GREEN FUNCTIONS, SEGRE NUMBERS, AND KING'S FORMULA

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Abstract. Let $\mathcal{J}$ be a coherent ideal sheaf on a complex manifold X with zero set Z, and let G be a plurisubharmonic function such that $G = \log |f| + \mathcal{O}(1)$ locally at Z, where f is a tuple of holomorphic functions that defines $\mathcal{J}$. We give a meaning to the Monge-Ampère products $(dd^cG)^k$ for $k = 0, 1, 2, \ldots$, and prove that the Lelong numbers of the currents $M_k^{\mathcal{J}} := \mathbf{1}_Z (dd^cG)^k$ at x coincide with the so-called Segre numbers of $\mathcal{J}$ at x, introduced independently by Tworzewski, Gaffney-Gassler, and Achilles-Manaresi. More generally, we show that $M_k^{\mathcal{J}}$ satisfy a certain generalization of the classical King formula.

1. Introduction

Let X be a complex manifold of dimension n and let $\mathcal{J} \to X$ be a coherent ideal sheaf with variety Z. Given a point $x \in X$, Tworzewski, [24], and Gaffney and Gassler, [14], have independently introduced a list of numbers, $e_0(\mathcal{J}, X, x), \ldots, e_n(\mathcal{J}, X, x)$, that we, following [14], call the *Segre numbers* at x. They are a generalization of the classical local intersection number at x in case the ideal $\mathcal{J}_x$ is a complete intersection. The definition in both papers is based on a local variant of the Stückrad-Vogel procedure, [23]. In [1, 2] is given an algebraic definition of these numbers generalizing the classical Hilbert-Samuel multiplicity of $\mathcal{J}$ at x.

In this paper we show that if $\mathcal{J}$ is generated by global bounded functions there is a canonical global representation of the Segre numbers of $\mathcal{J}$ as the Lelong numbers (of restrictions to Z) of Monge-Ampère masses of the *Green function $G = G_{\mathcal{J}}$ with poles along $\mathcal{J}$*. This function was introduced by Rashkovskii-Sigurdsson in [20, Definition 2.2] as a generalization of the classical Green function G_a with pole at a point $a \in X$. It is defined as the supremum over the class $\mathcal{F}_{\mathcal{J}}$ of all negative psh (plurisubharmonic) functions u on X that locally satisfy $u \le \log |f| + C$, where $f = (f_1, \ldots, f_m)$ is a tuple of local generators of $\mathcal{J}$ and C is a constant.

Note that even if X is hyperconvex there might not exist non-trivial functions in $\mathcal{F}_{\mathcal{J}}$. For example, if X is the ball in $\mathbb{C}$, and $\mathcal{J}$ is the radical ideal of functions vanishing at points $a_1, a_2, \ldots \in X$, then there are negative psh functions with poles at a_j if and only if a_j satisfy the the Blaschke condition. However, if $\mathcal{J}$ is globally generated by bounded functions f_j, then $\log |f| + C$ is itself in $\mathcal{F}_{\mathcal{J}}$ for some constant C. Then locally G is of the form

$$G = \log |f| + h, \tag{1.1}$$

Date: January 20, 2015.

1991 *Mathematics Subject Classification.* 32U35, 32U25, 32U40, 32B30, 14B05.

Key words and phrases. Green function, Segre numbers, Monge-Ampère products, King's formula.

The authors were partially supported by the Swedish Research Council.

where h is locally bounded, see [20, Theorem 2.8]. In particular, the unbounded locus of G equals Z and thus the Monge-Ampère type products

$$(dd^cG)^k, \quad k \leq p := \operatorname{codim} Z \tag{1.2}$$

are well-defined, see, e.g., [9, Theorem III.4.5]. Here and throughout $d^c = (i/2\pi)(\bar\partial - \partial)$. By *Demailly's comparison formula for Lelong numbers*, [10, Theorem 5.9],

$$\ell_x(dd^cG)^k = \ell_x(dd^c \log|f|)^k \tag{1.3}$$

for $x \in X$, where ℓ_x denotes the Lelong number at x. Moreover, recall that *King's formula*, [15], asserts that $(dd^c \log|f|)^p$ admits the Siu decomposition, [21],

$$(dd^c \log|f|)^p = \sum \beta_j [Z_j^p] + R, \tag{1.4}$$

cf. [10, Section 6]. Here $[Z_j^p]$ are the currents of integration along the irreducible components Z_j^p of codimension p of Z, β_j are the generic Hilbert-Samuel multiplicities of f along Z_j^p, see, e.g. [13, Chapter 4.3]. In fact, the remainder term R has integer Lelong numbers, see, e.g. [4, Theorem 1.1], and therefore the set where R has positive Lelong numbers is an analytic set of codimension $> p$. From (1.3) and (1.4) one deduces that

$$(dd^cG)^p = \sum \beta_j [Z_j^p] + R, \tag{1.5}$$

where β_j and Z_j^p are as above, and R has the same Lelong numbers as R in (1.4), cf. the proof of Theorem 2.8 in [20]. In particular, if Z is a point a, then $(dd^cG)^n = \sum \beta[a] + R$, where $[a]$ is the point evaluation at a and β is the Hilbert-Samuel multiplicity of $\mathcal{J}$. This generalizes the fact that $(dd^cG_a)^n = [a]$, [11, page 520]. The (Lelong numbers of the) Monge-Ampère products (1.2) are related to the integrability index of G (and thus the log-canoncial threshold of $\mathcal{J}$), see, e.g., [12, 19, 22]; in particular, Demailly-Pham [12] recently gave a sharp estimate of the integrability index of G in terms of the Lelong numbers of (1.2) for all $k \leq p$.

Recall that (1.2) can be defined inductively as

$$dd^c(G(dd^cG)^{k-1}). \tag{1.6}$$

In this paper we give meaning to $(dd^cG)^k$ for any k if G is any psh function of the form (1.1): Inductively we show that

$$G\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1}$$

has locally finite mass and define

$$(dd^cG)^k := dd^c(G\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1}),$$

see Proposition 4.1. When $k \leq p$ it follows from the dimension principle for closed positive currents, cf. Lemma 3.1 below, that $\mathbf{1}_Z(dd^cG)^{k-1} = 0$ and so our definition coincides with the classical one for $k \leq p$. Our definition is modeled on the paper [3] by the first author, in which currents $(dd^c \log|f|)^k$ are defined for all k inductively as above. In fact, $(dd^c \log|f|)^k$ can also be defined as a certain limit of smooth forms coming from regularizations of $\log|f|$:

$$\lim_{\epsilon\to 0}(dd^c \log(|f|^2+\epsilon)^{1/2})^k = (dd^c \log|f|)^k \tag{1.7}$$

for any k, see [3, Proposition 4.4]. However, one cannot hope for such a suggestive definition of $(dd^cG)^k$ in general, cf. Example 4.2. Also, our definition of $(dd^cG)^k$ does not coincide with the *non-pluripolar product* of dd^cG, as introduced in [6, 7], since our $(dd^cG)^k$ charges pluripolar sets in general, cf. the text after the proof of Proposition 4.1.

Our main result is the following generalization of (1.5). Let $\pi^+ \colon X^+ \to X$ be the normalization of the blow-up of X along $\mathcal{J}$ and let W_j be the various irreducible components of the exceptional divisor in X^+. Recall that the *(Fulton-MacPherson) distinguished varieties* of $\mathcal{J}$ are the subvarieties $\pi^+(W_j)$ of X, see, e.g., [16, Chapter 10.5]. In particular, the distinguished varieties of codimension p are precisely the irreducible components of Z of codimension p.

Theorem 1.1. *Let X be an n-dimensional complex manifold, let $\mathcal{J}$ be a coherent ideal sheaf on X generated by global bounded functions, and let G be the Green function with poles along $\mathcal{J}$. Moreover, let Z be the variety of $\mathcal{J}$ and Z_j^k the Fulton-MacPherson distinguished varieties of $\mathcal{J}$ of codimension k. Then*

$$M_k^{\mathcal{J}} := \mathbf{1}_Z (dd^cG)^k = \sum_j \beta_j^k [Z_j^k] + N_k^{\mathcal{J}} =: S_k^{\mathcal{J}} + N_k^{\mathcal{J}}, \tag{1.8}$$

where the β_j^k are positive integers and the $N_k^{\mathcal{J}}$ are positive closed currents. The numbers $n_k(\mathcal{J}, X, x) := \ell_x(N_k^{\mathcal{J}})$ are nonnegative integers that only depend on the integral closure class of $\mathcal{J}$ at x, and the set where $n_k(\mathcal{J}, X, x) \geq 1$ has codimension at least $k+1$.

The Lelong numbers at x of $M_k^{\mathcal{J}}$ and $\mathbf{1}_{X\setminus Z}(dd^cG)^k$ are precisely the Segre number $e_k(\mathcal{J}, X, x)$ and the polar multiplicity $m_k(\mathcal{J}, X, x)$, respectively, of $\mathcal{J}_x$.

For the notion of polar multiplicities see Section 2. Notice that $M_k^{\mathcal{J}} = 0$ if $k < \operatorname{codim} Z$ and that $N_p^{\mathcal{J}} = 0$, cf., Lemma 3.1 below. Also, notice that (1.8) is the Siu decomposition, [21], of $M_k^{\mathcal{J}}$.

Remark 1.2. If $\mathcal{J}$ is generated by a global tuple f, then Theorem 1.1 holds with G replaced by any psh function of the form (1.1). □

The analogous statement to Theorem 1.1 when G is replaced by $\log|f|$, where f is a tuple of global generators, was proved by the authors and Samuelsson Kalm and Yger in [4, Theorem 1.1]. The case $k = p$ corresponds to the classical King formula, (1.4). The main idea in the proof of Theorem 1.1 is to prove that for any psh G of the form (1.1),

$$\ell_x(\mathbf{1}_Z(dd^cG)^k) = \ell_x(\mathbf{1}_Z(dd^c \log|f|)^k), \ \ell_x(\mathbf{1}_{X\setminus Z}(dd^cG)^k) = \ell_x(\mathbf{1}_{X\setminus Z}(dd^c \log|f|)^k) \tag{1.9}$$

for $x \in X$, see Lemma 6.1 below. Using this the theorem follows from the corresponding result in [4]. In some sense, (1.9) can be seen as a generalization of Demailly's comparison formula, (1.3), to higher k, but for the very special class of psh functions of the form (1.1).

In [4], X is allowed to be singular. Given that there is a proper definition of G when X is singular so that (1.1) still holds, the results in this paper will extend as well.

Theorem 1.1 gives us a canonical representation of the Segre numbers of $\mathcal{J}$ in the case when $\mathcal{J}$ is generated by global bounded functions. Let X be a, say hyperconvex, domain in $\mathbb{C}^n$, and let $\mathcal{J}$ be a coherent ideal sheaf on X. If we exhaust X by reasonable relatively compact subsets X_ℓ, for each ℓ we then have currents $M_k^{\mathcal{J}_\ell}$, $\mathcal{J}_\ell = \mathcal{J}|_{X_\ell}$ whose Lelong numbers at each point are the Segre numbers. If for some reason these currents converge to currents $M_k^{\mathcal{J}}$, we would have a canonical representation of the Segre numbers of $\mathcal{J}$ on X, cf. Remark 4.3.

This paper is organized as follows. In Section 2 we recall the construction of Vogel cycles and Segre numbers. In Section 4 we show that the currents $(dd^cG)^k$ are well-defined and discuss some properties. The proof of Theorem 1.1 occupies Section 6. In Sections 3 and 5 we give some background on psh functions and positive currents needed for the proofs.

Acknowledgment. The work on this paper started when Pascal Thomas was visiting Göteborg. We are grateful to him for interesting and inspiring discussions on the subject. We would also like to thank Zbigniew Błocki and David Witt Nyström for valuable discussions.

2. Segre numbers

We will briefly recall the construction of Segre numbers from [24, 14]. Throughout we will assume that X is a complex manifold of dimension n and that $\mathcal{J}$ is a coherent ideal sheaf on X with variety Z. Fix a point $x \in X$. A sequence $h = (h_1, h_2, \dots, h_n)$ in the local ideal $\mathcal{J}_x$ is called a *Vogel sequence of $\mathcal{J}$ at x* if there is a neighborhood $\mathcal{U} \subset X$ of x where the h_j are defined, such that

$$\text{codim}\,\big[(\mathcal{U}\setminus Z)\cap(|H_1|\cap\cdots\cap|H_k|)\big] = k \text{ or } \infty,\ \ k = 1,\dots,n; \tag{2.1}$$

here $|H_\ell|$ are the supports of the divisors H_ℓ defined by h_ℓ. Notice that if $f_1, \dots, f_m$ generate $\mathcal{J}_x$, any generic sequence of n linear combinations of the f_j is a Vogel sequence at x. Set $X_0 = X$, let X_0^Z denote the irreducible components of X_0 that are contained in Z, and let $X_0^{X\setminus Z}$ be the remaining components[1] so that

$$X_0 = X_0^Z + X_0^{X\setminus Z}.$$

By the Vogel condition (2.1), H_1 intersects $X_0^{X\setminus Z}$ properly. Set

$$X_1 = H_1 \cdot X_0^{X\setminus Z}$$

and decompose analogously X_1 into the components X_1^Z contained in Z and the remaining components $X_1^{X\setminus Z}$, so that $X_1 = X_1^Z + X_1^{X\setminus Z}$. Define inductively $X_{k+1} = H_{k+1}\cdot X_k^{X\setminus Z}$, X_{k+1}^Z, and $X_{k+1}^{X\setminus Z}$. Then

$$V^h := X_0^Z + X_1^Z + \cdots + X_n^Z$$

is the *Vogel cycle*[2] associated with the Vogel sequence h. Let V_k^h denote the components of V^h of codimension k, i.e., $V_k^h = X_k^Z$. The irreducible components of V^h that appear in any Vogel cycle, associated with a generic Vogel sequence at x, are called *fixed* components in [14]. The remaining ones are called *moving*. It turns out that the fixed Vogel components of $\mathcal{J}$ coincide with the distinguished varieties of $\mathcal{J}$, see, e.g., see [14] or [4].

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[1]Since we assume X is smooth and connected, X_0^Z is empty unless $\mathcal{J} = 0$, in which case it equals X.

[2]If $\mathcal{J}$ is the pullback to X of the radical sheaf of an analytic set A, this is precisely Tworzewski's algorithm, [24]. The notion Vogel cycle was introduced by Massey [17, 18]. For a generic choice of Vogel sequence the associated Vogel cycle coincides with the *Segre cycle* introduced by Gaffney-Gassler, [14], see Lemma 2.2 in [14].

It is proved in [14] and in [24] that the multiplicities $e_k(\mathcal{J}, X, x) := \mathrm{mult}_x V_k^h$ and $m_k(\mathcal{J}, X, x) := \mathrm{mult}_x X_k^{X\setminus Z}$ are independent of h for a generic h, where however "generic" depends on x, cf., Remark 2.1; these numbers are called the *Segre numbers* and *polar multiplicities*, respectively.

Remark 2.1. Recall that if W is an analytic cycle in X, then the Lelong number at $x \in X$ of the current of integration $[W]$ along W is precisely the multiplicity $\mathrm{mult}_x W$ of W at x.

Assume that x is a point for which $n_k(\mathcal{J}, X, x) \geq 1$ for some k, where we use the notation from Theorem 1.1. Moreover, let V^h be a generic Vogel cycle such that $\mathrm{mult}_x V_k^h = e_k(x)$. Then $V_k^h = S_k^{\mathcal{J}} + W$, where we have identified $S_k^{\mathcal{J}}$ in Theorem 1.1 with the corresponding cycle and W is a positive cycle of codimension k, such that $\mathrm{mult}_x W = n_k(\mathcal{J}, X, x)$. Since $n_k(\mathcal{J}, X, y) \geq 1$ only on a set of codimension $\geq k+1$, at most points y on V_k^h we have that $e_k(\mathcal{J}, X, y) = \mathrm{mult}_y(S_k^{\mathcal{J}})$ and hence $\mathrm{mult}_y V_k^h > e_k(\mathcal{J}, X, y)$. As soon as there is a moving component at x it is thus impossible to find a Vogel cycle that realizes the Segre numbers in a whole neighborhood of x. $\square$

In [4] Theorem 1.1 with G replaced by $\log|f|$ was proved by showing that $M_k^f := \mathbf{1}_Z (dd^c \log|f|)^k$ can be seen as a certain average (of currents of integration) of Vogel cycles. The fixed Vogel components then appear as the leading part $S_k^{\mathcal{J}}$ in the Siu decomposition of M_k^f, whereas the remainder term N_k^f is a mean value of the moving parts.

3. Preliminaries

Let μ be a positive closed current on X. Recall that if W is any subvariety, then $\mathbf{1}_W \mu$ and $\mathbf{1}_{X\setminus W}\mu$ are positive closed currents as well; this is the Skoda-El Mir theorem, see, e.g., [9, Chapter III.2.A].

Lemma 3.1. *Let μ be a positive closed current of bidegree (p,p) that has support on a subvariety of codimension k. If $k > p$ then $\mu = 0$. If $k = p$, then $\mu = \alpha_1[W_1] + \cdots + \alpha_\nu [W_\nu]$ where W_j are the irreducible components of W and $\alpha_j \geq 0$.*

We refer to the first part of Lemma 3.1 as the *dimension principle*. A proof can be found in [9, Chapter III.2.C].

If b is psh and locally bounded and T is any positive closed current, then $T \wedge (dd^c b)^k$ is a well-defined positive current for any k, and if b_j is a decreasing sequence of bounded psh functions converging pointwise to b, then

$$T\wedge(dd^c b)^k = \lim_j T\wedge(dd^c b_j)^k, \quad T\wedge b(dd^c b)^k = \lim_j T\wedge b_j (dd^c b_j)^k, \;\; k \leq n. \tag{3.1}$$

See, e.g., [9, Theorem III.3.7]. The case $T \equiv 1$ was first proved by Bedford and Taylor, [5].

Proposition 3.2. *Assume that v, b are psh and that b is (locally) bounded.*
(i) For $k \leq n-1$,

$$v(dd^c b)^k$$

has locally finite mass; more precisely, for any compact sets L, K, such that $L \subset int(K)$, we have

$$\|v(dd^c b)^k\|_L \leq C_{K,L} \|v\|_K (\sup_K |b|)^k. \tag{3.2}$$

(ii) Moreover, if the unbounded locus of v has Hausdorff dimension $< 2n-1$, then

$$dd^c(v(dd^cb)^k) = dd^cv\wedge(dd^cb)^k. \tag{3.3}$$

If v_j is a decreasing sequence of psh functions converging pointwise to v, then

$$v_j(dd^cb)^k \to v(dd^cb)^k, \tag{3.4}$$

and

$$dd^cv_j\wedge(dd^cb)^k \to dd^cv\wedge(dd^cb)^k \tag{3.5}$$

in the current sense.

The first part of Proposition 3.2 follows immediately from Proposition 3.11 in [9, Chapter III]. Moreover, Proposition 4.9 in loc. cit. applied to $u_1 = v$ and $u_j = b$ implies (3.4) and (3.5). If we choose v_j smooth, then

$$dd^c(v_j(dd^cb)^k) = dd^cv_j\wedge(dd^cb)^k.$$

Thus (3.3) follows from (3.4) and (3.5). In fact, the assumption about the Hausdorff dimension is not necessary; an elegant and quite direct argument has been communicated to us by Z. Błocki, [8].

Corollary 3.3. *If b is psh and (locally) bounded on X and W is an analytic variety of positive codimension, then for each $k \geq 0$,*

$$\mathbf{1}_W(dd^cb)^k = 0. \tag{3.6}$$

Proof. It is enough to consider the case when W is a smooth hypersurface. The general case follows by stratification. Since it is a local statement, we may choose coordinates $z = (z', w)$ so that $W = \{w = 0\}$. Notice that in a set $|w| \leq r, |z'| \leq r'$, we have that $\mathbf{1}_W(dd^cb)^k$ is the value at $\lambda = 0$ of

$$-(|w|^{2\lambda} - 1)(dd^cb)^k.$$

Since $|w|^{2\lambda} - 1$ is psh, (3.6) follows from (3.2) since the total mass of $|w|^{2\lambda} - 1$ tends to 0 when $\lambda \to 0$. $\square$

Lemma 3.4. *If b is psh and (locally) bounded on X and $i\colon Y \to X$ is a smooth submanifold, then for $k \leq n$,*

$$[Y]\wedge(dd^cb)^k = i_*(dd^ci^*b)^k, \quad [Y]\wedge b(dd^cb)^k = i_*\big(i^*b(dd^ci^*b)^k\big). \tag{3.7}$$

Proof. First assume that b is smooth. Then

$$\int_X [Y]\wedge(dd^cb)^k\wedge\xi = \int_Y (dd^ci^*b)^k\wedge i^*\xi = \int_X i_*\big((dd^ci^*b)^k\big)\wedge\xi$$

and similarly

$$\int_X [Y]\wedge b(dd^cb)^k\wedge\xi = \int_X i_*\big(i^*b(dd^ci^*b)^k\big)\wedge\xi,$$

so that (3.7) holds in this case. Now let b be bounded and psh and let b_j be a decreasing sequence of smooth psh functions converging pointwise to b. Now (3.7) follows from the smooth case and (3.1). $\square$

4. Higher Monge-Ampère products

Let G be a psh function of the form (1.1). We will give meaning to

$$(dd^cG)^k \tag{4.1}$$

by inductively defining it as $(dd^cG)^0 = 1$ and

$$(dd^cG)^k := dd^c\big(G\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1}\big), \quad k \geq 1. \tag{4.2}$$

Proposition 4.1 below asserts that this definition makes sense and that $(dd^cG)^k$ are positive and closed. As pointed out in the introduction this definition coincides with the iterative definition (1.6) for $k \leq p$.

Proposition 4.1. *Let X be a complex manifold of dimension n, let f be a tuple of global functions of X, let G be a psh function of the form* (1.1)*, and let G_j be a decreasing sequence of smooth psh functions in X converging pointwise to G. Assume that* (4.1) *is inductively defined via* (4.2) *for a fixed k. Then*

$$G\mathbf{1}_{X\setminus Z}(dd^cG)^k := \lim_j G_j\mathbf{1}_{X\setminus Z}(dd^cG)^k$$

has locally finite mass and does not depend on the choice of sequence G_j. Moreover $(dd^cG)^{k+1} = dd^c(G\mathbf{1}_{X\setminus Z}(dd^cG)^k)$ is positive and closed.

The proof below relies heavily on the fact that G is of the form (1.1). It could be interesting to investigate whether Proposition 4.1 holds for a wider class of psh functions G with unbounded locus Z.

Proof. Let $\pi\colon \widetilde{X} \to X$ be a smooth modification such that $\pi^*\mathcal{J}$ is principal and its divisor is of the form

$$D = \sum \alpha_j D_j, \tag{4.3}$$

where D_j are smooth hypersurfaces with normal crossings. In particular, then $\pi^*f = f^0 f'$, where f^0 is a section of the line bundle L_D that defines D and f' is a non-vanishing tuple of sections of L_D^{-1}.

Locally on $\widetilde{X}$ we can choose a frame for L_D and in this frame we have, cf. (1.1),

$$\pi^*G = \log|f^0| + \log|f'| + \pi^*h =: \log|f^0| + b. \tag{4.4}$$

Since $\log|f^0|$ is pluriharmonic outside

$$|D| := \cup_j D_j$$

it follows that

$$b = \log|f'| + \pi^*h$$

is psh there; furthermore it is locally bounded at $|D|$. By a standard argument b has a unique (bounded) psh extension B across $|D|$. Notice that dd^cB is a global positive closed current on $\widetilde{X}$ and

$$dd^c\pi^*G = [D] + dd^cB.$$

Let G_j be a decreasing sequence of smooth psh functions converging pointwise to G. Since

$$dd^cG_j = \pi_*(dd^c\pi^*G_j) \to \pi_*\big(dd^c\pi^*G\big) = \pi_*\big([D] + dd^cB\big)$$

it follows that

$$dd^cG = \pi_*\big([D] + dd^cB\big).$$

Let us now assume that we have proved Proposition 4.1 as well as the equality

$$\big(dd^cG\big)^\ell = \pi_*\big([D]\wedge(dd^cB)^{\ell-1} + (dd^cB)^\ell\big) \tag{4.5}$$

for $\ell \le k$. We are to see that then:

(i) $G\mathbf{1}_{X\setminus Z}(dd^cG)^k := \lim_j G_j\mathbf{1}_{X\setminus Z}(dd^cG)^k$ *has locally finite mass.*

(ii) *If*

$$(dd^cG)^{k+1} := dd^c\big(G\mathbf{1}_{X\setminus Z}(dd^cG)^k\big),$$

then (4.5) *holds for* $\ell = k+1$.

As soon as (i) and (ii) are verified, Proposition 4.1 follows.

Notice that if μ is a closed positive current, then

$$\mathbf{1}_Z\pi_*\mu = \pi_*(\mathbf{1}_{|D|}\mu). \tag{4.6}$$

In view of Corollary 3.3 we have that

$$\mathbf{1}_{|D|}(dd^cB)^k = 0. \tag{4.7}$$

From the induction hypothesis (4.5), (4.6) and (4.7) we get

$$\mathbf{1}_{X\setminus Z}\big(dd^cG\big)^k = \pi_*(dd^cB)^k. \tag{4.8}$$

By Proposition 3.2, $(\pi^*G)(dd^cB)^k$ has locally finite mass, and

$$(\pi^*G_j)(dd^cB)^k \to (\pi^*G)(dd^cB)^k$$

if G_j is any decreasing sequence of psh functions that tends to G. If G_j are smooth we have by (4.8) that

$$G_j\mathbf{1}_{X\setminus Z}(dd^cG)^k = \pi_*\big((\pi^*G_j)(dd^cB)^k\big),$$

which tends to

$$G\mathbf{1}_{X\setminus Z}(dd^cG)^k = \pi_*\big((\pi^*G)(dd^cB)^k\big), \tag{4.9}$$

which has locally finite mass. Thus (i) is verified.

We now consider (ii). We claim that

$$dd^c\big(\pi^*G\wedge(dd^cB)^k\big) = [D]\wedge(dd^cB)^k + (dd^cB)^{k+1}. \tag{4.10}$$

Recall that locally $\pi^*G = v + B$, where $v = \log|f^0|$ and B is psh and bounded. Take smooth psh v_j that decrease to v. Then $v_j + B$ are psh and decrease to $v + B$ and thus, by Proposition 3.2,

$$v_j(dd^cB)^k + B(dd^cB)^k = (v_j + B)(dd^cB)^k \to (v + B)(dd^cB)^k.$$

It follows that

$$(v + B)(dd^cB)^k = v(dd^cB)^k + B(dd^cB)^k.$$

From Proposition 3.2 we get that

$$dd^c\big(v(dd^cB)^k\big) = [D]\wedge(dd^cB)^k,$$

which proves the claim. In view of (4.9) and (4.10) the statement (ii) now follows. □

For future reference we notice that

$$M_k^{\mathcal{J}} = \pi_*\big([D]\wedge(dd^cB)^{k-1}\big), \quad \mathbf{1}_{X\setminus Z}(dd^cG)^k = \pi_*(dd^cB)^k. \tag{4.11}$$

In fact $\mathbf{1}_{X\setminus Z}(dd^cG)^k$ equals the *non-pluripolar product* $\langle dd^cG\rangle^k$ as defined in [6, 7].

It follows from the proof above and Proposition 3.2 that if G_j is any decreasing sequence of psh functions converging pointwise to G, then $G_j\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1} \to G\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1}$ and

$$dd^c(G_j\wedge\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1}) = dd^cG_j\wedge\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1} \to (dd^cG)^k.$$

Recall that if G_j are psh functions that decrease to G, then

$$\lim_j (dd^cG_j)^k = (dd^cG)^k, \quad k \le p,$$

see, e.g., [9, Proposition III.4.9]. However, for $k > p$ one cannot hope for a definition of $(dd^cG)^k$ that is robust in this sense. In fact, even if G_j and $\widetilde{G}_j$ are sequences of smooth psh functions decreasing to G and $(dd^cG_j)^k$ and $(dd^c\widetilde{G}_j)^k$ converge to positive closed currents T and $\widetilde{T}$, respectively, T might be different from $\widetilde{T}$, as is illustrated by the following example.

Example 4.2. Let $\varphi = (w, zw)$. Then

$$dd^c\log|\varphi| = dd^c\log|w| + dd^c\log(1+|z|^2)^{1/2} = [w=0] + dd^c\alpha,$$

where $[w=0]$ denotes the current of integration along $\{w=0\}$ and $\alpha = \log(1+|z|^2)^{1/2}$. Thus by (4.2),

$$(dd^c\log|\varphi|)^2 = [w=0]\wedge dd^c\alpha.$$

Let $G_\epsilon = \log(|\varphi|^2+\epsilon)^{1/2}$ and $\widetilde{G}_\epsilon = \log(|w|^2+\epsilon)^{1/2}+\alpha$. Then G_ϵ and $\widetilde{G}_\epsilon$ are smooth psh functions that decrease towards $\log|\varphi|$ as ϵ tends to 0. On the one hand, by (1.7),

$$\lim_{\epsilon\to 0}(dd^cG_\epsilon)^2 = (dd^c\log|\varphi|)^2.$$

On the other hand, again using (1.7), but now for $(dd^c\log|w|)^2$,

$$(dd^c\widetilde{G}_\epsilon)^2 = (dd^c\log(|w|^2+\epsilon)^{1/2})^2 + 2dd^c\log(|w|^2+\epsilon)^{1/2}\wedge dd^c\alpha \longrightarrow 2[w=0]\wedge dd^c\alpha.$$

$\square$

Remark 4.3. Assume that X_ℓ is an exhaustion of X by relatively compact subsets such that the restriction $\mathcal{J}_\ell$ of $\mathcal{J}$ to X_ℓ is generated by global bounded functions. It would be interesting to know whether, or under what assumptions, the currents $M_k^{\mathcal{J}_\ell}$ then converge. Convergence would give us a global canonical representation of the Segre numbers of $\mathcal{J}$.

Assume that $\mathcal{J}$ is indeed generated by global bounded functions and let G_ℓ denote the Green function with poles along $\mathcal{J}_\ell$. Then, arguing as in the proof of Proposition 4.1 and using the notation from that proof,

$$\pi^*G_\ell = \log|f^0| + B_\ell,$$

where B_ℓ is psh and bounded, and moreover

$$(dd^cG_\ell)^k = \pi_*([D]\wedge(dd^cB_\ell)^{k-1} + (dd^cB_\ell)^k).$$

Assume that G_ℓ decrease towards G. Then B_ℓ decrease towards B, as defined in (4.4), and thus $\lim_\ell(dd^cG_\ell)^k = (dd^cG)^k$ in light of (3.1) and (4.5). $\square$

5. Lelong numbers

Let T be a positive closed (k,k)-current. If $k=n$, following [4, Section 5], we let

$$M_0^\xi \wedge T := \mathbf{1}_{\{x\}} T.$$

Otherwise

$$M_{n-k}^\xi \wedge T := \mathbf{1}_{\{x\}} \big((dd^c \log|\xi|)^{n-k} \wedge T \big);$$

here we inductively define

$$(dd^c \log|\xi|)^\ell \wedge T := \\ dd^c \big(\log|\xi| \wedge (dd^c \log|\xi|)^{\ell-1} \wedge T \big) = \lim_j dd^c \big(v_j \wedge (dd^c \log|\xi|)^{\ell-1} \wedge T \big),$$

where v_j is a decreasing sequence of smooth psh functions converging pointwise to $\log|\xi|$. Because of the dimension principle it is not necessary to insert $\mathbf{1}_{X\setminus\{x\}}$ in this definition, cf., Section 4. See Remark 5.1 below for another possible definition of $M_{n-k}^\xi \wedge T$. Clearly $M_{n-k}^\xi \wedge T$ is an (n,n)-current with support at x, and it is in fact equal to $\alpha[x]$, where α is the Lelong number of T at x, see, e.g, [4, Lemma 2.1].

Remark 5.1. As is pointed out in [4, Section 5] one can define $M^\xi \wedge T$ as the value at $\lambda = 0$ of the current-valued analytic function

$$\lambda \mapsto \frac{\bar\partial |\xi|^{2\lambda} \wedge \partial |\xi|^2}{2\pi i |\xi|^2} \wedge (dd^c \log|\xi|)^{n-k-1} \wedge T.$$

□

6. Proof of Theorem 1.1

We will prove the slightly more general formulation of Theorem 1.1 stated in Remark 1.2, i.e., we let G be any psh function of the form (1.1).

We still assume that $\pi\colon \widetilde{X} \to X$ is a smooth modification and use the notation from the proof of Proposition 4.1. Notice that L_D has a Hermitian metric such that $|f^0|_{L_D} = |\pi^* f|$. By the Poincaré-Lelong formula,

$$dd^c \log|\pi^* f| = [D] + \omega_f, \tag{6.1}$$

where ω_f is the first Chern form for L_D^{-1}.

Let us fix a local holomorphic frame so that $\log|f'|$ is a well-defined function as above. Since

$$\log|\pi^* f| = \log|f^0| + \log|f'|,$$

from (6.1) we have that

$$\omega_f = dd^c \log|f'|. \tag{6.2}$$

Let b be the psh bounded function outside $|D|$ defined in (4.4). If we choose another local frame for L_D, then $\log|f'|$ is changed to $\log|f'| + \alpha$ where α is pluriharmonic, and b is thus changed to $\tilde b := b + \alpha$. Moreover $\widetilde{B} := B + \alpha$ is the unique psh extension of $\tilde b$ across $|D|$, cf. the proof of Proposition 4.1. It follows that A, locally defined as

$$A := B - \log|f'|, \tag{6.3}$$

is a global upper semicontinuous extension of π^*h across $|D|$. Notice also that $A(dd^cB)^\ell$ is well-defined on $\widetilde{X}$ and, in light of (6.2) and (6.3), that

$$(dd^cB)^{k-1} - \omega_f^{k-1} = dd^c\Big(A\sum_{\ell=0}^{k-2}(dd^cB)^\ell\wedge\omega_f^{k-2-\ell}\Big).$$

Assume now that $Y \subset \widetilde{X}$ is a smooth submanifold and that $i\colon Y \to \widetilde{X}$ is the natural inclusion. Then i^*B is psh and bounded, $i^*\log|f'|$ is smooth, and, in the same way as above, i^*A is a global upper semi-continuous function on Y and

$$(dd^ci^*B)^{k-1} - i^*\omega_f^{k-1} = dd^c\Big(i^*A\sum_{\ell=0}^{k-2}(dd^ci^*B)^\ell\wedge i^*\omega_f^{k-2-\ell}\Big). \tag{6.4}$$

In view of Lemma 3.4, (6.4) implies that

$$[Y]\wedge\Big((dd^cB)^{k-1} - \omega_f^{k-1}\Big) = dd^ci_*\Big(i^*A\sum_{\ell=0}^{k-2}(dd^ci^*B)^\ell\wedge i^*\omega_f^{k-2-\ell}\Big).$$

The currents $(dd^c\log|f|)^k$ and M_k^f are defined in a completely analogous way as $(dd^cG)^k$ and $M_k^{\mathcal{J}}$, just replacing G by $\log|f|$, cf., the introduction and the end of Section 2 and also [4]. Arguing as in the proof of Proposition 4.1, we get, cf., (4.11), that

$$M_k^f = \pi_*([D]\wedge\omega_f^{k-1}), \quad \mathbf{1}_{X\setminus Z}(dd^c\log|f|)^k = \pi_*\omega_f^k$$

Lemma 6.1. *The currents $M_k^{\mathcal{J}}$ and M_k^f have the same Lelong number at each point $x \in X$. Moreover, the currents $\mathbf{1}_{X\setminus Z}(dd^cG)^k$ and $\mathbf{1}_{X\setminus Z}(dd^c\log|f|)^k$ have the same Lelong number at each point $x \in X$.*

Proof. Let us fix a point $x \in X$ and let ξ be a tuple of functions that defines the maximal ideal $\mathfrak{m}_x$ at x. We can choose the modification $\pi\colon \widetilde{X} \to X$ so that also $\pi^*\mathfrak{m}_x$ is principal, i.e., $\pi^*\xi = \xi^0\xi'$, where ξ^0 is a section of a line bundle L_E that defines the exceptional divisor E, and ξ' is a non-vanishing tuple of sections of L_E^{-1}. Let us assume that

$$E = \sum_\kappa \beta_\kappa E_\kappa, \tag{6.5}$$

where E_κ are irreducible with simple normal crossings and β_κ are integers. We may also assume that, for each j, cf., (4.3), either $D_j \subset |E|$ or all E_κ intersect D_j properly and that

$$E_\kappa^{D_j} := E_\kappa \cap D_j$$

are smooth. Let ω_ξ be the first Chern form of L_E^{-1} with respect to the metric induced by ξ, so that

$$\omega_\xi = dd^c\log|\xi'|,$$

cf., (6.2), and

$$dd^c\log|\pi^*\xi| = [E] + \omega_\xi.$$

Let $i_j\colon D_j \to \widetilde{X}$ be the injection of D_j as a submanifold of $\widetilde{X}$. It follows from (4.3), (4.11) and Lemma 3.4 that

$$M_k^{\mathcal{J}} = \sum_j \alpha_j\pi_*(i_j)_*\big((dd^c(i_j)^*B)^{k-1}\big). \tag{6.6}$$

In order to prove the first part of the lemma, it is enough to consider one single term in (6.6) and verify that

$$T_k^{\mathcal{J}} := \pi_* i_*\big((dd^c i^* B)^{k-1}\big)$$

and

$$T_k^f := \pi_* i_*\big(i^* \omega_f^{k-1}\big)$$

have the same Lelong numbers, where we write $D = D_j$ and $i = i_j$ for simplicity.

Let us first assume that $k = n$. If $D \subset |E|$, then $T_n^{\mathcal{J}}$ and T_n^f both have support at x. In view of (6.4), with $Y = D$, we have that

$$T_k^{\mathcal{J}} - T_k^f = dd^c \pi_* i_* \Big(i^* A \sum_{\ell=1}^{k-2} (dd^c i^* B)^\ell \wedge i^* \omega_f^{k-2-\ell}\Big) =: dW,$$

where W has support at x. By Stokes' theorem thus

$$\int (T_n^{\mathcal{J}} - T_n^f) = \int dW = 0,$$

which means that $T_n^{\mathcal{J}}$ and T_n^f have the same Lelong number at x. If D is not contained in E, then $i^{-1}E$ has positive codimension in D and therefore,

$$\mathbf{1}_{\{x\}} T_n^{\mathcal{J}} = \pi_* i_* (\mathbf{1}_{|i^{-1}E|} (dd^c i^* B)^{n-1}) = 0$$

by Corollary 3.3. In the same way we see that $\mathbf{1}_{\{x\}} T_n^f = 0$.

Let us now assume that $k < n$. If $D \subset |E|$, then $T_k^{\mathcal{J}}$ and T_k^f are positive closed (k, k)-currents with support at x, so by the dimension principle they both vanish. We can therefore assume that $i^* \pi^* \xi$ does not vanish identically on D; by assumption it then defines a smooth divisor E^D on D. Locally on D,

$$\log |i^* \pi^* \xi| = \log |i^* \xi^0| + \log |i^* \xi'|,$$

and thus

$$dd^c \log |i^* \pi^* \xi| = [E^D] + i^* \omega_\xi, \tag{6.7}$$

where $[E^D]$ is the Lelong current on D associated to E^D. If v_j are as in Section 5, then

$$dd^c i^* \pi^* v_j \to [E^D] + i^* \omega_\xi.$$

Now

$$dd^c (v_j T_k^{\mathcal{J}}) = \pi_* i_* \big(dd^c i^* \pi^* v_j \wedge (dd^c i^* B)^{k-1}\big)$$

so that

$$dd^c \log |\xi| \wedge T_k^{\mathcal{J}} = \pi_* i_* \big(([E^D] + i^* \omega_\xi) \wedge (dd^c i^* B)^{k-1}\big)$$

by Proposition 3.2 and (6.7). Moreover, since $\pi_* i_* \big([E^D] \wedge (dd^c i^* B)^{k-1}\big)$ has support at x, by the dimension principle,

$$dd^c \log |\xi| \wedge T_k^{\mathcal{J}} = \pi_* i_* \big(i^* \omega_\xi \wedge (dd^c i^* B)^{k-1}\big).$$

By induction we get

$$(dd^c \log |\xi|)^{n-k} \wedge T_k^{\mathcal{J}} = \pi_* i_* \big(([E^D] + i^* \omega_\xi) \wedge i^* \omega_\xi^{n-k-1} \wedge (dd^c i^* B)^{k-1}\big).$$

Therefore, by Corollary 3.3,

$$M_{n-k}^\xi \wedge T_k^{\mathcal{J}} = \mathbf{1}_{\{x\}} (dd^c \log |\xi|)^{n-k} \wedge T_k^{\mathcal{J}} = \pi_* i_* \big([E^D] \wedge i^* \omega_\xi^{n-k-1} \wedge (dd^c i^* B)^{k-1}\big).$$

Let $\iota_\kappa\colon E^D_\kappa \to D$ be the natural injection. By (6.5) and Lemma 3.4 we have that

$$M^\xi_{n-k}\wedge T^{\mathcal{J}}_k = \sum_\kappa \beta_\kappa \pi_* i_* (\iota_\kappa)_*\big((\iota_\kappa)^* i^* \omega_\xi^{n-k-1}\wedge (dd^c(\iota_\kappa)^* i^* B)^{k-1}\big).$$

By analogous arguments,

$$M^\xi_{n-k}\wedge T^{f}_k = \sum_\kappa \beta_\kappa \pi_* i_* (\iota_\kappa)_*\big((\iota_\kappa)^* i^* \omega_\xi^{n-k-1}\wedge (\iota_\kappa)^* i^* \omega_f^{k-1}\big).$$

For simplicity in notation let us assume that E^D has just one irreducible component and let $\iota : E^D \to D$ be the natural injection. By (6.4) applied to E^D we have that

$$M^\xi_{n-k}\wedge T^{\mathcal{J}}_k - M^\xi_{n-k}\wedge T^{f}_k =$$
$$dd^c \pi_* i_* \iota_* \Big(\iota^* i^* A \ \iota^* i^* \omega_\xi^{n-k-1}\wedge \sum_{\ell=0}^{k-2} (dd^c \iota^* i^* B)^\ell \iota^* i^* \omega_f^{k-1-\ell}\Big) =: dW,$$

where W has support at x. It follows by Stokes' theorem that the integral of this current is zero, and thus the Lelong numbers at x of $T^{\mathcal{J}}_k$ and T^f_k coincide. Thus the first part of the lemma is proved.

By analogous arguments we get that $\pi_*(dd^c B)^k$ and $\pi_*(\omega_f)^k$ have the same Lelong number at x, which proves the second part of the lemma, cf. (4.11) and (6.5).

□

We can now conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Let D^ℓ_j be the irreducible components of D such that $\pi(D^\ell_j)$ have codimension ℓ. Then

$$M^{\mathcal{J}}_k = \pi_*\big([D]\wedge (dd^c B)^{k-1}\big) = \pi_*\big(\sum_{\ell\le k}\sum_j ([D^\ell_j]\wedge(dd^c B)^{k-1}\big)$$

since terms with $\ell > k$ vanish because of the dimension principle. We claim that

$$M^{\mathcal{J}}_k = \pi_*\big(\sum_j ([D^k_j]\wedge (dd^c B)^{k-1}\big) + \pi_*\big(\sum_{\ell<k}\sum_j ([D^\ell_j]\wedge(dd^c B)^{k-1}\big) =: S^{\mathcal{J}}_k + N^{\mathcal{J}}_k \tag{6.8}$$

is the Siu decomposition of $M^{\mathcal{J}}_k$. First notice that since

$$\pi_*\big([D^k_j]\wedge (dd^c B)^{k-1}\big)$$

is a (k,k)-current with support on the set $Z := \pi(D^k_j)$ of codimension k it must be of the form $\alpha[Z]$ where α is a constant, see Lemma 3.1.

It is now enough to see that if W is a subvariety of codimension k, then $\mathbf{1}_W N^{\mathcal{J}}_k = 0$, i.e.,

$$\mathbf{1}_W \pi_*\big([D^\ell_j]\wedge (dd^c B)^{k-1}\big) = 0$$

if $\ell < k$. Let $i\colon D^\ell_j \to \widetilde{X}$ be the natural injection. By Lemma 3.4 we have

$$\mathbf{1}_W \pi_*\big([D^\ell_j]\wedge (dd^c B)^{k-1}\big) = \mathbf{1}_W\big(\pi_* i_*\big(dd^c i^* B\big)^{k-1}\big) = \pi_* i_*\big(\mathbf{1}_{(\pi\circ i)^{-1}(W)}(dd^c i^* B)^{k-1}\big).$$

Notice that since $\pi(D^\ell_j)$ is irreducible and not contained in W it follows that $\pi^{-1}(W)\cap D^\ell_j$ has positive codimension in D^ℓ_j, and hence $\mathbf{1}_{(\pi\circ i)^{-1}(W)}(dd^c i^* B)^{k-1} = 0$ in view of Corollary 3.3.

Thus (6.8) is the Siu decomposition. Since $M^{\mathcal{J}}_k$ and M^f_k have the same Lelong number at each point by Lemma 6.1 and the set where $N^{\mathcal{J}}_k$ and N^f_k have positive

Lelong number have codimension $> k$ we conclude that $S_k^{\mathcal{J}} = S_k^f$, see Remark 2.1. Since also $\mathbf{1}_{X\setminus Z}(dd^c G)^k$ and $\mathbf{1}_{X\setminus Z}(dd^c \log|f|)^k$ have the same Lelong numbers at x by Lemma 6.1, Theorem 1.1 follows from the analogous result, Theorem 1.1, for M^f in [4].

□

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