V\cap W}\mu \tag{2.8}$$

holds. If α is a smooth form, then

$$\mathbf{1}_V(\alpha\wedge\mu) = \alpha\wedge\mathbf{1}_V\mu. \tag{2.9}$$

Moreover, if $\varphi\colon W \to X$ is a holomorphic mapping as in Lemma 2.3 and μ has compact support, then

$$\mathbf{1}_V\varphi_*\mu = \varphi_*\big(\mathbf{1}_{\varphi^{-1}V}\mu\big). \tag{2.10}$$

We will need the following observation, which can proved in the same way as Lemma 2.8 in [7], using (2.10).

Lemma 2.5. *If μ has the form (2.5), then*

$$\mathbf{1}_V\mu = \sum_{\operatorname{supp}\tau_\ell\subset\varphi^{-1}V} \varphi_*\tau_\ell.$$

One can just as well take the sum over all ℓ such that the elementary supports of τ_ℓ are contained in $\varphi^{-1}V$.

For future reference we also include

Lemma 2.6. *If $T \in \mathcal{PM}_X$ and $T' \in \mathcal{PM}_{X'}$, then $T\otimes T' \in \mathcal{PM}_{X\times X'}$.*

See, e.g., [7, Lemma 2.12]. It is easy to verify that

$$\mathbf{1}_{V\times V'}T\otimes T' = \mathbf{1}_V T \otimes \mathbf{1}_{V'}T'. \tag{2.11}$$

2.2. **The sheaves $\mathcal{PM}_X^Z$ and $\mathcal{W}_X^Z$.** Let $Z \subset X$ be a (reduced) subspace of pure dimension, and denote by $\mathcal{PM}_X^Z$ the subsheaf of $\mathcal{PM}_X$ of currents that have support on Z. We say that $\mu \in \mathcal{PM}_X^Z$ has the *standard extension property*, SEP, on Z if $\mathbf{1}_W\mu = 0$ for each subvariety $W \subset \mathcal{U}\cap Z$ of positive codimension, where $\mathcal{U}$ is any open set in X. Let $\mathcal{W}_X^Z$ be the subsheaf of $\mathcal{PM}_X^Z$ of currents with the SEP on Z. If $Z = X$ we omit the superscript and write $\mathcal{W}_X$.

Example 2.7. An elementary current in $\Omega \subset \mathbb{C}^N$ with elementary support H is in $\mathcal{W}_\Omega^H$. □

We will need the following lemma.

Lemma 2.8. *Let X and Y be analytic spaces and let p be a point in Y.*
(i) If $\pi\colon X\times Y\to X$ is the natural projection and $\mu\in\mathcal{W}^{X\times\{p\}}_{X\times Y}$, then $\pi_*\mu\in\mathcal{W}_X$.
(ii) If μ is in $\mathcal{W}_X$ and $\nu\in\mathcal{PM}_Y$ has support at p, then $\mu\otimes\nu$ is in $\mathcal{W}^{X\times\{p\}}_{X\times Y}$.

Proof. Let W be a subvarity of $\mathcal{U}\subset X$ of positive codimension. If μ has support and the SEP on $X\times\{p\}$, then

$$\mathbf{1}_{\pi^{-1}W}\mu=\mathbf{1}_{W\times Y}\mathbf{1}_{X\times\{p\}}\mu=\mathbf{1}_{W\times\{p\}}\mu=0,$$

cf. (2.8). Thus $\mathbf{1}_W\pi_*\mu=\pi_*(\mathbf{1}_{\pi^{-1}W}\mu)=0$, and so part (i) follows. Part (ii) follows from (2.11). In fact, assume that the hypothesis is fulfilled. If $W\subset\mathcal{U}\cap X$ has positive codimension, then $\mathbf{1}_{W\times\{p\}}\mu\otimes\nu=\mathbf{1}_W\mu\times\mathbf{1}_{\{p\}}\nu=0$, since $\mathbf{1}_W\mu=0$. □

2.3. **Almost semi-meromorphic currents.** The results and definitions in this and the next subsection are taken from [7, Section 4]. We say that a current on X is *semi-meromorphic* if it is of the form $\omega[1/f]$, where f is a generically nonvanishing holomorphic section of a line bundle $L\to X$ and ω is a smooth form with values in L. For simplicity we will often omit the brackets [ ] indicating principal value. Since $\omega[1/f]=[1/f]\omega$ when ω is smooth we can write just ω/f.

Following [4, 7] we say that a current a is *almost semi-meromorphic* in X, $a\in ASM(X)$, if there is a modification $\pi\colon X'\to X$ such that

$$a=\pi_*(\omega/f), \tag{2.12}$$

where ω/f is semi-meromorphic in X'. If $\mathcal{U}\subset X$ is an open subset, then the restriction $a_{\mathcal{U}}$ of $a\in ASM(X)$ to $\mathcal{U}$ is in $ASM(\mathcal{U})$. Moreover, $ASM(X)$ is contained in $\mathcal{W}(X)$.

Given a modification $\pi\colon X'\to X$, let $\operatorname{sing}(\pi)\subset X'$ be the (analytic) set where π is not a biholomorphism. By the definition it has positive codimension. Let $Z\subset X'$ be the zero set of f. Notice that $a\in ASM(X)$ is smooth outside $\pi(Z\cup\operatorname{sing}(\pi))$, which has positive codimension in X. Let $ZSS(a)$, the *Zariski-singular support* of a, be the smallest Zariski-closed set $V\subset X$ such that a is smooth outside V.

Example 2.9. If f is a holomorphic function in X such that $Z(f)$ has positive codimension, then clearly $[1/f]$ is almost semi-meromorphic and $ZSS(a)=Z(f)$. □

Example 2.10. We claim that $b=\partial|\zeta|^2/2\pi i|\zeta|^2$ is almost semi-meromorphic in $\mathbb{C}^n$. In fact, let $\pi\colon Y\to\mathbb{C}^n$ be the blow-up at the origin. Then, outside the exceptional divisor, $\pi^*b=\omega/s$, where s is a holomorphic section of the line bundle L_D that defines the exceptional divisor D and ω is an L_D-valued smooth $(1,0)$-form on Y. It is readily verified that $b=\pi_*(\omega/s)$. In fact, it clearly holds outside the origin, and since both sides are locally integrable, the equality holds in the current sense. Thus $b\in ASM(\mathbb{C}^n)$. □

We now recall one of the main results, Theorem 4.8, in [7]:

Theorem 2.11. *Assume that $a\in ASM(X)$. For each $\mu\in\mathcal{PM}(X)$ there is a unique pseudomeromorphic current T in X that coincides with $a\wedge\mu$ in $X\setminus ZSS(a)$ and such that $\mathbf{1}_{ZSS(a)}T=0$.*

The proof is highly nontrivial and relies on the fact that one can find a representation (2.12) of a such that f is nonvanishing in $X'\setminus\pi^{-1}ZSS(a)$ ([7, Lemma 4.7]).

Lemma 2.4 implies that

$$T = \lim_{\epsilon\to 0} \chi(|h|^2 v/\epsilon) a\wedge\mu \tag{2.13}$$

if h is a holomorphic tuple such that $Z(h) = ZSS(a)$. We will denote the extension T by $a\wedge\mu$ as well.

The definition of $a \wedge \mu$ is local, so that it commutes with restrictions to open subsets of X.

Proposition 2.12. *Assume that $a \in ASM(X)$. If W is an analytic subset of $\mathcal{U} \subset X$ and $\mu \in \mathcal{PM}(\mathcal{U})$, then*

$$\mathbf{1}_W (a\wedge\mu) = a\wedge\mathbf{1}_W \mu. \tag{2.14}$$

Clearly $\mathcal{W}_X^Z$ is closed under multiplication by smooth forms. We also have

Proposition 2.13. *Each $a \in ASM(X)$ induces a linear mapping*

$$\mathcal{W}_X^Z \to \mathcal{W}_X^Z, \quad \mu \mapsto a \wedge \mu. \tag{2.15}$$

Proposition 2.14. *Assume that $a_1, a_2 \in ASM(X)$ and $\mu \in \mathcal{PM}_X$. Then*

$$a_1\wedge a_2\wedge\mu = (-1)^{\deg a_1 \deg a_2} a_2\wedge a_1\wedge\mu.$$

In particular, one of the a_j may be a smooth form. It follows that (2.15) is $\mathcal{E}$-linear.

Example 2.15. Assume that μ is in $\mathcal{W}_X$. In view of (2.14), $\mu' = [1/h]\mu$ is in $\mathcal{W}_X$ as well. If h is generically nonvanishing, then $h\mu' = h[1/h]\mu = \mathbf{1}_{\{h\neq 0\}}\mu = \mu$. $\square$

2.4. **Residues of almost semi-meromorphic currents.** We shall now consider the effect of ∂ and $\bar\partial$ on almost semi-meromorphic currents.

Proposition 2.16. *If $a \in ASM(X)$, then $\partial a \in ASM(X)$ and*

$$\bar\partial a = b + r, \tag{2.16}$$

where $b = \mathbf{1}_{X\setminus ZSS(a)} \bar\partial a$ is in $ASM(X)$ and $r = \mathbf{1}_{ZSS(a)}\bar\partial a$ has support on $ZSS(a)$.

Clearly the decomposition (2.16) is unique. We call $r = r(a)$ the *residue (current)* of a.

Notice that current $\bar\partial(1/f)$ is the residue of the principal value current $1/f$. Similarly, the residue currents introduced, e.g., in [16, 2, 5] can be considered as residues of certain almost semi-meromorphic currents, generalizing $1/f$, cf. [7, Example 4.18].

As a consequence of Theorem 2.11 we can define products of $\bar\partial$, and residues, of almost semi-meromorphic currents and pseudomeromorphic currents.

Definition 2.17. For $a \in ASM(X)$ and $\mu \in \mathcal{PM}_X$ we define

$$\bar\partial a\wedge\mu := \bar\partial(a\wedge\mu) - (-1)^{\deg a} a\wedge\bar\partial\mu, \tag{2.17}$$

where $a\wedge\mu$ and $a\wedge\bar\partial\mu$ are defined as in Theorem 2.11. Moreover we define

$$r(a) \wedge \mu := \mathbf{1}_{ZSS(a)}\bar\partial a \wedge \mu.$$

Thus $\bar\partial a\wedge\mu$ is defined so that the Leibniz rule holds. It is easily checked that

$$r(a) \wedge \mu = \lim_{\epsilon\to 0} \bar\partial\chi(|h|^2 v/\epsilon) a\wedge\mu, \tag{2.18}$$

if $Z(h) = ZSS(a)$. In particular this gives a way of defining products of $\bar\partial$ and residues of almost semi-meromorphic currents. For example, (1.1) can be defined by inductively applying (2.17) and Theorem 2.11, cf. [14].

Example 2.18. Let b be the almost semi-meromorphic current from Example 2.10. If $n=1$, then $\bar\partial b$ is the current of integration $[0]$ at the origin. If $n>1$, then $\bar\partial b$ is almost semi-meromorphic since then $r(b)$ must vanish in view of the dimension principle. For $k\le n$ we can form the products $B_k := b\wedge(\bar\partial b)^{k-1}$. It is just a product of almost semi-meromorphic currents since no residues appear because of the dimension principle. However, it is well-known that $\bar\partial B_n = [0]$. This is in fact a compact way of writing the Bochner-Martinelli formula, see, e.g., [1]. □

3. Regularity of pseudomeromorphic currents

Throughout the rest of the paper X is a complex manifold of dimension n. We shall now discuss regularity properties of pseudomeromorphic currents on X. To this end we first have to consider local images of analytic sets under holomorphic mappings that are not necessarily proper. Recall that if $\varphi: Y\to X$ is a holomorphic mapping between manifolds and Y is connected, then generically φ attains its optimal rank, $\operatorname{rank}\varphi$, i.e., $\operatorname{rank}_y\varphi\le\operatorname{rank}\varphi$ for all y with equality outside an analytic variety of positive codimension.

Definition 3.1. Let X be a complex manifold. We say that a compact set $V\subset X$ is a *cqa (compact quasianalytic)* set if there are a (not necessarily connected) complex manifold Y, a holomorphic map $\varphi: Y\to X$, and a compact set $K\subset Y$, such that $V=\varphi(K)$. We say that the *dimension* of V, $\dim V$, is $\le d$ if there are such φ and K such that $\operatorname{rank}_y\varphi\le d$ for all $y\in K$.

If $\dim V\le d$, then the *codimension* of V is $\ge \dim X-d$. If d is as in Definition 3.1, K has nonempty interior, and $\operatorname{rank}_y\varphi = d$ generically on the interior of K, then we say that $\dim V = d$.

Remark 3.2. Our definition of a cqa set is closely related to the theory of subanalytic sets in the real setting, see, e.g., [8]. However we have not been able to rely directly on this theory. □

Example 3.3. Clearly, any compact set $K\subset X$ is a cqa set; however the dimension according to Definition 3.1 might not be the expected. For example, in view of Example 3.9 below, a point set with a limit point is not a cqa set of dimension 0. □

Since we do not require Y to be connected, any finite union of cqa sets of dimension $\le d$ is a cqa set of dimension $\le d$.

Remark 3.4. If $\varphi: Y\to X$ is a holomorphic map of rank n, X is a submanifold of M, and $i: X\to M$ is the inclusion, then $\operatorname{rank} i\circ\varphi = n$. Thus if $V\subset X$ is a cqa set of dimension $\le n$, then so is $i(V)\subset M$. □

Remark 3.5. We may allow Y to be singular in Definition 3.1. Indeed, assume that $V=\varphi(K)$, where $\varphi: Y\to X$ is a holomorphic map of optimal rank d and Y is an analytic variety. Let $\pi:\widetilde Y\to Y$ be a desingularization of Y. Then $\widetilde K := \pi^{-1}(K)\subset\widetilde Y$ is compact and $\tilde\varphi := \varphi\circ\pi: \widetilde Y\to X$ is a holomorphic map of optimal rank d, and thus $V=\tilde\varphi(\widetilde K)$ is a cqa set of dimension $\le d$ acccording to Definition 3.1. □

The notion of a cqa set generalizes the notion of (a compact part of) a variety.

Example 3.6. Assume that $Z\subset X$ is a subvariety of pure dimension ℓ. Then $i: Z\to X$ has optimal rank ℓ and thus any compact $K\subset Z$ is a cqa set of dimension $\le\ell$. If K has non-empty interior, then $\dim K=\ell$. □

There exists a cqa set that is not contained in an analytic variety of the same dimension. The following example, which is a complex variant of an example due to Osgood, see, e.g., [8, Ex. 2.4], was pointed out to us by Jean-Pierre Demailly.

Example 3.7. Let $u_1, u_2, u_3 : \mathbb{C} \to \mathbb{C}$ be entire functions that are algebraically independent, e.g., let $u_i(z) = e^{a_i z}$, where a_1, a_2, a_3 are linearly independent over $\mathbb{Q}$. Moreover let $\varphi : \mathbb{C}^2 \to \mathbb{C}^3$ be the map

$$\varphi(z,w) = \big(u_1(z)w, u_2(z)w, u_3(z)w\big)$$

and let $V = \varphi(\overline{\mathcal{V}})$, where $\mathcal{V}$ is a relatively compact neighborhood of $0 \in \mathbb{C}^2$. Then $V \subset \mathbb{C}^3$ is a cqa set of dimension 2 since $\operatorname{rank}\varphi = 2$. We claim that V is not contained in any 2-dimensional subvariety of an open set in $\mathbb{C}^3$ that contains V. To prove this assume, to the contrary, that there is a holomorphic function $g \not\equiv 0$ in a neighborhood of V such that $V \subset \{g = 0\}$. Then $g(0) = 0$. Let

$$g(x) = \sum_{m \in \mathbb{N}^3} a_m \, x_1^{m_1} x_2^{m_2} x_3^{m_3}$$

be the Taylor expansion of g at $0 \in \mathbb{C}^3$. Since $g \not\equiv 0$, there is at least one index m such that $a_m \neq 0$. Let d denote the sum $m_1 + m_2 + m_3$ for this m. The assumption $V = \varphi(\overline{\mathcal{V}}) \subset \{g = 0\}$ implies that

$$0 = g \circ \varphi(z,w) = \sum_{m \in \mathbb{N}^3} a_m \, u_1(z)^{m_1} u_2(z)^{m_2} u_3(z)^{m_3} w^{m_1+m_2+m_3}$$

for $(z,w) \in \mathcal{V}$. Identifying the coefficient of w^d we get

$$\sum_{m_1+m_2+m_3=d} a_m \, u_1(z)^{m_1} u_2(z)^{m_2} u_3(z)^{m_3} = 0,$$

which contradicts the algebraic independence of u_1, u_2, and u_3 and thus proves the claim. $\square$

However, in a sense, a cqa set of dimension $\leq d$ is generically contained in an analytic variety of dimension d.

Lemma 3.8. *Assume that $V \subset X$ is a cqa set of dimension $\leq d$. Then there is a cqa set $V' \subset V$ of dimension $\leq d-1$, such for each $x \in V \setminus V'$ there is a neighborhood $\mathcal{U} \subset X$ of x and a finite union $W \subset \mathcal{U}$ of submanifolds of dimension $\leq d$ such that $V \cap \mathcal{U} \subset W$.*

If $d = 0$, then V' should be interpreted as the empty set; more generally, a cqa set of dimension ≤ -1 equals the empty set.

Proof. Let $V = \varphi(K)$, where $\varphi : Y \to X$ is a holomorphic map of generic rank $\leq d$, Y is a complex manifold, and $K \subset Y$ is compact. Let $Y' = \{y \in Y, \operatorname{rank}_y \varphi \leq d-1\}$. Then Y' is a subvariety of Y, and it follows, cf. Remark 3.5, that $V' := \varphi(Y' \cap K)$ is a cqa set of dimension $\leq d-1$.

If $V' = V$ the lemma is trivial. Otherwise, take $x \in V \setminus V'$ and let $Z = \varphi^{-1}(x) \cap K$. If $y \in Z$ then $y \notin Y'$, and since Y' is closed there is a neighborhood $\mathcal{V}_y \subset Y$ of y such that φ has constant rank d in $\mathcal{V}_y$. After possibly shrinking $\mathcal{V}_y$, we may assume, in view of the constant rank theorem, that $\varphi(\mathcal{V}_y)$ is a submanifold of dimension d of some neighborhood $\mathcal{U}_y$ of x in X. By compactness, Z is contained in a finite union $\cup \mathcal{V}_{y_j}$ of such sets. Let $\mathcal{U}_{y_j}$ be the associated neighborhoods of x.

Since K is compact and φ is continuous there is a neighborhood $\mathcal{U} \subset \cap \mathcal{U}_{y_j}$ of x such that the closure of $\varphi^{-1}\mathcal{U} \cap K$ is contained in a finite union $\cup \mathcal{V}_{y_j}$ of such sets $\mathcal{V}_y$. It follows that $V \cap \mathcal{U}$ is contained in $W = \varphi\big(\cup \mathcal{V}_{y_j}\big) \cap \mathcal{U}$. □

Example 3.9. It follows from Lemma 3.8 that a cqa set of dimension 0 is a compact part of a variety of dimension 0 and thus a discrete point set, i.e., without limit points. □

Remark 3.10. If $V = \varphi(K)$, where $\varphi : Y \to X$ has constant rank d, then V' is empty in the proof above, and thus V is contained in a subvariety of X of dimension d. □

Example 3.11. Let φ be as in Example 3.7, with the choice $u_i(z) = e^{a_i z}$. Then

$$\frac{\partial \varphi_i}{\partial z} = a_i e^{a_i z} w, \ \frac{\partial \varphi_i}{\partial w} = e^{a_i z}$$

so it follows that $\mathrm{rank}_{(z,w)} = 1$ if $w = 0$ and $\mathrm{rank}_{(z,w)} = 2$ otherwise. Thus, the set Y' in the proof of Lemma 3.8 equals $\{w = 0\}$ and $V' = \varphi(Y') = \{0\}$. Therefore the quasi-analytic set $V = \varphi(\overline{\mathcal{V}})$ is "locally analytic" outside 0. □

We have the following version of the dimension principle.

Proposition 3.12. *(i) If a pseudomeromorphic current μ of bidegree $(*, p)$ has its support contained in a cqa set of codimension $\geq p + 1$, then $\mu = 0$.*
(ii) If $\mu \in \mathcal{W}_X$ has support on a cqa set of positive codimension, then $\mu = 0$.

Proof. Assume that the support of μ is contained in the cqa set V of codimension $\geq p+1$. In view of Lemma 3.8 and the usual dimension principle, see Section 2.1, then μ must have its support contained in a cqa set V' of codimension $\geq p+2$. Repeating the argument, (i) follows by a finite induction. The statement (ii) is verified in a similar way. □

Example 3.13. Let us use the notation in Example 3.7. Let χ be a cutoff function in $\mathbb{C}^2$ that is 1 in a neighborhood of 0 and 0 outside $\mathcal{V}$ and let $\mu := \varphi_* \chi$. Then

$$\mu.1 = \int_{\mathbb{C}^2} \chi \neq 0$$

so μ is a pseudomeromorphic nonvanishing current with compact support in the cqa set V in Example 3.7. It follows from Proposition 3.12 (ii) that μ is not in $\mathcal{W}_{\mathbb{C}^3}$. However, note that $\mathbf{1}_W \mu = 0$ for all germs of proper subvarieties W at $0 \in \mathbb{C}^3$. In fact, $\mathbf{1}_W \mu = \varphi_*(\mathbf{1}_{\varphi^{-1}W} \chi) = 0$ by the dimension principle, since $\varphi^{-1}W$ has positive codimension in Y in view of Example 3.7. □

We are now ready for our main result of this section.

Theorem 3.14. *Let μ be a pseudomeromorphic current with compact support on a complex manifold X of dimension n. Then there is a cqa set $V \subset X$ of dimension $\leq n - 1$ such that μ is smooth in $X \setminus V$.*

Proof. Note that the case $n = 0$ is trivial.

We may assume that $\mu = \varphi_* \tau$, where $\varphi : \mathcal{U} \to X$ is a holomorphic map, $\mathcal{U} \subset \mathbb{C}^N$ is open, and τ is an elementary current of the form (2.4) with compact support $K \subset \mathcal{U}$. For each multi-index $I = \{i_1, \dots, i_k\} \subset \{1, \dots, N\}$, let

$$E_I = \{t_{i_1} = \cdots = t_{i_k} = 0\} = E_{i_1} \cap \cdots \cap E_{i_k},$$

where $E_i = \{t_i = 0\}$. Moreover, let

$$E'_I = \{y \in \mathcal{U};\ \operatorname{rank}_y \varphi|_{E_I} < n\},$$

where $\varphi|_{E_I}$ denotes the restriction of φ to E_I. Notice that $E'_\emptyset = \{y \in \mathcal{U}, \operatorname{rank}_y \varphi < n\}$. Let $E' = \cup_I E'_I$ and let $V = \varphi(E' \cap K)$. Then V is a cqa set in view of Remark 3.5 and $\dim V = \operatorname{rank} \varphi|_{E'} \le n - 1$.

We claim that the restriction to $X \setminus V$ of μ is smooth. Let χ be any smooth cutoff function with support in $X \setminus V$. We have to prove that $\chi\mu$ is smooth. To this end, consider $y \in \varphi^{-1}(\operatorname{supp}\chi) \cap K$. Let $I_y = \{i, y \in E_i\}$, i.e., I_y is the maximal I, under inclusion, such that $y \in E_I$. Then there is a neighborhood $\mathcal{V}_y$ such that $\mathcal{V}_y \cap E_i = \emptyset$ for all $i \notin I_y$. If $I_y = \{i_1, \dots, i_k\}$, it follows, possibly after reordering the variables, that τ is of the form

$$\tau = \beta \wedge \Big[\frac{1}{t_{i_1}^{m_{i_1}}}\Big] \cdots \Big[\frac{1}{t_{i_\ell}^{m_{i_\ell}}}\Big] \bar\partial \Big[\frac{1}{t_{i_{\ell+1}}^{m_{i_{\ell+1}}}}\Big] \wedge \dots \wedge \bar\partial \Big[\frac{1}{t_{i_k}^{m_{i_k}}}\Big],$$

where β is smooth in $\mathcal{V}_y$.

Since $y \notin E'$, possibly after shrinking $\mathcal{V}_y$ we can assume that $\mathcal{V}_y \cap E' = \emptyset$, which, in particular, implies that $\varphi|_{E_{I_y}}$ has rank n in $E_{I_y} \cap \mathcal{V}_y$. It follows that

$$d\varphi_1 \wedge \dots \wedge d\varphi_n \wedge dt_{i_1} \wedge \dots \wedge dt_{i_k} \neq 0$$

in $E_{I_y} \cap \mathcal{V}_y$ if $\varphi = (\varphi_1, \dots, \varphi_n)$. By the inverse function theorem, after possibly shrinking $\mathcal{V}_y$ further, we can thus choose a coordinate system in $\mathcal{V}_y$ so that $\varphi_1, \dots, \varphi_n, t_{i_1}, \dots, t_{i_k}$ are the first $n + k$ coordinates. Let $\sigma_1, \dots, \sigma_{N-n-k}$ be a choice of complementary coordinate functions. Then

$$\varphi : (\varphi_1, \dots, \varphi_n, t_{i_1} \dots t_{i_k}, \sigma_1, \dots, \sigma_{N-n-k}) \mapsto (\varphi_1, \dots, \varphi_n),$$

i.e., φ is just the projection onto the first n coordinates.

Let χ_y be a smooth cutoff function that is 1 in a neighborhood of y and has compact support in $\mathcal{V}_y$. Then

$$\varphi_*(\chi_y \tau) = \int_{t_i, \sigma_j} \chi_y \beta \wedge \Big[\frac{1}{t_{i_1}^{m_{i_1}}}\Big] \cdots \Big[\frac{1}{t_{i_\ell}^{m_{i_\ell}}}\Big] \bar\partial \Big[\frac{1}{t_{i_{\ell+1}}^{m_{i_{\ell+1}}}}\Big] \wedge \dots \wedge \bar\partial \Big[\frac{1}{t_{i_k}^{m_{i_k}}}\Big],$$

which is smooth.

Since $\varphi^{-1}(\operatorname{supp}\chi) \cap K$ is compact, there are finitely many y and $\mathcal{V}_y$ as above, such that $\cup \mathcal{V}_y$ is a neighborhood of $\varphi^{-1}(\operatorname{supp}\chi) \cap K$. It follows that there is a finite number of smooth cutoff functions χ_y with compact support in $\mathcal{V}_y$ such that $\{\chi_y\}$ is a partition of unity on $\varphi^{-1}(\operatorname{supp}\chi) \cap K$. Thus

$$\chi\mu = \varphi_*(\varphi^* \chi \tau) = \sum \varphi_*(\chi_y \varphi^* \chi \tau)$$

is smooth, since each term in the rightmost expression is. □

Recall that the *singular support* of a current μ is the smallest closed set $V \subset X$ such that μ is smooth outside V.

Corollary 3.15. *If $\mu \in \mathcal{W}_X$ vanishes outside its singular support, then it vanishes identically.*

Proof. By Theorem 3.14 the singular support of μ is contained in a cqa set V of dimension $\le n - 1$ and thus by assumption μ vanishes outside V. Since μ is in $\mathcal{W}_X$ it vanishes identically in view of Proposition 3.12 (ii). □

4. Regularity properties of currents in $\mathcal{PM}_X^Z$ and $\mathcal{W}_X^Z$

Our first result is a local description of $\mathcal{PM}_X^Z$ when Z is smooth.

Proposition 4.1. *Let μ be a pseudomeromorphic current on a complex manifold X. Assume that μ has support on a submanifold $Z \subset X$ of codimension p. If we choose local coordinates $z_1 \dots, z_{n-p}, w_1 \dots, w_p$ in $\mathcal{U} \subset\subset X$ so that $Z = \{w_1 = \dots = w_p = 0\}$, then, in $\mathcal{U}$, μ has a unique finite expansion*

$$\mu = \sum_r \sideset{}{'}\sum_{|I|=r} \sum_{m\in\mathbb{N}^p} \mu_{I,m}(z) \otimes \bar\partial \frac{1}{w_p^{m_p+1}} \wedge \dots \wedge \bar\partial \frac{1}{w_1^{m_1+1}} \wedge dw_{I_1} \wedge \dots \wedge dw_{I_r}, \tag{4.1}$$

where $\mu_{I,m}$ are pseudomeromorphic currents on Z.
Moreover, $\bar\partial\mu = 0$ if and only if $\bar\partial\mu_{I,m} = 0$ for each I, m, and μ is in $\mathcal{W}_X^Z$ if and only if $\mu_{I,m}$ is in $\mathcal{W}_Z$ for each I, m.

Notice that the right hand side of (4.1) indeed defines a current μ in $\mathcal{PM}_X^Z$ if $\mu_{I,m}$ are in $\mathcal{PM}_Z$ in view of Lemma 2.6.

Proof. In view of [7, Theorem 3.5] it suffices to consider the case where the terms in (4.1) vanishes except for $r = p$, i.e., $I = (1, \dots, p)$. Therefore, let us assume from now on that this is the case. Let $\mu_m = \mu_{I,m}$, $dw = dw_1 \wedge \dots \wedge dw_p$, $w^m = w_1^{m_1} \cdots w_p^{m_p}$, and

$$\bar\partial\frac{1}{w^{m+\mathbf{1}}} = \bar\partial \frac{1}{w_p^{m_p+1}} \wedge \dots \wedge \bar\partial \frac{1}{w_1^{m_1+1}}.$$

It is readily checked that if $\phi_\ell(z)$ are test forms on $Z \cap \mathcal{U}$, then

$$\int_{z,w} \big(\phi_\ell(z) \otimes w^\ell\big) \wedge \big(\mu_m(z) \otimes \bar\partial\frac{1}{w^{m+\mathbf{1}}} \wedge dw\big) = \delta_{\ell,m}(2\pi i)^p \int_z \phi_\ell(z) \wedge \mu_m(z), \tag{4.2}$$

where $\delta_{\ell,m}$ is the Kronecker symbol. Let $\pi : \mathbb{C}^n \to \mathbb{C}^{n-p}$ be the projection

$$(z_1, \dots, z_{n-p}, w_1, \dots, w_p) \mapsto (z_1, \dots, z_{n-p}).$$

As a consequence of (4.2) we have that if μ has a representation (4.1), then

$$\pi_*(w^m\mu) = (2\pi i)^p \mu_m. \tag{4.3}$$

Thus the representation (4.1) of μ is unique if it exists.

Now assume that μ is given, and let

$$T = \frac{1}{(2\pi i)^p} \sum_{m\in\mathbb{N}^p} \mu_m(z) \otimes \bar\partial \frac{1}{w^{m+\mathbf{1}}} \wedge dw,$$

where μ_m are defined by (4.3). Since μ has locally finite order this sum is finite and thus defines an element in $\mathcal{PM}_X^Z$. We claim that

$$\mu = T. \tag{4.4}$$

To prove (4.4), first notice that for each j, $dw_j \wedge \mu = dw_j \wedge T = 0$ for degree reasons and $d\bar w_j \wedge \mu = d\bar w_j \wedge T = 0$ by(2.7), so we only have to check the equality for test forms ϕ with no differentials with respect to w. A Taylor expansion with respect to w of such a form ϕ gives that

$$\phi = \sum_{|\ell|<M} \phi_\ell(z) \otimes w^\ell + \mathcal{O}(\bar w) + \mathcal{O}(|w|^M),$$

where $\mathcal{O}(\bar{w})$ denotes terms with some factor $\bar{w}_j$ and M is chosen so large that $\mathcal{O}(|w|^M)\mu = \mathcal{O}(|w|^M)T = 0$ in $\mathcal{U}$. Since $\bar{w}_j\mu = \bar{w}_jT = 0$, cf. (2.7), it follows that we just have to check (4.4) for test forms like $\phi = \phi_\ell(z) \otimes w^\ell$. However, it follows immediately from (4.2) and (4.3) that $\mu.\phi = T.\phi$ for such ϕ, which proves (4.4) and the first part of the proposition.

Since $\bar{\partial}(1/w^{m+\mathbf{1}})\wedge dw$ is $\bar{\partial}$-closed it follows by the uniqueness that $\bar{\partial}\mu_m = 0$ for all m if (and only if) $\bar{\partial}\mu = 0$. The last statement follows from Lemma 2.8 (ii). □

This gives us the following extension of Theorem 3.14.

Corollary 4.2. *Assume that μ is a pseudomeromorphic current on a complex manifold X with compact support in $Z \cap \mathcal{U}$, where $\mathcal{U}$ and Z are as in Propostion 4.1. Then there is a cqa set $V \subset Z \cap \mathcal{U}$ of codimension $\geq p+1$ in $\mathcal{U}$ and such that*

$$\mu = \alpha \wedge \tilde{\mu},$$

in $\mathcal{U} \setminus V$, where α is a smooth form in $X \setminus V$ and $\tilde{\mu}$ is a pseudomeromorphic current of bidegree $(0,p)$ with compact support in $Z \cap \mathcal{U}$.

Proof. Note that the case $\dim X = 0$ is trivial.

Consider the representation (4.1) of μ. As in the proof of Proposition 4.1 it suffices to consider terms in the representation (4.1) where $r = p$; let us use the notation from that proof. Choose $M \in \mathbb{N}^p$ such that $M_j \geq m_j$ for all j in (4.1). Let

$$\tilde{\mu} = \bar{\partial}\frac{1}{w^{M+1}}\wedge dw$$

and let

$$\alpha = \sum_{m\in\mathbb{N}^p} w^{M-m}\mu_m(z).$$

Then clearly $\mu = \alpha \wedge \tilde{\mu}$ in $\mathcal{U}$.

Since μ has compact support in $\mathcal{U}\cap Z$, each μ_m has compact support in $\mathcal{U}\cap Z$ and thus by Theorem 3.14 there are cqa sets $V_m \subset Z$ of strictly positive codimension, such that μ_m is smooth outside V_m. Now α is smooth in $\mathcal{U}\setminus V \times \mathbb{C}^p_w$, where $V := \cup V_m$ is a cqa set of codimension $\geq p+1$ in X. Multiplying $\tilde{\mu}$ by a suitable cutoff function in $\mathcal{U}$ and replacing α by a smooth form on $X\setminus V$ that coincides with α on the support of μ, we get the desired representation of μ in $\mathcal{U} \setminus V$. □

The main result in this section is the following local characterization of elements in $\mathcal{W}_X^Z$ in terms of elementary currents.

Theorem 4.3. *Assume that μ is a pseudomeromorphic current on a complex manifold X with support on the subvariety Z of dimension d. Then $\mu \in \mathcal{W}_X^Z$ if and only if there is a locally finite representation*

$$\mu = \sum_\ell \varphi_*\tau_\ell, \tag{4.5}$$

where φ is a holomorphic mapping, such that, for each ℓ, the elementary support of τ_ℓ is contained in $\varphi^{-1}Z$, and the restriction $\tilde{\varphi}_\ell$ of φ to the elementary support of τ_ℓ has generic rank d.

For the proof we need the following lemmas.

Lemma 4.4. *Assume that $\mu = \varphi_*\tau$, where $\varphi : \Omega \to X$ and τ is an elementary current on Ω with elementary support H. Moreover, assume that the restriction of φ to H has generic rank d. Let $W \subset X$ be a subvariety of dimension $\leq d-1$. Then $\mathbf{1}_W\mu = 0$.*

Proof. In view of Remark 2.2 we may assume that H is an irreducible subvariety of Ω. Assume that $\varphi^{-1}W \cap H = H$. Then, since $\varphi|_H$ has generic rank d, by the constant rank theorem, there is an open subset $\mathcal{V}$ of H such that $\varphi(\mathcal{V})$ is a complex manifold of dimension d. It follows that $W \supset \varphi(\mathcal{V})$ has dimension $\geq d$, which contradicts that W has dimension $\leq d-1$. Since H is irreducible, we conclude that $\varphi^{-1}W \cap H$ is a subvariety of H of positive codimension. Since τ has the SEP on H, cf. Example 2.7, it follows that $\mathbf{1}_W\mu = \varphi_*\big(\mathbf{1}_{\varphi^{-1}W\cap H}\tau\big) = 0$. □

The next lemma is a generalization of Proposition 3.12 (ii).

Lemma 4.5. *If $\mu \in \mathcal{W}_X^Z$ has support on a cqa set $V \subset Z$ of positive codimension, then $\mu = 0$.*

Proof. Let d be the dimension of Z. By Lemma 3.8 there is a cqa set $V' \subset V$ of dimension $\leq d-2$ such that locally $V \setminus V'$ is contained in a variety of dimension $\leq d-1$. Since μ has the SEP on Z it follows that $\operatorname{supp}\mu \subset V'$. By repeating this argument we get that μ vanishes, cf. the proof of Proposition 3.12. □

Proof of Theorem 4.3. Let $\varphi\colon \mathcal{U} \to X$ be a holomorphic mapping and let τ be elementary with compact support in $\mathcal{U}$. Moreover assume that the restriction of φ to the elementary support H of τ has generic rank d and that $H \subset \varphi^{-1}Z$. Then clearly $\varphi_*\tau$ has support on Z. Let $\mathcal{V}$ be an open subset of X and let $W \subset \mathcal{V} \cap Z$ be a subvariety of positive codimension. We claim that $\mathbf{1}_W\mu = 0$. To prove this it suffices to show that $\mathbf{1}_W\chi\mu = 0$ for each smooth cutoff function χ with compact support in $\mathcal{V}$. This however follows from Lemma 4.4 applied to $\hat\varphi_*(\varphi^*\chi\tau)$, where $\hat\varphi : \varphi^{-1}\mathcal{V} \to \mathcal{V}$ is the restriction of φ to $\varphi^{-1}\mathcal{V}$. Hence $\varphi_*\tau$ is in $\mathcal{W}_X^Z$ and thus the "if"-part of the proposition is proved.

For the converse assume that μ is in $\mathcal{W}_X^Z$. With no loss of generality we can assume that μ has compact support, so that we have a finite representation like (4.5), without any special assumption on the φ and τ_ℓ. In view of Lemma 2.5 (and its proof) we may also assume that all the elementary supports of the τ_ℓ are contained in $\varphi^{-1}Z$. Consider τ_ℓ such that $\tilde\varphi_\ell$ has generic rank $\geq d+1$. Since $\varphi_*\tau_\ell$ is contained in Z of dimension d, $\varphi_*\tau_\ell$ vanishes by Lemma 4.4. Thus we may assume from now on that $\operatorname{rank}\tilde\varphi_\ell \leq d$ for all ℓ. Now write $\mu = \mu' + \mu''$, where μ' is the sum of all $\varphi_*\tau_\ell$ for which $\operatorname{rank}\tilde\varphi_\ell = d$. Then μ' is in $\mathcal{W}_X^Z$ by the first part of the proof. Hence so is μ''.

If $\operatorname{rank}\tilde\varphi_\ell \leq d-1$, then $\operatorname{supp}\varphi_*\tau_\ell$ is contained in a cqa set $\varphi(H)$ of dimension $\leq d-1$. Thus $\operatorname{supp}\mu''$ is as well, and hence it vanishes in view of Lemma 4.5. Hence $\mu = \mu'$. □

As an immediate consequence we get:

Corollary 4.6. *If μ_j is in $\mathcal{W}_{X_j}^{Z_j}$, $j = 1, 2$, then $\mu_1 \otimes \mu_2$ is in $\mathcal{W}_{X_1\times X_2}^{Z_1\times Z_2}$.*

5. Stalkwise injectivity of $\mathcal{PM}_X$

Theorem 5.1. *Let X be a complex manifold. The sheaf $\mathcal{PM}_X$ is stalkwise injective.*

The corresponding statement for general currents $\mathcal{C}_X$ is a classical result due to Malgrange. For a quite simple proof of this by integral formulas, see [3, Section 2].

Theorem 5.1 means: *If E_k are holomorphic vector bundles and*

$$\cdots \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0) \to \mathcal{S} \to 0 \tag{5.1}$$

is a locally free resolution of a coherent sheaf $\mathcal{S}$ over X, then the induced sheaf complex

$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{S}, \mathcal{PM}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{O}(E_0), \mathcal{PM}) \xrightarrow{f_1^*} \mathcal{H}om_{\mathcal{O}}(\mathcal{O}(E_1), \mathcal{PM}) \xrightarrow{f_2^*} \cdots$$

is exact.

The exactness at the first two places is trivial, so we are to prove that the equation $f_k^* u = \mu$ can be (locally) solved in $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}(E_{k-1}), \mathcal{PM})$ for each μ in $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}(E_k), \mathcal{PM})$ such that $f_{k+1}^* \mu = 0$, $k = 1, 2, \ldots$.

Note that Theorem 5.1 is equivalent to that $\mathcal{PM}_X^{\ell,k}$ is stalkwise injective for each ℓ, k.

Example 5.2. Let f be a single generically non-vanishing holomorphic function. Then

$$0 \to \mathcal{O} \xrightarrow{f} \mathcal{O} \to \mathcal{O}/(f) \to 0$$

is a free resolution of $\mathcal{S} = \mathcal{O}/(f)$. The condition $f_2^* \mu = 0$ is vacuous in this case so the stalkwise injectivity means that the equation $f\nu = \mu$ is locally solvable for any pseudomeromorphic μ, which is precisely the content of [7, Proposition 3.1] (in case X is smooth). □

We postpone the proof of Theorem 5.1 to Section 6 and first discuss some consequences. To this end we need some facts about residue currents as well as solvability of the $\bar\partial$-equation for pseudomeromorphic currents.

5.1. **Residues associated to a locally free resolution.** Consider a locally free resolution (5.1) of the coherent sheaf $\mathcal{S}$ on X, let $E = \oplus E_k$ and $f = f_1 + f_2 + \cdots$. We equip E with a superstructure so that $E_+ = \oplus E_{2j}$ and $E_- = \oplus E_{2j+1}$. Then both f and $\bar\partial$ are odd mappings on the sheaf $\mathcal{C}(E)$ of E-valued currents, and thus so is $\nabla = f - \bar\partial$. Let $\nabla_{\mathrm{End}\, E}$ be the induced mapping on endomorphisms on E, see [5] for more details.

Let us choose Hermitian metrics on the vector bundles E_k, and let U and R be the associated $\mathrm{End}\, E$-valued principal value, and residue currents, respectively, as defined in [5, Section 2], so that $\nabla_{\mathrm{End}\, E} U = I_E - R$. It follows from the construction that U is almost semi-meromorphic on X and that R is the residue of U, cf. Section 2.4. Thus R has support on $Z := ZSS(U)$, which by construction is precisely the analytic set where $\mathcal{S}$ is not locally free, or equivalently, the set where the complex $\big(\mathcal{O}(E_\bullet), f_\bullet\big)$ is not pointwise exact.

If $\chi \sim \chi_{[1,\infty)}$ as before and $\chi_\epsilon = \chi(|h|^2/\epsilon)$, where h is a holomorphic tuple whose common zero set is Z, then $U_\epsilon := \chi_\epsilon U$ is smooth for $\epsilon > 0$ and $U_\epsilon \to U$ when $\epsilon \to 0$; in fact, $\chi(|f_1|^2/\epsilon)$ will do. We can define the smooth form R_ϵ so that $\nabla_{\mathrm{End}\, E} U_\epsilon = I_E - R_\epsilon$. Then clearly $R_\epsilon \to R$ when $\epsilon \to 0$. Since $\nabla_{\mathrm{End}\, E} U = I_E$ outside Z it follows that

$$R_\epsilon = (1 - \chi_\epsilon) I_E + \bar\partial \chi_\epsilon \wedge U. \tag{5.2}$$

Let U_k^ℓ and R_k^ℓ be the components of U and R, respectively, that take values in $\mathrm{Hom}\,(E_\ell, E_k)$. By [5, Theorem 3.1], $R_k^\ell = 0$ when $\ell \geq 1$. Thus we can write R_k rather then R_k^0.

Example 5.3. For the resolution in Example 5.2, we have $U = 1/f$ and $R = \bar\partial(1/f)$. □

5.2. **The $\bar\partial$-equation for pseudomeromorphic currents.** Let us recall how one can solve the $\bar\partial$-equation by means of simple integral formulas. From Example 2.18 we know that

$$B' := \sum_{k=1}^n b' \wedge (\bar\partial b')^{k-1}$$

is almost semi-meromorphic in $\mathbb{C}^n_\eta$ if $b' = \partial|\eta|^2/2\pi i|\eta|^2$.

Thus $B' \otimes 1$ is almost semi-meromorphic in $\mathbb{C}^n_\eta \times \mathbb{C}^n_\xi$, and by a linear change of coordinates we find that $B := \eta^* B'$ is almost semi-meromorphic in $\mathbb{C}^n_\zeta \times \mathbb{C}^n_z$, if $\eta(\zeta, z) = \zeta - z$. If μ is any current with compact support in $\mathbb{C}^n_\zeta$, one can define the convolution operator

$$\mathcal{K}\mu(z) = \int_\zeta B_{n,n-1}(\zeta, z) \wedge \mu(\zeta), \tag{5.3}$$

where $B_{n,n-1}$ denotes the component of bidegree $(n, n-1)$, for instance by replacing B by the regularization $B_\epsilon = \chi(|\zeta - z|^2/\epsilon)B$ and taking the limit when $\epsilon \to 0$. More formally, $\mathcal{K}\mu = p_*(B_{n,n-1} \wedge \mu \otimes 1)$, where p is the natural projection $(\zeta, z) \mapsto z$. If μ is pseudomeromorphic, then also $\mu \otimes 1$ is, cf. Lemma 2.6, and thus $B \wedge \mu \otimes 1$ is just multiplication by the almost semi-meromorphic current B, see Theorem 2.11. It follows that $\mathcal{K}\mu$ is pseudomeromorphic if μ is.

The top degree term $B_{n,n-1}$ is the classical Bochner-Martinelli kernel. The other terms in B will play an important role below. It is well-known that

$$\mu = \bar\partial \mathcal{K}\mu + \mathcal{K}\bar\partial\mu. \tag{5.4}$$

Proposition 5.4. *If X is a complex manifold, then*

$$0 \to \Omega^p_X \to \mathcal{PM}^{p,0}_X \xrightarrow{\bar\partial} \mathcal{PM}^{p,1}_X \xrightarrow{\bar\partial} \cdots$$

is a fine resolution of Ω^p_X.

Here Ω^p_X denotes the sheaf of holomorphic p-forms. This proposition is implicitly proved in [4] but for the reader's convenience we supply a simple direct argument.

Proof. Since the case $k = 0$ is well-known let us assume that μ is pseudomeromorphic of bidegree (p, k), $k \geq 1$, and $\bar\partial\mu = 0$. Fix a point $x \in X$ and let χ be a cutoff function in a coordinate neighborhood of x that is identically 1 in a neighborhood of x. We can then apply (5.4) to $\chi\mu$ and so we get that $\chi\mu = \bar\partial\mathcal{K}(\chi\mu) + \mathcal{K}(\bar\partial\chi \wedge \mu)$. Now $\mathcal{K}(\chi\mu)$ is pseudomeromorphic in view of Proposition 5.5 below. Furthermore, $\mathcal{K}(\bar\partial\chi \wedge \mu)$ is smooth where $\chi = 1$ since B only has singularities at the diagonal. Since this term in addition is $\bar\partial$-closed near x it is locally of the form $\bar\partial\psi$ for some smooth ψ. It follows that there is a local pseudomeromorphic solution at x to $\bar\partial\nu = \mu$. □

Proposition 5.5. *The integral operator $\mathcal{K}$ in* (5.3) *maps pseudomeromorphic currents on $\mathbb{C}^n$ with compact support into $\mathcal{W}(\mathbb{C}^n) \subset \mathcal{PM}(\mathbb{C}^n)$.*

This is an immediate consequence of

Proposition 5.6. *If A is almost semi-meromorphic on $X \times Y$, $\mu \in \mathcal{PM}(X)$ has compact support, and $\pi\colon X \times Y \to Y$ is the natural projection, then $\pi_*(A\wedge\mu \otimes 1)$ is in $\mathcal{W}(Y)$.*

Proof. By Theorem 2.11, $\pi_*(A\wedge\mu \otimes 1)$ is in $\mathcal{W}(Y)$. Assume that $V \subset \mathcal{U} \subset Y$ has positive codimension. Then, in view of (2.10), (2.14) and (2.11), we have

$$\mathbf{1}_V \pi_*\big(A\wedge(\mu\otimes 1)\big) = \pi_*\big(\mathbf{1}_{X\times V}(A\wedge(\mu\otimes 1))\big) = \pi_*\big(A\wedge \mathbf{1}_{X\times V}(\mu\otimes 1)\big) = \pi_*\big(A\wedge(\mathbf{1}_X\mu\otimes \mathbf{1}_V 1)\big) = 0,$$

since $\mathbf{1}_V 1 = 0$. □

5.3. **A generalization of the Dickenstein-Sessa decomposition.** Let Z be a reduced analytic variety of pure codimension ν. A (p,ν)-current μ on X is a *Coleff-Herrera current* with respect to Z, $\mu \in \mathcal{CH}^Z_p$, if $\bar\partial\mu = 0$, $\bar\psi\mu = 0$ for all holomorphic functions ψ vanishing on Z, and μ has the SEP with respect to Z; see, e.g., [9, Section 6.2]. Let $(\mathcal{C}^Z_{p,k}, \bar\partial)$ be the Dolbeault complex of $(p,*)$-currents on X with support on Z. Dickenstein and Sessa proved in [11, 12][2], see also [3, 9], that Coleff-Herrera currents are canonical representatives in moderate cohomology, i.e.,

$$\mathcal{K}er_{\bar\partial}\, \mathcal{C}^Z_{p,\nu} = \mathcal{CH}^Z_p \oplus \bar\partial \mathcal{C}^Z_{p,\nu-1}; \tag{5.5}$$

in other words, each $\bar\partial$-closed current μ with support on Z has a unique decomposition

$$\mu = \mu_1 + \bar\partial\gamma, \tag{5.6}$$

where μ_1 is in $\mathcal{CH}^Z_p$ and γ has support on Z.

Let $\mathcal{S}$ be a coherent sheaf over X and let (5.1) be a locally free resolution. Combining Theorem 5.1 and Proposition 5.4 we find that

$$\mathcal{M}_{\ell,k} = \mathcal{H}om_{\mathcal{O}}\big(\mathcal{O}(E_\ell, \mathcal{PM}^{p,k})\big)$$

is a double complex with vanishing cohomology except at $\ell = 0$ and $k = 0$, where the kernels are $\mathcal{H}om_{\mathcal{O}}(\mathcal{S}, \mathcal{PM}^{p,k})$ and $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}(E_\ell), \Omega^p)$, respectively. The same holds if $\mathcal{PM}^{p,\bullet}$ are replaced by the sheaves of general currents $\mathcal{C}^{p,\bullet}$, in view of the well-known local solvability of $\bar\partial$ for $\mathcal{C}$, and Malgrange's theorem. By standard cohomological algebra we get

Theorem 5.7. *If $\mathcal{S}$ is a coherent sheaf over a complex manifold X and (5.1) is a locally free resolution, then there are canonical isomorphisms*

$$\mathcal{E}xt^k_{\mathcal{O}}(\mathcal{S}, \Omega^p) \simeq \mathcal{H}^k(\mathcal{H}om_{\mathcal{O}}(\mathcal{O}(E_\ell), \Omega^p), f^*_\bullet) \simeq \mathcal{H}^k(\mathcal{H}om_{\mathcal{O}}(\mathcal{S}, \mathcal{PM}^{p,\bullet}), \bar\partial) \simeq \mathcal{H}^k(\mathcal{H}om_{\mathcal{O}}(\mathcal{S}, \mathcal{C}^{p,\bullet}), \bar\partial), \quad k \geq 1. \tag{5.7}$$

The novelty in (5.7) is the representation of $\mathcal{E}xt^k_{\mathcal{O}}(\mathcal{S}, \Omega^p)$ by Dolbeault cohomology for the smaller sheaves of currents $\mathcal{PM}$. In particular we have the decompositions

$$\mathcal{K}er\,_{\bar\partial}\mathcal{H}om_{\mathcal{O}}(\mathcal{S}, \mathcal{C}^{p,k}) = \mathcal{H}^k(\mathcal{H}om_{\mathcal{O}}(\mathcal{S}, \mathcal{PM}^{p,\bullet}), \bar\partial) \oplus \bar\partial\mathcal{H}om_{\mathcal{O}}(\mathcal{S}, \mathcal{C}^{p,k-1}). \tag{5.8}$$

That is, each $\bar\partial$-closed μ in $\mathcal{H}om(\mathcal{S}, \mathcal{PM}^{p,k})$ has a decomposition (5.6) where μ_1 is determined modulo $\bar\partial\mathcal{H}om(\mathcal{S}, \mathcal{PM}^{p,k-1})$ and γ is in $\mathcal{H}om(\mathcal{S}, \mathcal{C}^{p,k-1})$.

[2] In [11] the Dickenstein-Sessa decomposition (5.5) was proved for complete intersections Z and in [12, Proposition 5.2] for arbitrary Z of pure dimension.

Remark 5.8. From [3, Theorem 7.1], see also [3, Remark 4], it follows that the second mapping in (5.7) is realized by

$$\xi \mapsto \xi \cdot R_k,$$

for ξ in $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}(E_k), \Omega^p)$ such that $f_{k+1}^*\xi = 0$. □

Let us now assume that $\mathcal{S} = \mathcal{O}/\mathcal{J}$, where $\mathcal{J}$ is an ideal sheaf of pure codimension ν, and let Z be the associated zero set. It turns out that $\mathcal{CH}_p^Z$ is precisely the sheaf of $\bar\partial$-closed currents in $\mathcal{PM}_{p,\nu}^Z$, see e.g., [3]. Taking $k = \nu$ in the last equality in (5.7) we get, in view of the dimension principle, that

$$\mathcal{H}^k(\mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{J}, \mathcal{PM}^{p,\bullet}), \bar\partial) = \mathcal{K}er\,_{\bar\partial}\mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{J}, \mathcal{PM}^{p,\nu}) = \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{J}, \mathcal{CH}_p^Z),$$

cf., e.g., [3, Theorem 1.5] and [9]. Notice that $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{J}, \mathcal{CH}_p^Z)$ is the sheaf of Coleff-Herrera currents μ such that $\mathcal{J}\mu = 0$.

Let $\mathcal{I} \subset \mathcal{O}$ be the radical ideal associated with Z, i.e., the sheaf of functions that vanish on Z. If μ is any current of bidegree (p, ν) with support on Z, i.e., in $\mathcal{C}_{p,\nu}^Z$, then locally $\mathcal{J}\mu = 0$ if $\mathcal{J} = \mathcal{I}^m$ for sufficiently large m. Applying (5.8) to $\mathcal{J} = \mathcal{I}^m$ for $m = 1, 2, \ldots$, and $k = \nu$, we get the Dickenstein-Sessa decomposition (5.5).

Notice that $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{J}, \mathcal{PM}^{p,k})$ is the subsheaf of μ in $\mathcal{PM}^{p.k}$ such that $\mathcal{J}\mu = 0$. In particular such μ must have support on Z. Arguing as in the case $k = \nu$ above we get from (5.8) the following extension of (5.5) for general k.

Corollary 5.9 (Generalized Dickenstein-Sessa decomposition). *If μ is a $\bar\partial$-closed (p, k)-current with support on Z, then there is a decomposition* (5.6)*, where μ_1 is in $\mathcal{K}er_{\bar\partial}\mathcal{PM}_{p,k}^Z$, determined modulo $\bar\partial\mathcal{PM}_{p,k-1}^Z$, and γ has support on Z.*

In [17] Samuelsson Kalm proves a generalization of this decomposition, where $\mathcal{PM}_{p,k}^Z$ are replaced by certain subsheaves.

6. Proof of Theorem 5.1

We first consider the case when $\mathcal{S} = \mathcal{O}/(f)$ as in Examples 5.2 and 5.3. We will provide an argument in this special case that admits an extension to a proof of Theorem 5.1. Recall from Example 2.15 that if μ is in $\mathcal{W}_X$, then $f(1/f)\mu = \mathbf{1}_{\{f\neq 0\}}\mu = \mu$. Notice that also $f\bar\partial\big((1/f)\mu\big) = \bar\partial\big(f(1/f)\mu\big) = \bar\partial\mu$. In view of (5.4) and Proposition 5.5 (and the dimension principle if μ is $(*, 0)$), an arbitrary pseudomeromorphic current with compact support can be written

$$\mu = \mu_1 + \bar\partial\mu_2,$$

where μ_j are in $\mathcal{W}_X$. If

$$\nu = \frac{1}{f}\mu_1 + \bar\partial\big(\frac{1}{f}\mu_2\big),$$

thus $f\nu = \mu$.

We now turn our attention to a general locally free resolution (5.1). If $k \ge 1$, $\mu \in \mathcal{H}om_{\mathcal{O}}(\mathcal{O}(E_k), \mathcal{PM})$, and $f_{k+1}^*\mu = 0$ in a neighborhood of a point x, we must then find a pseudomeromorphic current $\nu \in \mathcal{H}om_{\mathcal{O}}(\mathcal{O}(E_{k-1}), \mathcal{PM})$ in a neighborhood of x such that $f_k^*\nu = \mu$.

With no loss of generality we may assume that μ has bidegree $(n, *)$, cf. [7, Theorem 3.5], and compact support in an open ball $\mathcal{U} \subset \mathbb{C}^n$ with center x, and that $f^*\mu = 0$. We will construct integral operators $\mathcal{A}$ and $\mathcal{F}$ for such μ such that

$$\mu = f^*\mathcal{A}\mu + \mathcal{F}\mu \tag{6.1}$$

and $\mathcal{A}\mu$ and $\mathcal{F}\mu$ are pseudomeromorphic. If $\mathcal{F}\mu = 0$, then $\nu = \mathcal{A}\mu$ thus solves our problem. This is in fact the case if the support of μ is discrete. In general, unfortunately $\mathcal{F}\mu$ does not vanish, or at least we cannot prove it. However, we can prove that $\mathcal{F}\mu$ has substantially "smaller" support than μ, see Lemma 6.2 below. In particular, $\operatorname{supp}(\mathcal{F}\mu) \subset \operatorname{supp}\mu$. Since $f^*\mu = 0$, (6.1) implies that $f^*\mathcal{F}\mu = 0$. Therefore we can apply (6.1) to $\mathcal{F}\mu$, and then

$$\mu = f^*\big(\mathcal{A}\mu + \mathcal{A}\mathcal{F}\mu\big) + \mathcal{F}^2\mu.$$

Again $f^*\mathcal{F}^2\mu = 0$ so we can iterate and in view of Lemma 6.4 below we obtain a solution $\nu = \mathcal{A}(\mu + \mathcal{F}\mu + \mathcal{F}^2\mu + \cdots)$ to $f^*\nu = \mu$ after a finite number of steps. Thus Theorem 5.1 follows. It thus remains to construct integral operators $\mathcal{A}$ and $\mathcal{F}$ with the desired properties.

6.1. **The integral operators $\mathcal{A}$ and $\mathcal{F}$ in $\mathcal{U}$.** Let us recall some facts from [1, Section 9] about integral representation in $\mathcal{U}$. Let $F \to \mathcal{U}$ be a holomorphic vector bundle and assume that $g = g_{0,0} + \cdots + g_{n,n}$ is a smooth form in $\mathcal{U}_\zeta \times \mathcal{U}_z$, where lower indices denote bidegree, such that g takes values in $\operatorname{Hom}(F_\zeta, F_z)$ at the point (ζ, z). We will also assume that g has no holomorphic differentials[3] with respect to z. Let δ_ζ denote interior multiplication with the vector field

$$2\pi i \sum_1^n \zeta_j \frac{\partial}{\partial \zeta_j}$$

and let $\nabla_\zeta = \delta_\zeta - \bar\partial$. We say that g is a *weight* (with respect to F) if $\nabla_\zeta g = 0$ and if in addition $g_{0,0} = I_F$, the identity mapping on F, on the diagonal in $\mathcal{U} \times \mathcal{U}$.

From now on we only consider the components of the form B from Section 5.2 above with no holomorphic differentials with respect to z. For simplicity we denote it by B as well. Let g be a weight with respect to F. For test forms $\phi(\zeta)$ of bidegree $(0,*)$ in $\mathcal{U}$ with values in F we have the Koppelman formula

$$\phi(z) = \bar\partial \int_\zeta (g\wedge B)_{n,n-1}\wedge\phi + \int_\zeta (g\wedge B)_{n,n-1}\wedge\bar\partial\phi + \int_\zeta g_{n,n}\wedge\phi, \quad z \in \mathcal{U}. \tag{6.2}$$

The case when F is a trivial line bundle is proved in [1, Section 9] and the general case is verified in exactly the same way.

Consider now our (locally) free resolution (5.1) in $\mathcal{U}$, choose Hermitian metrics on the vector bundles E_k, and let U_ϵ and R_ϵ be the associated currents as in Section 5.1 above. Let H be a Hefer morphism with respect to E that is holomorphic in both ζ and z. See, e.g., [5, Section 5] for the definition and basic properties of Hefer morphisms; in particular H is an $\operatorname{End} E$-valued holomorphic form. Then

$$g_\epsilon := f(z)HU_\epsilon + HU_\epsilon f + HR_\epsilon$$

is a smooth weight with respect to E. Here $f, U_\epsilon, R_\epsilon$ stands for $f(\zeta), U_\epsilon(\zeta), R_\epsilon(\zeta)$. Let g_ϵ^k be the component of g_ϵ that is a weight with respect to E_k. For test forms ϕ of bidegree $(0,*)$ with values in E_k we have then, in view of (6.2), the representation

$$\phi(z) = \bar\partial \int_\zeta (g_\epsilon^k\wedge B)_{n,n-1}\wedge\phi + \int_\zeta (g_\epsilon^k\wedge B)_{n,n-1}\wedge\bar\partial\phi + \int_\zeta (g_\epsilon^k)_{n,n}\wedge\phi. \tag{6.3}$$

[3] We are only interested here in integral formulas for forms of bidegree $(0,*)$ and therefore we can take $d\zeta_j$ instead of $d\eta_j$ in [1].

By the way, the last term vanishes unless ϕ has bidegree $(0,0)$, since g_ϵ^k contains no anti-holomorpic differentials with respect to z so that $(g_\epsilon^k)_{n,n}$ must have bidegree (n,n) with respect to ζ.

Let R^k and R_ϵ^k be the components of R and R_ϵ, respectively, that take values in $\mathrm{Hom}\,(E_k, E_*)$, and define U^k and U_ϵ^k analogously. Let H^k be the component of H that takes values in $\mathrm{Hom}\,(E_*, E_k)$. Then

$$g_\epsilon^k = f_{k+1}(z)H^{k+1}U_\epsilon^k + H^k U_\epsilon^{k-1} f_k + H^k R_\epsilon^k. \tag{6.4}$$

Now assume that μ is a pseudomeromorphic (n,q)-current with compact support in $\mathcal{U}$ and taking values in E_k^* for $k \geq 1$. Integrating μ against (6.3) for test forms ϕ with values in E_k we get

$$\mu(\zeta) = \int_z (g_\epsilon^k \wedge B)^*_{n,n-1}\wedge\bar\partial\mu + \bar\partial\int_z (g_\epsilon^k\wedge B)^*_{n,n-1}\wedge\mu + \int_z (g_\epsilon^k)^*_{n,n}\wedge\mu$$

(up to signs). Assuming that $f_{k+1}^*\mu = 0$ and plugging in (6.4) we get

$$\begin{aligned}\mu(\zeta) = f_k^*(\zeta)\int_z (H^k U_\epsilon^{k-1}B)^*_{n,n-1}\wedge\bar\partial\mu + \bar\partial\Big(f_k^*(\zeta)\wedge\int_z (H^k U_\epsilon^{k-1}B)^*_{n,n-1}\wedge\mu\Big) + \\ f_k^*(\zeta)\int_z (H^k U_\epsilon^{k-1})^*_{n,n}\wedge\mu + \int_z (H^k R_\epsilon^k\wedge B)^*_{n,n-1}\wedge\bar\partial\mu \\ \bar\partial\int_z (H^k R_\epsilon^k)\wedge B)^*_{n,n-1}\wedge\mu + \int_z (H^k R_\epsilon^k)^*_{n,n}\wedge\mu.\end{aligned} \tag{6.5}$$

To simplify notation we now suppress the lower indices, and instead tacitly understand that we only consider products of terms such that the total bidegrees add up to the desired one. We can then write (6.5) more suggestively as

(6.6)

$$\begin{aligned}\mu(\zeta) = f_k^*(\zeta)(U_\epsilon^{k-1})^*(\zeta)\int_z (H^k)^*\wedge B\wedge\bar\partial\mu + \bar\partial\Big(f_k^*(\zeta)\wedge(U_\epsilon^{k-1})^*(\zeta)\int_z (H^k)^*\wedge B\wedge\mu\Big) + \\ f_k^*(\zeta)(U_\epsilon^{k-1})^*(\zeta)\int_z (H^k)^*\wedge\mu + (R_\epsilon^k)^*(\zeta)\int_z (H^k)^*\wedge B\wedge\bar\partial\mu + \\ \bar\partial\Big((R_\epsilon^k)^*(\zeta)\int_z (H^k)^*\wedge B\wedge\mu\Big) + (R_\epsilon^k)^*(\zeta)\int_z (H^k)^*\wedge\mu.\end{aligned}$$

Since f_k^* and $\bar\partial$ have odd order with respect to the superstructure, cf. Section 5.1, they anti-commute and thus we can we can write (6.6) as

$$\mu(\zeta) = f_k^*(\zeta)\mathcal{A}_\epsilon\mu(\zeta) + \mathcal{F}_\epsilon\mu(\zeta),$$

where

$$\begin{aligned}\mathcal{A}_\epsilon\mu = (U_\epsilon^{k-1})^*(\zeta)\int_z (H^k)^*\wedge B\wedge\bar\partial\mu - \\ \bar\partial\Big((U_\epsilon^{k-1})^*(\zeta)\int_z (H^k)^*\wedge B\wedge\mu\Big) + (U_\epsilon^{k-1})^*(\zeta)\int_z (H^k)^*\wedge\mu\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_\epsilon\mu(\zeta) = (R_\epsilon^k)^*(\zeta)\int_z (H^k)^*\wedge B\wedge\bar\partial\mu + \\ \bar\partial\Big((R_\epsilon^k)^*(\zeta)\int_z (H^k)^*\wedge B\wedge\mu\Big) + (R_\epsilon^k)^*(\zeta)\int_z (H^k)^*\wedge\mu.\end{aligned}$$

Lemma 6.1. *Each term in $\mathcal{A}_\epsilon\mu$ and $\mathcal{F}_\epsilon\mu$ tends to a pseudomeromorphic current when $\epsilon\to 0$.*

We denote the limits of $\mathcal{A}_\epsilon\mu$ and $\mathcal{F}_\epsilon\mu$ by $\mathcal{A}\mu$ and $\mathcal{F}\mu$, respectively.

Proof. In view Proposition 5.6,

$$\gamma := \int_z (H^k)^*\wedge B\wedge\bar\partial\mu$$

is in $\mathcal{W}(\mathcal{U})$, since B is almost semi-meromorphic and H is smooth. Since U is almost semi-meromorphic, by Theorem 2.11 we can form the pseudomeromorphic current $T=(U^{k-1})^*\wedge\gamma$, which is in $\mathcal{W}(\mathcal{U})$ in view of Proposition 2.13 (with $Z=X=\mathcal{U}$). Since $U_\epsilon^{k-1}=\chi(|h|^2/\epsilon)U^{k-1}$, where $Z(h)=ZSS(U)=ZSS(U_k)$, cf. [5, Section 2], it follows that $(U_\epsilon^{k-1})^*(\zeta)\wedge\gamma\to T$, cf. (2.13). Thus the first term in $\mathcal{A}_\epsilon\mu$ tends to a pseudomeromorphic current in $\mathcal{U}$. Moreover, from the definition (5.2) for R_ϵ it follows that the limit of the first term in $\mathcal{F}_\epsilon$ equals $r(U^k)\wedge\gamma=R^k\wedge\gamma$, cf. (2.18).

Since $\bar\partial$ preserves pseudomeromorphicity, the same argument works for the other terms in $\mathcal{A}_\epsilon\mu$ and $\mathcal{F}_\epsilon$. □

Recall that since (5.1) is exact the current R^k vanishes when $k\geq 1$, cf. Section 5.1. Unfortunately, from this we cannot conclude that the limit $\mathcal{F}\mu$ vanishes in general; cf. [7, Example 4.23]. However, as we now shall see, the support of $\mathcal{F}\mu$ is small in the following sense:

Lemma 6.2. (i) *The support of $\mathcal{F}\mu$ is contained in the support of μ.*

(ii) *Assume that μ has compact support on a submanifold $Z\subset\mathcal{V}$ of codimension $\geq p$, where $\mathcal{V}$ is an open subset of $\mathcal{U}$. Then there is a cqa set $V\subset Z$ of codimension $\geq p+1$ such that* $\operatorname{supp}(\mathcal{F}\mu)\subset V$.

Proof. First notice that $(R_\epsilon^k)^*(\zeta)(H^k)^*\wedge\mu$ is a smooth form times the tensor product of $(R_\epsilon^k)^*$ and μ. It follows that the last term in the definition of $\mathcal{F}_\epsilon\mu$ tends to 0, since $R^k=0$. We thus have to deal with the first two terms.

To prove (i) we note that if $\mu=0$ close to $x\in\mathcal{U}$, then

$$\int_z (H^k)^*\wedge B\wedge\bar\partial\mu$$

is smooth close to x, since B is smooth outside the diagonal in $\mathcal{U}\times\mathcal{U}$. Thus, close to x, the first term in $\mathcal{F}_\epsilon\mu$ tends to $(R^k)^*$ times a smooth form and thus the limit vanishes since $R^k=0$. The second term in $\mathcal{F}_\epsilon\mu$ tends to 0 for the same reason.

To prove (ii), let us consider the limit

$$T\mu=\lim_{\epsilon\to 0}(R_\epsilon^k)^*(\zeta)\wedge(H^k)^*\wedge B\wedge\mu, \tag{6.7}$$

where, as before, we use the simplified notation and in fact only take into account terms of $(R_\epsilon^k)^*(\zeta)(H^k)^*\wedge B$ of total bidegree $(n,n-1)$. Note that $T\mu$ is the product of a residue of an almost semimeromorphic current $(R^k)^*$ and a pseudomeromorphic current, cf. Definition 2.17, and thus is pseudomeromorphic. Let

$$\mathcal{T}\mu=\int_z T\mu.$$

Then

$$\mathcal{F}\mu=\mathcal{T}\bar\partial\mu+\bar\partial(\mathcal{T}\mu).$$

Thus it is enough to prove (ii) for $\mathcal{T}\mu$ instead of $\mathcal{F}\mu$.

Lemma 6.3. *Assume that μ has compact support on a subvariety $W \subset \mathcal{V}$ of codimension p and*

$$\mu = \alpha \wedge \tilde{\mu}, \tag{6.8}$$

where α is smooth and $\tilde{\mu}$ has support on W and bidegree $(*, p)$. Then $\mathcal{T}\mu = 0$.

Proof. Notice, in view of the proof of (i) above, that (i) holds for $\mathcal{T}$ instead of $\mathcal{F}$. Therefore suffices to show that $\mathcal{T}\mu = 0$ in $\mathcal{V}$. Outside the diagonal in $\mathcal{V} \times \mathcal{V}$, the current B is smooth, and hence $T\mu$ vanishes, as it is a smooth form times the tensor product of $(R^k)^*$ and μ, and $R^k = 0$. If μ is of the form (6.8), therefore (6.7) is a smooth form α times a pseudomeromorphic current with support on $(\mathcal{V} \times W) \cap \Delta$ that is a subvariety of $\mathcal{V} \times \mathcal{V}$ of codimension $\geq n + p$. On the other hand the antiholomorphic degree is $n - 1 + p$. Thus $T\mu$ must vanish in view of the dimension principle. It follows that $\mathcal{T}\mu$ vanishes. □

We can now conclude the proof of (ii) for $\mathcal{T}$. We can cover $\mathcal{V}$ by finitely many neighborhoods $\mathcal{V}_j$ such that $\mathcal{V}_j$ and $Z \cap \mathcal{V}_j$ are as in Proposition 4.1. Moreover we can find smooth cutoff functions χ_j with support in $\mathcal{V}_j$ such that $\mu = \sum_j \chi_j \mu$. Then by Corollary 4.2 there are cqa sets $V_j \subset \mathcal{V}_j \cap Z$ of codimension $\geq p+1$ such that $\chi_j \mu$ is of the form (6.8) in $\mathcal{V}_j \setminus V_j$.

Fix j, pick $x \in Z \setminus V_j$, let $\mathcal{W} \subset \mathcal{V}_j \setminus V_j$ be a neighborhood of x, and let χ be a cutoff function with compact support in $\mathcal{W}$ that is 1 in a neighborhood of x. Then $\chi\chi_j\mu$ is of the form (6.8) and thus $\mathcal{T}(\chi\chi_j\mu) = 0$ by Lemma 6.3. Next, since $(1-\chi)\chi_j\mu = 0$ in $\mathcal{W}$, (i) implies that $\mathcal{T}\big((1-\chi)\chi_j\mu\big) = 0$ in $\mathcal{W}$. Since $\mathcal{T}$ is linear,

$$\mathcal{T}(\chi_j\mu) = \mathcal{T}(\chi\chi_j\mu) + \mathcal{T}\big((1-\chi)\chi_j\mu\big) = 0$$

in $\mathcal{W}$. Since x was arbitrary we conclude that $\operatorname{supp}(\mathcal{T}(\chi_j\mu)) \subset V_j$. Now the finite union $V = \cup_j V_j$ is a cqa set of codimension $\leq d$ and $\operatorname{supp}(\mathcal{T}\mu) \subset V$. □

Lemma 6.4. *Given $m \in \mathbb{N}$, there is a constant c_m such that if μ is a pseudomeromorphic current with support on a cqa set of dimension $\leq m$, then $\mathcal{F}^j\mu$ vanishes if $j \geq c_m$.*

In fact, it follows from the proof below that we can choose c_m as $2^{m+1} - 1$.

Proof. First assume that $m = 0$. By Example 3.9, a cqa set of dimension 0 is a variety of dimension 0, and thus $\mathcal{F}\mu$ vanishes by Lemma 6.2 (ii). It follows that the lemma holds in this case with $c_0 = 1$.

Now assume that the lemma holds for $m = \ell$. Moreover, assume that μ is a pseudomeromorphic current with support on a cqa set $V \subset \mathcal{U}$ of dimension $\ell + 1$. Let $V' \subset V$ be a cqa set of dimension $\leq \ell$ as in Lemma 3.8. We claim that $\mathcal{F}^{c_\ell+1}\mu$ has support on V'. Taking this for granted we get that

$$\mathcal{F}^{c_\ell}(\mathcal{F}^{c_\ell+1}\mu) = 0,$$

by the induction hypothesis. Thus the lemma holds for $m = \ell+1$ with $c_{\ell+1} = 2c_\ell + 1$, and hence by induction for all m.

It remains to prove the claim. Take $x \in V \setminus V'$, let $\mathcal{V} \subset \mathcal{U}$ be a neighborhood of x as in Lemma 3.8, so that $V \cap \mathcal{V} \subset W = \cup W_j$, where the $W_j \subset \mathcal{V}$ are submanifolds of dimension $\leq \ell + 1$, and let χ be a cutoff function with compact support in $\mathcal{V}$ that is 1 in a neighborhood $\tilde{\mathcal{V}}$ of x. Let $\mu_j = \mathbf{1}_{W_j}\mu$. Then

$$\chi\mu = \sum_j \chi\mu_j + \nu,$$

where ν is a pseudomeromorphic current with

$$\operatorname{supp}\nu \subset W_{\text{sing}} \cap \operatorname{supp}\chi =: A;$$

by Example 3.6 A is a cqa set of dimension $\le \ell$. By Lemma 6.2 (i) $\operatorname{supp}(\mathcal{F}\nu) \subset A$, and by Lemma 6.2 (iii) there are cqa sets $V_j \subset W$ of dimension $\le \ell$ such that $\operatorname{supp}(\mathcal{F}\mu_j) \subset V_j$. Thus, since $\mathcal{F}$ is linear,

$$\operatorname{supp}\big(\mathcal{F}(\chi\mu)\big) \subset \bigcup_j V_j \cup A =: \widetilde{A}.$$

Since a finite union of cqa sets of dimension $\le \ell$ is a cqa set of dimension $\le \ell$, $\widetilde{A}$ is a cqa set of dimension $\le \ell$. Therefore, using that the lemma holds for $m = \ell$,

$$\mathcal{F}^{c_\ell}\big(\mathcal{F}(\chi\mu)\big) = 0.$$

Next, since $(1-\chi)\mu = 0$ in $\tilde{\mathcal{V}}$, Lemma 6.2 (i) gives that $\mathcal{F}^\kappa\big((1-\chi)\mu\big) = 0$ in $\tilde{\mathcal{V}}$ for any $\kappa \ge 1$. We conclude that

$$\mathcal{F}^{c_\ell+1}\mu = \mathcal{F}^{c_\ell}\big(\mathcal{F}(\chi\mu)\big) + \mathcal{F}^{c_\ell+1}\big((1-\chi)\mu\big) = 0$$

in $\tilde{\mathcal{V}}$. Since x was arbitrary this proves the claim. □

References

[1] M. Andersson: Integral representation with weights I. *Math. Ann.*, **326**, (2003), 1–18.

[2] M. Andersson: Residue currents and ideals of holomorphic functions. *Bull. Sci. Math.*, **128**, (2004), 481–512.

[3] M. Andersson: Coleff-Herrera currents, duality, and Noetherian operators. *Bull. Soc. Math. France*, Bull. Soc. Math. France 139 (2011), 535–554.

[4] M. Andersson, H. Samuelsson: A Dolbeault-Grothendieck lemma on complex spaces via Koppelman formulas. *Invent. Math.* **190** (2012), 261–297.

[5] M. Andersson, E. Wulcan: Residue currents with prescribed annihilator ideals. *Ann. Sci. École Norm. Sup.*, **40** (2007), 985–1007.

[6] M. Andersson, E. Wulcan: Decomposition of residue currents. *J. Reine Angew. Math.*, **638** (2010), 103–118.

[7] M. Andersson, E. Wulcan: Direct images of semi-meromorphic currents. *Ann. Inst. Fourier (Grenoble)*, **68** (2018), 875–900.

[8] E. Bierstone & P. Milman: Semianalytic and subanalytic sets. *Publ. math. I.H.É.S.* **67** (1988), 5–42.

[9] J-E. Björk: Residues and $\mathcal{D}$-modules. *The legacy of Niels Henrik Abel* 605651, Springer, Berlin, 2004.

[10] N. Coleff & M. Herrera: *Les courants résiduels associés à une forme méromorphe. (French) [The residue currents associated with a meromorphic form] Lecture Notes in Mathematics* **633**. Springer, Berlin, 1978. x+211 pp.

[11] A. Dickenstein, C. Sessa: *Invent. Math.*, **80** (1985), 417–434.

[12] A. Dickenstein, C. Sessa: Résidus de formes méromorphes et cohomologie modérée, *Géométrie complexe, Paris, 1992* Actualités Sci. Indust. **1438**, 1996, 35–59.

[13] M. Herrera & D. Liebermann: Residues and principal values on complex spaces. *Math. Ann.*, **194** (1971), 259–294.

[14] R. Lärkäng, H. Samuelsson Kalm: Various approaches to products of residue currents. *J. Funct. Anal.* **264** (2013), 118–138.

[15] B. Malgrange: Sur les fonctions différentiables et les ensembles analytiques *Bull. Soc. Math. France* **91** (1963), 113–127

[16] M. Passare & A. Tsikh & A. Yger: Residue currents of the Bochner-Martinelli type. *Publ. Mat.* **44** (2000), 85–117.

[17] H. Samuelsson Kalm: Integral representation of moderate cohomology. *preprint, arXiv:1703.03661,* to appear in Ann. Fac. Sci. Toulouse Math.

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