EXPLICIT SERRE DUALITY ON COMPLEX SPACES

JEAN RUPPENTHAL & HÅKAN SAMUELSSON KALM & ELIZABETH WULCAN

Abstract. In this paper we use recently developed calculus of residue currents together with integral formulas to give a new explicit analytic realization, as well as a new analytic proof, of Serre duality on any reduced pure n-dimensional paracompact complex space X. At the core of the paper is the introduction of certain fine sheaves $\mathscr{B}_X^{n,q}$ of currents on X of bidegree (n,q), such that the Dolbeault complex $(\mathscr{B}_X^{n,\bullet},\bar\partial)$ becomes, in a certain sense, a dualizing complex. In particular, if X is Cohen-Macaulay then $(\mathscr{B}_X^{n,\bullet},\bar\partial)$ is an explicit fine resolution of the Grothendieck dualizing sheaf.

1. Introduction

Let X be a complex n-dimensional manifold and let $F \to X$ be a complex vector bundle. Let $\mathcal{E}^{0,q}(X,F)$ denote the space of smooth F-valued $(0,q)$-forms on X and let $\mathcal{E}_c^{n,q}(X,F^*)$ denote the space of smooth compactly supported (n,q)-forms on X with values in the dual vector bundle F^*. Serre duality, [29], can be formulated analytically as follows: *There is a non-degenerate pairing*

$$H^q\left(\mathcal{E}^{0,\bullet}(X,F),\bar\partial\right) \times H^{n-q}\left(\mathcal{E}_c^{n,\bullet}(X,F^*),\bar\partial\right) \to \mathbb{C}, \tag{1.1}$$

$$([\varphi]_{\bar\partial},[\psi]_{\bar\partial}) \mapsto \int_X \varphi\wedge\psi,$$

provided that $H^q(\mathcal{E}^{0,\bullet}(X,F),\bar\partial)$ and $H^{q+1}(\mathcal{E}^{0,\bullet}(X,F),\bar\partial)$ are Hausdorff considered as topological vector spaces. If we set $\mathscr{F} := \mathscr{O}(F)$ and $\mathscr{F}^* := \mathscr{O}(F^*)$ and let Ω_X^n denote the sheaf of holomorphic n-forms on X, then one can, via the Dolbeault isomorphism, rephrase Serre duality more algebraically: There is a non-degenerate pairing

$$H^q(X,\mathscr{F}) \times H_c^{n-q}(X,\mathscr{F}^*\otimes\Omega_X^n) \to \mathbb{C}, \tag{1.2}$$

realized by the cup product, provided that $H^q(X,\mathscr{F})$ and $H^{q+1}(X,\mathscr{F})$ are Hausdorff. In this formulation Serre duality has been generalized to complex spaces, see, e.g., Hartshorne [19], [20], and Conrad [15] for the algebraic setting and Ramis-Ruget [27] and Andreotti-Kas [11] for the analytic. In fact, if X is a pure n-dimensional paracompact complex space that in addition is Cohen-Macaulay, then again there is a perfect pairing (1.2) if we construe Ω_X^n as the *Grothendieck dualizing sheaf* that we will get back to shortly. If X is not Cohen-Macaulay things get more involved and $H_c^{n-q}(X,\mathscr{F}^*\otimes\Omega_X^n)$ is replaced by $\mathrm{Ext}_c^{-q}(X;\mathscr{F},\mathbf{K}^\bullet)$, where $\mathbf{K}^\bullet$ is the *dualizing complex* in the sense of Ramis-Ruget [27], that is a certain complex of $\mathscr{O}_X$-modules with coherent cohomology.

Date: October 18, 2016.

2000 *Mathematics Subject Classification.* 32A26, 32A27, 32B15, 32C30.

The first author was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), grant RU 1474/2 within DFG's Emmy Noether Programme.

The second and the third author were partially supported by the Swedish Research Council.

To our knowledge there is no such explicit analytic realization of Serre duality as (1.1) in the case of singular spaces. In fact, *verbatim* the pairing (1.1) cannot realize Serre duality in general since the Dolbeault complex $(\mathscr{E}_X^{0,\bullet}, \bar\partial)$[1] in general does not provide a resolution of $\mathscr{O}_X$. In this paper we replace the sheaves of smooth forms by certain fine sheaves of currents $\mathscr{A}_X^{0,q}$ and $\mathscr{B}_X^{n,n-q}$ that are smooth on X_{reg} and such that (1.1) with $\mathscr{E}^{0,\bullet}$ and $\mathscr{E}^{n,\bullet}$ replaced by $\mathscr{A}^{0,\bullet}$ and $\mathscr{B}^{n,\bullet}$, respectively, indeed realizes Serre duality.

We will say that a complex $(\mathscr{D}_X^\bullet, \delta)$ of fine sheaves is a *dualizing Dolbeault complex* for a coherent sheaf $\mathscr{F}$ if $(\mathscr{D}_X^\bullet, \delta)$ has coherent cohomology and if there is a non-degenerate pairing $H^q(X, \mathscr{F}) \times H^{n-q}(\mathscr{D}_c^\bullet(X), \delta) \to \mathbb{C}$. The relation to the Ramis-Ruget dualizing complex is not completely clear to us, but we still find this terminology convenient. For instance, $(\mathscr{B}_X^{n,\bullet}, \bar\partial)$ is a dualizing Dolbeault complex for $\mathscr{O}_X$.

At this point it is appropriate to mention that Ruget in [28] shows, using Coleff-Herrera residue theory, that there is an injective morphism $\mathbf{K}_X^\bullet \to \mathscr{C}_X^{n,\bullet}$, where $\mathscr{C}_X^{n,\bullet}$ is the sheaf of germs of currents on X of bidegree $(n, \bullet)$.

—

Let X be a reduced complex space of pure dimension n. Recall that every point in X has a neighborhood V that can be embedded into some pseudoconvex domain $D \subset \mathbb{C}^N$, $i\colon V \to D$, and that $\mathscr{O}_V \cong \mathscr{O}_D/\mathscr{J}_V$, where $\mathscr{J}_V$ is the radical ideal sheaf in D defining $i(V)$. Similarily, a (p, q)-form φ on V_{reg} is said to be smooth on V if there is a smooth (p, q)-form $\tilde\varphi$ in D such that $\varphi = i^*\tilde\varphi$ on V_{reg}. It is well known that the so defined smooth forms on V define an intrinsic sheaf $\mathscr{E}_X^{p,q}$ on X. The currents of bidegree (p, q) on X are defined as the dual of the space of compactly supported smooth $(n-p, n-q)$-forms on X. More concretely, given a local embedding $i\colon V \to D$, for any (p, q)-current μ on V, $\tilde\mu := i_*\mu$ is a current of bidegree $(p+N-n, q+N-n)$ in D with the property that $\tilde\mu.\xi = 0$ for every test form ξ in D such that $i^*\xi|_{V_{reg}} = 0$. Conversely, if $\tilde\mu$ is a current in D with this property, then it defines a current on V (with a shift in bidegrees). We will often suggestively write $\int \mu \wedge \xi$ for the action of the current μ on the test form ξ.

A current μ on X is said to have the *standard extension property* (SEP) with respect to a subvariety $Z \subset X$ if for all open $\mathcal{U} \subset X$, $\chi(|h|/\epsilon)\mu|_{\mathcal{U}} \to \mu|_{\mathcal{U}}$ as $\epsilon \to 0$, where $\mu|_{\mathcal{U}}$ denotes the restriction of μ to $\mathcal{U}$, χ is a smooth regularization of the characteristic function of $[1, \infty) \subset \mathbb{R}$, and h is any holomorphic tuple that does not vanish identically on any irreducible component of $Z \cap \mathcal{U}$. If $Z = X$ we simply say that μ has the SEP on X. In particular, two currents with the SEP on X are equal on X if and only if they are equal on X_{reg}.

We will say that a current μ on X has *principal value-type singularities* if μ is locally integrable outside a hypersurface and has the SEP on X. Notice that if μ has principal value-type singularities and h is a generically non-vanishing holomorphic tuple such that μ is locally integrable outside $\{h = 0\}$, then the action of μ on a test form ξ can be computed as

$$\lim_{\epsilon \to 0} \int_X \chi(|h|/\epsilon)\mu \wedge \xi,$$

where the integral now is an honest integral of an integrable form on the manifold X_{reg}.

[1]See below for the definition of $\mathscr{E}_X^{p,q}$; the sheaf of smooth (p, q)-forms on X.

By using integral formulas and residue theory, Andersson and the second author introduced in [6] fine sheaves $\mathscr{A}_X^{0,q}$ (i.e., modules over $\mathcal{E}_X^{0,0}$) of $(0,q)$-currents with the SEP on X, containing $\mathcal{E}_X^{0,q}$, and coinciding with $\mathcal{E}_{X_{reg}}^{0,q}$ on X_{reg}, such that the associated Dolbeault complex yields a resolution of $\mathscr{O}_X$, see [6, Theorem 1.2]. Notice that it follows that $H^q(\mathscr{A}^{0,\bullet}(X),\bar\partial) \simeq H^q(X,\mathscr{O}_X)$. Moreover, by a standard construction it then follows that each cohomology class in $H^q(\mathscr{A}^{0,\bullet}(X),\bar\partial)$ has a smooth representative; cf. Section 7 below. Similar to the construction of the $\mathscr{A}$-sheaves in [6] we introduce our sheaves $\mathscr{B}_X^{n,q}$ of (n,q)-currents and show that these currents have the SEP on X, that $\mathcal{E}_X^{n,q} \subset \mathscr{B}_X^{n,q}$, and that $\mathscr{B}_X^{n,q}$ coincides with $\mathcal{E}_X^{n,q}$ on X_{reg}; cf. Proposition 4.3. Moreover, by Theorem 4.4, $\bar\partial\colon \mathscr{B}_X^{n,q} \to \mathscr{B}_X^{n,q+1}$, where of course $\bar\partial$ is defined by duality: $\int \bar\partial\mu \wedge \xi := \pm \int \mu \wedge \bar\partial \xi$ for currents μ and test forms ξ on X. By adapting the constructions in [6] to the setting of (n,q)-forms we get the following semi-global homotopy formula for $\bar\partial$.

Theorem 1.1. *Let V be a pure n-dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, let $D' \Subset D$, and put $V' = V \cap D'$. There are integral operators*

$$\check{\mathscr{K}}\colon \mathscr{B}^{n,q}(V) \to \mathscr{B}^{n,q-1}(V'), \quad \check{\mathscr{P}}\colon \mathscr{B}^{n,q}(V) \to \mathscr{B}^{n,q}(V'),$$

such that if $\psi \in \mathscr{B}^{n,q}(V)$, then the homotopy formula

$$\psi = \bar\partial \check{\mathscr{K}}\psi + \check{\mathscr{K}}(\bar\partial\psi) + \check{\mathscr{P}}\psi$$

holds on V'.

The integral operators $\check{\mathscr{K}}$ and $\check{\mathscr{P}}$ are given by kernels $k(z,\zeta)$ and $p(z,\zeta)$ that are respectively integrable and smooth on $Reg(V_z) \times Reg(V'_\zeta)$ and that have principal value-type singularities at the singular locus of $V \times V'$. In particular, one can compute $\check{\mathscr{K}}\psi$ and $\check{\mathscr{P}}\psi$ as

$$\check{\mathscr{K}}\psi(\zeta) = \lim_{\epsilon\to 0}\int_{V_z} \chi(|h(z)|/\epsilon) k(z,\zeta)\wedge\psi(z), \quad \check{\mathscr{P}}\psi(\zeta) = \lim_{\epsilon\to 0}\int_{V_z} \chi(|h(z)|/\epsilon) p(z,\zeta)\wedge\psi(z),$$

where χ is as above, h is a holomorphic tuple such that $\{h=0\} = V_{sing}$, and where the limit is understood in the sense of currents. We use our integral operators to prove the following result.

Theorem 1.2. *Let X be a reduced complex space of pure dimension n. The cohomology sheaves $\omega_X^{n,q} := \mathscr{H}^q(\mathscr{B}_X^{n,\bullet},\bar\partial)$ of the sheaf complex*

$$0 \to \mathscr{B}_X^{n,0} \xrightarrow{\bar\partial} \mathscr{B}_X^{n,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{B}_X^{n,n} \to 0 \tag{1.3}$$

are coherent. If X is Cohen-Macaulay, then

$$0 \to \omega_X^{n,0} \hookrightarrow \mathscr{B}_X^{n,0} \xrightarrow{\bar\partial} \mathscr{B}_X^{n,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{B}_X^{n,n} \to 0 \tag{1.4}$$

is exact.

In fact, our proof of Theorem 1.2 shows that if $V \subset X$ is identified with an analytic codimension κ subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, then $\omega_V^{n,q} \cong \mathscr{E}xt^{\kappa+q}(\mathscr{O}_D/\mathcal{J}_V, \Omega_D^N)$, where Ω_D^N is the canonical sheaf on D. Hence, we get a concrete analytic realization of these $\mathscr{E}xt$-sheaves.

The sheaf $\omega_V^{n,0}$ of $\bar\partial$-closed currents in $\mathscr{B}_V^{n,0}$ is in fact equal to the sheaf of $\bar\partial$-closed meromorphic currents on V in the sense of Henkin-Passare [21, Definition 2], cf. [6, Example 2.8]. This sheaf was introduced earlier by Barlet in a different way in [12]; cf. also [21, Remark 5]. In case X is Cohen-Macaulay $\mathscr{E}xt^{\kappa}(\mathscr{O}_D/\mathcal{J}_V, \Omega_D^N)$ is by definition the Grothendieck dualizing sheaf. Thus, (1.4) can be viewed as a concrete analytic fine resolution of the Grothendieck dualizing sheaf in the Cohen-Macaulay case.

—

Let φ and ψ be sections of $\mathscr{A}_X^{0,q}$ and $\mathscr{B}_X^{n,q'}$ respectively. Since φ and ψ then are smooth on the regular part of X, the exterior product $\varphi|_{X_{reg}} \wedge \psi|_{X_{reg}}$ is a smooth $(n, q+q')$-form on X_{reg}. In Theorem 5.1 we show that $\varphi|_{X_{reg}} \wedge \psi|_{X_{reg}}$ has a natural extension across X_{sing} as a current with principal value-type singularities; we denote this current by $\varphi \wedge \psi$. Moreover, it turns out that the Leibniz rule $\bar\partial(\varphi\wedge\psi) = \bar\partial\varphi\wedge\psi + (-1)^q\varphi\wedge\bar\partial\psi$ holds. Now, if $q' = n-q$ and ψ (or φ) has compact support, then $\int \varphi\wedge\psi$ (i.e., the action of $\varphi\wedge\psi$ on 1) gives us a complex number. Since the Leibniz rule holds we thus get a pairing, a *trace map*, on cohomology level:

$$Tr\colon H^q\left(\mathscr{A}^{0,\bullet}(X), \bar\partial\right) \times H^{n-q}\left(\mathscr{B}_c^{n,\bullet}(X), \bar\partial\right) \to \mathbb{C},$$

$$Tr([\varphi]_{\bar\partial}, [\psi]_{\bar\partial}) = \int_X \varphi\wedge\psi,$$

where $\mathscr{A}^{0,q}(X)$ denotes the global sections of $\mathscr{A}_X^{0,q}$ and $\mathscr{B}_c^{n,q}(X)$ denotes the global sections of $\mathscr{B}_X^{n,q}$ with compact support. It causes no problems to insert a locally free sheaf: If $F\to X$ is a vector bundle, $\mathscr{F} = \mathscr{O}(F)$ the associated locally free sheaf, and $\mathscr{F}^* = \mathscr{O}(F^*)$ the dual sheaf, then the trace map gives a pairing $\mathscr{F}\otimes\mathscr{A}^{0,q}(X)\times\mathscr{F}^*\otimes\mathscr{B}_c^{n,n-q}(X)\to\mathbb{C}$.

Theorem 1.3. *Let X be a paracompact reduced complex space of pure dimension n and $\mathscr{F}$ a locally free sheaf on X. If $H^q(X,\mathscr{F})$ and $H^{q+1}(X,\mathscr{F})$, considered as topological vector spaces, are Hausdorff (e.g., finite dimensional), then the pairing*

$$H^q\left(\mathscr{F}\otimes\mathscr{A}^{0,\bullet}(X), \bar\partial\right) \times H^{n-q}\left(\mathscr{F}^*\otimes\mathscr{B}_c^{n,\bullet}(X), \bar\partial\right) \to \mathbb{C}, \quad ([\varphi],[\psi]) \mapsto \int_X \varphi\wedge\psi$$

is non-degenerate.

Since the $\mathscr{A}$-cohomology has smooth representatives, it follows that if X is compact and ψ is a smooth $\bar\partial$-closed (n,q)-form on X, then there is a $u\in\mathscr{B}^{n,q-1}(X)$ (in particular u is smooth on X_{reg}) such that $\bar\partial u = \psi$ if and only if $\int_X \varphi\wedge\psi = 0$ for all smooth $\bar\partial$-closed $(0,n-q)$-forms φ.

Notice also that, by [6, Theorem 1.2], the complex $(\mathscr{F}\otimes\mathscr{A}_X^{0,\bullet}, \bar\partial)$ is a fine resolution of $\mathscr{F}$ and so, via the Dolbeault isomorphism, Theorem 1.3 gives us a non-degenerate pairing

$$H^q(X,\mathscr{F}) \times H^{n-q}(\mathscr{F}^*\otimes\mathscr{B}_c^{n,\bullet}(X), \bar\partial) \to \mathbb{C}.$$

The complex $(\mathscr{F}^*\otimes\mathscr{B}_X^{n,\bullet}, \bar\partial)$ is thus a concrete analytic dualizing Dolbeault complex for $\mathscr{F}$. If X is Cohen-Macaulay, then $(\mathscr{F}^*\otimes\mathscr{B}_X^{n,\bullet}, \bar\partial)$ is, by Theorem 1.2 above, a fine resolution of the sheaf $\mathscr{F}^*\otimes\omega_X^{n,0}$ and so Theorem 1.3 yields in this case a non-degenerate pairing

$$H^q(X,\mathscr{F}) \times H_c^{n-q}(X, \mathscr{F}^*\otimes\omega_X^{n,0}) \to \mathbb{C}.$$

In Section 7 we show that this pairing also can be realized as the cup product in Čech cohomology.

Remark 1.4. By [27, Théorème 2] there is another non-degenerate pairing

$$H_c^q(X,\mathscr{F}) \times \mathrm{Ext}^{-q}(X;\mathscr{F},\mathbf{K}_X^\bullet) \to \mathbb{C}$$

if $H^q_c(X,\mathscr{F})$ and $H^{q+1}_c(X,\mathscr{F})$ are Hausdorff. In view of this we believe that one can show that, under the same assumption, the pairing

$$H^q\left(\mathscr{F}\otimes\mathscr{A}_c^{0,\bullet}(X),\bar\partial\right)\times H^{n-q}\left(\mathscr{F}^*\otimes\mathscr{B}^{n,\bullet}(X),\bar\partial\right)\to\mathbb{C},\quad ([\varphi],[\psi])\mapsto\int_X\varphi\wedge\psi$$

is non-degenerate but we do not pursue this question in this paper.

Acknowledgment: We would like to thank Mats Andersson for valuable discussions and comments that have simplified some proofs significantly. We would also like to thank the referee for many important comments.

2. Preliminaries

Our considerations here are local or semi-global so let V be a pure n-dimensional analytic subset of a pseudoconvex domain $D\subset\mathbb{C}^N$. Throughout we let $\kappa=N-n$ denote the codimension of V.

2.1. **Pseudomeromorphic currents on a complex space.** In $\mathbb{C}_z$ the principal value current $1/z^m$ can be defined, e.g., as the limit as $\epsilon\to 0$ in the sense of currents of $\chi(|h(z)|/\epsilon)/z^m$, where χ is a smooth regularization of the characteristic function of $[1,\infty)\subset\mathbb{R}$ and h is a holomorphic function vanishing at $z=0$, or as the value at $\lambda=0$ of the analytic continuation of the current-valued function $\lambda\mapsto|h(z)|^{2\lambda}/z^m$. Regularizations of the form $\chi(|h|/\epsilon)\mu$ of a current μ occur frequently in this paper and throughout χ will denote a smooth regularization of the characteristic function of $[1,\infty)\subset\mathbb{R}$. The *residue current* $\bar\partial(1/z^m)$ can be computed as the limit of $\bar\partial\chi(|h(z)|/\epsilon)/z^m$ or as the value at $\lambda=0$ of $\lambda\mapsto\bar\partial|h(z)|^{2\lambda}/z^m$. Since tensor products of currents are well-defined we can form the current

$$\tau=\bar\partial\frac{1}{z_1^{m_1}}\wedge\cdots\wedge\bar\partial\frac{1}{z_r^{m_r}}\wedge\frac{\gamma(z)}{z_{r+1}^{m_{r+1}}\cdots z_n^{m_n}}\tag{2.1}$$

in $\mathbb{C}^n_z$, where $m_1,\dots,m_r$ are positive integers, $m_{r+1},\dots,m_n$ are nonnegative integers, and γ is a smooth compactly supported form. Notice that τ is anti-commuting in the residue factors $\bar\partial(1/z_j^{m_j})$ and commuting in the principal value factors $1/z_k^{m_k}$. A current of the form (2.1) is called an *elementary pseudomeromorphic current* and we say that a current μ on V is *pseudomeromorphic*, $\mu\in\mathcal{PM}(V)$, if it is a locally finite sum of pushforwards $\pi_*\tau=\pi^1_*\cdots\pi^\ell_*\tau$ under maps

$$V^\ell\xrightarrow{\pi^\ell}\cdots\xrightarrow{\pi^2}V^1\xrightarrow{\pi^1}V^0=V,$$

where each π^j is either a modification, a simple projection $V^j=V^{j-1}\times Z\to V^{j-1}$, or an open inclusion, and τ is an elementary pseudomeromorphic current on V^ℓ. The sheaf of pseudomeromorphic currents on V is denoted $\mathcal{PM}_V$. Since the class of elementary currents is closed under $\bar\partial$ and $\bar\partial$ commutes with push-forwards it follows that $\mathcal{PM}_V$ is closed under $\bar\partial$. Pseudomeromorphic currents were originally introduced in [9] but with a more restrictive definition; simple projections were not allowed. In this paper we adopt the definition of pseudomeromorphic currents in [6].

Example 2.1. Let $f\in\mathscr{O}(V)$ be generically non-vanishing and let α be a smooth form on V. Then α/f is a semi-meromorphic form on V and it defines a *semi-meromorphic current*, also denoted α/f, on V by

$$\xi\mapsto\lim_{\epsilon\to0}\int_V\chi(|h|/\epsilon)\frac{\alpha}{f}\wedge\xi,\tag{2.2}$$

where ξ is a test form on V and $h \in \mathscr{O}(V)$ is generically non-vanishing and vanishes on $\{f = 0\}$. That (2.2) indeed gives a well-defined current is proved in [22]; the existence of the limit in (2.2) relies on Hironaka's theorem on resolution of singularities. Let $\pi\colon \tilde{V} \to V$ be a smooth modification such that $\{\pi^* f = 0\}$ is a normal crossings divisor. Locally on $\tilde{V}$ one can thus choose coordinates so that $\pi^* f$ is a monomial. One can then show that the semi-meromorphic current α/f is the push-forward under π of elementary pseudomeromorphic currents (2.1) with $r = 0$; hence, $\alpha/f \in \mathcal{PM}(V)$.

The $(0,1)$-current $\bar\partial(1/f) \in \mathcal{PM}(V)$ is the residue current of f. Since the action of $1/f$ on test forms is given by (2.2) with $\alpha = 1$ it follows from Stokes' theorem that

$$\bar\partial\frac{1}{f}.\,\xi = \lim_{\epsilon\to 0}\int_V \frac{\bar\partial\chi(|h|/\epsilon)}{f}\wedge\xi.$$

□

One crucial property of pseudomeromorphic currents is the following, see, e.g., [6, Proposition 2.3].

Dimension principle. *Let $\mu \in \mathcal{PM}(V)$ and assume that μ has support on the subvariety $Z \subset V$. If $\dim V - \dim Z > q$ and μ has bidegree $(*, q)$, then $\mu = 0$.*

Pseudomeromorphic currents can be "restricted" to analytic subsets. In fact, following [9], if $\mu \in \mathcal{PM}(V)$ and $Z \subset V$ is an analytic subset, then $\mu|_{V\setminus Z}$ has a natural pseudomeromorphic extension to V denoted $\mathbf{1}_{V\setminus Z}\mu$. Thus, $\mathbf{1}_Z\mu := \mu - \mathbf{1}_{V\setminus Z}\mu$ is a pseudomeromorphic current on V with support on Z. In [9], $\mathbf{1}_{V\setminus Z}\mu$ is defined as $|h|^{2\lambda}\mu|_{\lambda=0}$, where h is a holomorphic tuple such that $\{h = 0\} = Z$, but it can also be defined as $\lim_{\epsilon\to 0}\chi(|h|/\epsilon)\mu$; cf. [10] and [24, Lemma 6]. It follows that if $\mu = \pi_*\tau$, then $\mathbf{1}_Z\mu = \pi_*(\mathbf{1}_{\pi^{-1}(Z)}\tau)$. Notice that a pseudomeromorphic current μ has the SEP if and only if $\mathbf{1}_Z\mu = 0$ for all germs of analytic sets Z with positive codimension. We will denote by $\mathcal{W}_V$ the subsheaf of $\mathcal{PM}_V$ of currents with the SEP. It is closed under multiplication by smooth forms and if $\pi\colon \tilde{V} \to V$ is either a modification or a simple projection then $\pi_*\colon \mathcal{W}(\tilde{V}) \to \mathcal{W}(V)$.

A natural subclass of $\mathcal{W}(V)$ is the class of *almost semi-meromorphic currents* on V; a current μ on V is said to be almost semi-meromorphic if there is a smooth modification $\pi\colon \tilde{V} \to V$ and a semi-meromorphic current $\tilde\mu$ on $\tilde{V}$ such that $\pi_*\tilde\mu = \mu$, see [6]. Notice that almost semi-meromorphic currents are generically smooth and have principal value-type singularities. Let μ be an almost semi-meromorphic current. Following [10], we let $ZSS(\mu)$ (the Zariski-singular support of μ) be the smallest Zariski-closed set outside of which μ is smooth. The following result can be found in [10]; the last part is [6, Proposition 2.7].

Proposition 2.2. *Let a be an almost semi-meromorphic current on V and let $\mu \in \mathcal{PM}(V)$. Then there is a unique pseudomeromorphic current $a\wedge\mu$ on V that coincides with $a \wedge \mu$ outside of $ZSS(a)$ and such that $\mathbf{1}_{ZSS(a)}a \wedge \mu = 0$. If $\mu \in \mathcal{W}(V)$, then $a \wedge \mu \in \mathcal{W}(V)$.*

If $\mu \in \mathcal{PM}(V_z)$ and $\nu \in \mathcal{PM}(W_\zeta)$ then we will denote the current $(\mu\otimes 1)\wedge(1\otimes\nu)$ on $V_z \times W_\zeta$ by $\mu(z)\wedge\nu(\zeta)$, or sometimes $\mu\wedge\nu$ if there is no risk of confusion, and refer to it as the *tensor product* of μ and ν. From [10] we have that $\mu(z)\wedge\nu(\zeta) \in \mathcal{PM}(V\times W)$ and that $\mu(z) \wedge \nu(\zeta) \in \mathcal{W}(V \times W)$ if $\mu \in \mathcal{W}(V)$ and $\nu \in \mathcal{W}(W)$.

We will also have use for the following slight variation of [5, Theorem 1.1 (ii)].

Proposition 2.3. *Let $Z \subset V$ be a pure dimensional analytic subset and let $\mathcal{J}_Z \subset \mathscr{O}_V$ be the ideal sheaf of holomorphic functions vanishing on Z. Assume that $\tau \in \mathcal{PM}(V)$ has the SEP with respect to Z and that $h\tau = dh \wedge \tau = 0$ for all $h \in \mathcal{J}_Z$. Then there is a current $\mu \in \mathcal{PM}(Z)$ with the SEP such that $\iota_*\mu = \tau$, where $\iota\colon Z \hookrightarrow V$ is the inclusion.*

Proof. Let $i\colon V \hookrightarrow D$ be the inclusion. By [5, Theorem 1.1 (i)] we have that $i_*\tau \in \mathcal{PM}(D)$. It is straightforward to verify that $i_*\tau$ has the SEP with respect to Z considered now as a subset of D and that $hi_*\tau = dh \wedge i_*\tau = 0$ for all $h \in \mathcal{J}_Z$, where we now consider $\mathcal{J}_Z$ as the ideal sheaf of Z in D. Hence, it is sufficient to show the proposition when V is smooth. To this end, we will see that there is a current μ on Z such that $\iota_*\mu = \tau$; then the proposition follows from [5, Theorem 1.1 (ii)].

The existence of such a μ is equivalent to that $\tau.\xi = 0$ for all test forms ξ such that $\iota^*\xi = 0$ on Z_{reg}. By, e.g., [6, Proposition 2.3] and the assumption on τ it follows that $\bar{h}\tau = d\bar{h} \wedge \tau = h\tau = dh \wedge \tau = 0$ for every $h \in \mathcal{J}_Z$. Using this it is straightforward to check that if $x \in Z_{reg}$ and ξ is a smooth form such that $\iota^*\xi = 0$ in a neighborhood of x, then $\xi \wedge \tau = 0$ in a neighborhood of x. Thus, if g is a holomorphic tuple in V such that $\{g = 0\} = Z_{sing}$, then $\chi(|g|/\epsilon)\tau.\xi = 0$ for any test form ξ such that $\iota^*\xi = 0$ on Z_{reg}. Since τ has the SEP with respect to Z it follows that $\tau.\xi = 0$ for all test forms ξ such that $\iota^*\xi = 0$ on Z_{reg}. ☐

2.2. **Residue currents.** We briefly recall the the construction in [8] of a residue current associated to a generically exact complex of Hermitian vector bundles.

Let $\mathcal{J}_V$ be the radical ideal sheaf in D associated with $V \subset D$. Possibly after shrinking D somewhat there is a free resolution

$$0 \to \mathscr{O}(E_m) \xrightarrow{f_m} \cdots \xrightarrow{f_2} \mathscr{O}(E_1) \xrightarrow{f_1} \mathscr{O}(E_0) \tag{2.3}$$

of $\mathscr{O}_D/\mathcal{J}_V$, where E_k are trivial vector bundles, E_0 is the trivial line bundle, f_k are holomorphic mappings, and $m \le N$. The resolution (2.3) induces a complex of vector bundles

$$0 \to E_m \xrightarrow{f_m} \cdots \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0$$

that is pointwise exact outside V. For $r \ge 1$, let V^r be the set where $f_{\kappa+r}\colon E_{\kappa+r} \to E_{\kappa+r-1}$ does not have optimal rank[2], and let $V^0 := V_{sing}$. Then

$$\cdots \subset V^{k+1} \subset V^k \subset \cdots \subset V^1 \subset V^0 \subset V. \tag{2.4}$$

By the uniqueness of minimal free resolutions, these sets are in fact independent of the choice of resolution (2.3) of $\mathscr{O}_V = \mathscr{O}_D/\mathcal{J}_V$, i.e., they are invariants of that sheaf, and they somehow measure the singularities of V. Since V has pure dimension it follows from [17, Corollary 20.14] that

$$\dim V^r < n - r, \quad r \ge 0.$$

Hence, $V^n = \emptyset$ and so f_N has optimal rank everywhere; we may thus assume that $m \le N-1$ in (2.3). Recall that V is Cohen-Macaulay if and only if there a resolution (2.3) with $m = \kappa$ of $\mathscr{O}_V$, see, e.g., [17, Chapter 18]. Notice that $V^r = \emptyset$ for $r \ge 1$ if and only if V is Cohen-Macaulay.

[2]For $j \le \kappa$, the set where f_j does not have optimal rank is V.

Assuming V has positive codimension, given Hermitian metrics on the E_j, following [8], one can construct a smooth form $u = \sum_{k\geq 1} u_k$ in $D \setminus V$, where u_k is a $(0, k-1)$-form taking values in E_k, such that

$$f_1 u_1 = 1, \quad f_{k+1}u_{k+1} = \bar\partial u_k,\ k = 1, \dots, m-1, \quad \bar\partial u_m = 0 \quad \text{in } D \setminus V. \tag{2.5}$$

The form u has an extension as an almost semimeromorphic current

$$\lim_{\epsilon\to 0} \chi(|F|/\epsilon)u =: U = \sum_{k\geq 1} U_k, \tag{2.6}$$

where F is a holomorphic tuple in D vanishing on V and U_k is a $(0, k-1)$-current taking values in E_k; one should think of U as a generalization of the meromorphic current $1/f$ in D when $V = f^{-1}(0)$ is a hypersurface.[3] The residue current $R = \sum_k R_k$ associated with V is then defined by

$$R_k = \bar\partial U_k - f_{k+1}U_{k+1},\ k = 1, \dots, m-1, \quad R_m = \bar\partial U_m. \tag{2.7}$$

Hence, R_k is a pseudomeromorphic $(0, k)$-current in D with values in E_k, and from (2.5) it follows that R has support on V. By the dimension principle, thus $R = R_\kappa + \cdots + R_m$. Notice that if V is Cohen-Macaulay and (2.3) ends at level κ, then $R = R_\kappa$ and $\bar\partial R = 0$. By [8, Theorem 1.1] we have that if $h \in \mathscr{O}_D$ then

$$hR = 0 \quad \text{if and only if} \quad h \in \mathcal{J}_V. \tag{2.8}$$

Example 2.4. Let $V = f^{-1}(0)$ be a hypersurface in D. Then $0 \to \mathscr{O}(E_1) \xrightarrow{f} \mathscr{O}(E_0)$ is a resolution of $\mathscr{O}/\langle f\rangle$, where E_1 and E_0 are auxiliary trivial line bundles. The associated current U then becomes $(1/f) \otimes e_1$, where e_1 is a holomorphic frame for E_1, and the associated residue current R is $\bar\partial(1/f) \otimes e_1$.

Let $g_1, \dots, g_\kappa \in \mathscr{O}(D)$ be a regular sequence. Then the Koszul complex associated to the g_j is a free resolution of $\mathscr{O}_D/\langle g_1, \dots, g_\kappa\rangle$. The associated residue current R then becomes the Coleff-Herrera product

$$\bar\partial\frac{1}{g_1} \wedge \cdots \wedge \bar\partial\frac{1}{g_\kappa},$$

introduced in [14], times an auxiliary frame element, see [2, Theorem 1.7]. □

2.3. **Structure forms of a complex space.** Assume first that V is a reduced hypersurface, i.e., $V = f^{-1}(0) \subset D \subset \mathbb{C}^N$, $N = n+1$, where $f \in \mathscr{O}(D)$ and $df \neq 0$ on V_{reg}. Let ω' be a meromorphic $(n, 0)$-form in $D \subset \mathbb{C}^{n+1}_z$ such that

$$df \wedge \omega' = 2\pi i\, dz_1 \wedge \cdots \wedge dz_{n+1} \quad \text{on} \quad V_{reg}.$$

Then $\omega := i^*\omega'$, where $i\colon V \hookrightarrow D$ is the inclusion, is a meromorphic form on V that is uniquely determined by f; ω is the Poincaré residue of the meromorphic form $2\pi i dz_1 \wedge \cdots \wedge dz_{n+1}/f(z)$. For brevity we will sometimes write dz for $dz_1 \wedge \cdots \wedge dz_N$. Leray's residue formula can be formulated as

$$\int \bar\partial\frac{1}{f} \wedge dz \wedge \xi = \lim_{\epsilon\to 0} \int_V \chi(|h|/\epsilon)\omega \wedge i^*\xi, \tag{2.9}$$

[3]In [9] U was originally defined as the analytic continuation to $\lambda = 0$ of $|F|^{2\lambda}u$. However, in view of [10, Section 4] this definition coincides with (2.6), see also [24, Lemma 6].

where ξ is a $(0,n)$-test form in D, the left hand side is the action of $\bar\partial(1/f)$ on $dz\wedge\xi$ and h is a holomorphic tuple such that $\{h=0\}=V_{sing}$. If we consider ω as a meromorphic current on V we can rephrase (2.9) as

$$\bar\partial\frac{1}{f}\wedge dz = i_*\omega. \tag{2.10}$$

Assume now that $V \overset{i}{\hookrightarrow} D\subset\mathbb{C}^N$ is an arbitrary pure n-dimensional analytic subset. From Section 2.2 we have, given a free resolution (2.3) of $\mathscr{O}_D/\mathcal{J}_V$ and a choice of Hermitian metrics on the involved bundles E_j, the associated residue current R that plays the role of $\bar\partial(1/f)$. By the following result, which is an abbreviated version of [6, Proposition 3.3], there is an almost semi-meromorphic current ω on V such that $R\wedge dz = i_*\omega$; such a current will be called a *structure form* of V.

Proposition 2.5. *Let* (2.3) *be a Hermitian free resolution of $\mathscr{O}_D/\mathcal{J}_V$ in D and let R be the associated residue current. Then there is a unique almost semi-meromorphic current*

$$\omega=\omega_0+\omega_1+\cdots+\omega_{n-1}$$

on V, where ω_r is smooth on V_{reg}, has bidegree (n,r), and takes values in $E_{\kappa+r}|_V$, such that

$$R\wedge dz_1\wedge\cdots\wedge dz_N = i_*\omega. \tag{2.11}$$

Moreover,

$$f_\kappa|_V\omega_0=0,\quad f_{\kappa+r}|_V\omega_r=\bar\partial\omega_{r-1},\quad r\geq 1,$$

in the sense of currents on V, and there are $(0,1)$-forms α_k, $k\geq 1$, that are smooth outside V^k and that take values in $Hom(E_{\kappa+k-1}|_V, E_{\kappa+k}|_V)$, such that

$$\omega_k=\alpha_k\omega_{k-1},\quad k\geq 1.$$

It is sometimes useful to reformulate (2.11) suggestively as

$$R\wedge dz_1\wedge\cdots\wedge dz_N=\omega\wedge[V], \tag{2.12}$$

where $[V]$ is the current of integration along V.

The following result will be useful for us when defining our dualizing complex.

Proposition 2.6 (Lemma 3.5 in [6])**.** *If ψ is a smooth (n,q)-form on V, then there is a smooth $(0,q)$-form ψ' on V with values in $E_p^*|_V$ such that $\psi=\omega_0\wedge\psi'$.*

2.4. **Koppelman formulas in $\mathbb{C}^N$.** We recall some basic constructions from [1] and [3]. Let $D\Subset\mathbb{C}^N$ be a domain (not necessarily pseudoconvex at this point), let $k(z,\zeta)$ be an integrable $(N,N-1)$-form in $D\times D$, and let $p(z,\zeta)$ be a smooth (N,N)-form in $D\times D$. Assume that k and p satisfy the equation of currents

$$\bar\partial k(z,\zeta)=[\Delta^D]-p(z,\zeta) \tag{2.13}$$

in $D\times D$, where $[\Delta^D]$ is the current of integration along the diagonal. Applying this current equation to test forms $\psi(z)\wedge\varphi(\zeta)$ it is straightforward to verify that for any compactly supported (p,q)-form φ in D one has the following Koppelman formula

$$\varphi(z)=\bar\partial_z\int_{D_\zeta}k(z,\zeta)\wedge\varphi(\zeta)+\int_{D_\zeta}k(z,\zeta)\wedge\bar\partial\varphi(\zeta)+\int_{D_\zeta}p(z,\zeta)\wedge\varphi(\zeta).$$

In [1] Andersson introduced a very flexible method of producing solutions to (2.13). Let $\eta=(\eta_1,\ldots,\eta_N)$ be a holomorphic tuple in $D\times D$ that defines the diagonal and let

Λ_η be the exterior algebra spanned by $T^*_{0,1}(D\times D)$ and the $(1,0)$-forms $d\eta_1,\dots,d\eta_N$. On forms with values in Λ_η interior multiplication with $2\pi i\sum \eta_j\partial/\partial\eta_j$, denoted δ_η, is defined; put $\nabla_\eta=\delta_\eta-\bar\partial$.

Let s be a smooth $(1,0)$-form in Λ_η such that $|s|\lesssim|\eta|$ and $|\eta|^2\lesssim|\delta_\eta s|$ and let $B=\sum_{k=1}^N s\wedge(\bar\partial s)^{k-1}/(\delta_\eta s)^k$. It is proved in [1] that then $\nabla_\eta B=1-[\Delta^D]$. Identifying terms of top degree we see that $\bar\partial B_{N,N-1}=[\Delta^D]$ and we have found a solution to (2.13). For instance, if we take $s=\partial|\zeta-z|^2$ and $\eta=\zeta-z$, then the resulting B is sometimes called the full Bochner-Martinelli form and the term of top degree is the classical Bochner-Martinelli kernel.

A smooth section $g(z,\zeta)=g_{0,0}+\cdots+g_{N,N}$ of Λ_η, defined for $z\in D_1\subset D$ and $\zeta\in D_2\subset D$, such that $\nabla_\eta g=0$ and $g_{0,0}|_{\Delta^D\cap D'}=1$, where $D':=D_1\cap D_2$, is called a *weight* in $D_1\times D_2$. It follows that $\nabla_\eta(g\wedge B)=g-[\Delta^D]$ and, identifying terms of bidegree $(N,N-1)$, we get that

$$\bar\partial(g\wedge B)_{N,N-1}=[\Delta^D]-g_{N,N} \tag{2.14}$$

in $D'\times D'$. Hence $(g\wedge B)_{N,N-1}$ and $g_{N,N}$ give a solution to (2.13) in $D'\times D'$.

If D is pseudoconvex and K is a holomorphically convex compact subset, then one can find a weight g in $D'\times D$ for some neighborhood $D'\subset D$ of K such that $z\mapsto g(z,\zeta)$ is holomorphic in D', which in particular means that there are no differentials of the form $d\bar z_j$, and $\zeta\mapsto g(z,\zeta)$ has compact support in D; see, e.g., Example 2 in [3].

2.5. **Koppelman formulas for $(0,q)$-forms on a complex space.** We briefly recall from [6] the construction of Koppelman formulas for $(0,q)$-forms on $V\subset D$. The basic idea is to use the currents U and R discussed in Section 2.2 to construct a weight that will yield an integral formula of division/interpolation type in the same spirit as in, e.g., [13, 25].

Let (2.3) be a resolution of $\mathscr{O}_D/\mathscr{J}_V$, where as before $\mathscr{J}_V$ is the sheaf in D associated to $V\overset{i}{\hookrightarrow}D$. One can find, see [3, Proposition 5.3], holomorphic Λ_η-valued Hefer morphisms $H^\ell_k\colon E_k\to E_\ell$ of bidegree $(k-\ell,0)$ such that $H^k_k=I_{E_k}$ and

$$\delta_\eta H^\ell_k=H^\ell_{k-1}f_k(\zeta)-f_{\ell+1}(z)H^{\ell+1}_k,\quad k>1. \tag{2.15}$$

Let F be a holomorpic tuple in D such that $\{F=0\}=V$, let $U^\epsilon=\chi(|F|/\epsilon)u$, and let

$$R^\epsilon:=1-\sum f_kU^\epsilon_k+\bar\partial U^\epsilon,$$

so that $R^\epsilon=\sum_{k\ge0}R^\epsilon_k$, where $R^\epsilon_0=1-\chi(|F|/\epsilon)$ and $R^\epsilon_k=\bar\partial\chi(|F|/\epsilon)\wedge u$ for $k\ge1$. Then $\lim_{\epsilon\to0}U^\epsilon=U$ and $\lim_{\epsilon\to0}R^\epsilon=R$, cf. (2.6) and (2.7), and moreover

$$\gamma^\epsilon:=\sum_{k=0}^N H^0_kR^\epsilon_k(\zeta)+f_1(z)\sum_{k=1}^N H^1_kU^\epsilon_k(\zeta). \tag{2.16}$$

is a weight in $D'\times D'$ for $\epsilon>0$. Let g be an arbitrary weight in $D'\times D'$. Then $\gamma^\epsilon\wedge g$ is again a weight in $D'\times D'$ and we get

$$\bar\partial(\gamma^\epsilon\wedge g\wedge B)_{N,N-1}=[\Delta^D]-(\gamma^\epsilon\wedge g)_{N,N} \tag{2.17}$$

in the current sense in $D'\times D'$, cf. (2.14). Let us proceed formally and, also, let us temporarily assume that V is Cohen-Macaulay and that (2.3) ends at level κ, so that R is $\bar\partial$-closed. Then, multiplying (2.17) with $R(z)\wedge dz$ and using (2.8) so that $f_1(z)R(z)=0$, we get that

(2.18)
$$\bar\partial\left(R(z)\wedge dz\wedge (HR^\epsilon(\zeta)\wedge g\wedge B)_{N,N-1}\right)=R(z)\wedge dz\wedge[\Delta^D]-R(z)\wedge dz\wedge(HR^\epsilon(\zeta)\wedge g)_{N,N},$$

where $HR^\epsilon=\sum_{k=0}^N H_k^0R_k^\epsilon$, cf. (2.16). In view of (2.12) we have $R(z)\wedge dz\wedge[\Delta^D]=\omega\wedge[\Delta^V]$, where $[\Delta^V]$ is the integration current along the diagonal $\Delta^V\subset V\times V\subset D\times D$, and formally letting $\epsilon\to 0$ in (2.18) we thus get

$$\bar\partial\Big(\omega(z)\wedge[V_z]\wedge(HR(\zeta)\wedge g\wedge B)_{N,N-1}\Big)=\omega\wedge[\Delta^V]-\omega(z)\wedge[V_z]\wedge(HR(\zeta)\wedge g)_{N,N}. \tag{2.19}$$

To see what this means we will use (2.12). Notice first that, since H, R, g, and B takes values in Λ_η, one can factor out $d\eta=d\eta_1\wedge\cdots\wedge d\eta_N$ from $(HR(\zeta)\wedge g\wedge B)_{N,N-1}$ and $(HR(\zeta)\wedge g)_{N,N}$. After making these factorization in (2.19) we may replace $d\eta$ by $C_\eta(z,\zeta)d\zeta$, where $C_\eta(z,\zeta)=N!\det(\partial\eta_j/\zeta_k)$, since $\omega(z)\wedge[V_z]$ has full degree in dz_j. More precisely, let $\epsilon_1,\dots,\epsilon_N$ be a basis for an auxiliary trivial complex vector bundle over $D\times D$ and replace all occurrences of $d\eta_j$ in H, g, and B by ϵ_j. Denote the resulting forms by $\hat H$, $\hat g$, and $\hat B$ respectively and let

$$k(z,\zeta)=C_\eta(z,\zeta)\epsilon_N^*\wedge\cdots\wedge\epsilon_1^*\lrcorner\sum_{k=0}^n\hat H_{p+k}^0\omega_k(\zeta)\wedge(\hat g\wedge\hat B)_{n-k,n-k-1} \tag{2.20}$$

$$p(z,\zeta)=C_\eta(z,\zeta)\epsilon_N^*\wedge\cdots\wedge\epsilon_1^*\lrcorner\sum_{k=0}^n\hat H_{p+k}^0\omega_k(\zeta)\wedge\hat g_{n-k,n-k}. \tag{2.21}$$

Notice that k and p have bidegrees $(n,n-1)$ and (n,n) respectively. In view of (2.12) we can replace $(HR\wedge g\wedge B)_{N,N-1}$ and $(HR\wedge g)_{N,N}$ with $[V_\zeta]\wedge k(z,\zeta)$ and $[V_\zeta]\wedge p(z,\zeta)$ respectively in (2.19). It follows that

$$\bar\partial\big(\omega(z)\wedge k(z,\zeta)\big)=\omega\wedge[\Delta^V]-\omega(z)\wedge p(z,\zeta)$$

holds in the current sense at least on $V_{reg}\times V_{reg}$. The formal computations above can be made rigorous, see [6, Section 5], and combined with Proposition 2.6 we get Proposition 2.7 below; notice that $\omega=\omega_0$ and $\bar\partial\omega=0$ since we are assuming that V is Cohen-Macaulay and that (2.3) ends at level κ.

The following result will be the starting point of the next section and it holds without any assumption about Cohen-Macaulay.

Proposition 2.7 (Lemma 5.3 in [6]). *With $k(z,\zeta)$ and $p(z,\zeta)$ defined by* (2.20) *and* (2.21) *respectively we have*

$$\bar\partial k(z,\zeta)=[\Delta^V]-p(z,\zeta)$$

in the sense of currents on $V_{reg}\times V_{reg}$.

Remark 2.8. In [6] it is assumed that g is a weight in $D'\times D$, where $D'\Subset D$ and $\zeta\mapsto g(z,\zeta)$ has compact support in D, but the proof goes through for any weight.

The integral operators $\mathscr{K}$ and $\mathscr{P}$ for forms in $\mathcal{W}^{0,q}$ introduced in [6] are defined as follows. Let g in (2.20) and (2.21) be a weight in $D'\times D$, where $D'\Subset D$ and $\zeta\mapsto g(z,\zeta)$ has compact support in D, cf. Section 2.4, and let $\mu\in\mathcal{W}^{0,q}(D)$. Since ω and B are almost semi-meromorphic $k(z,\zeta)$ and $p(z,\zeta)$ are also almost semi-meromorphic and it follows from Proposition 2.2 that $k(z,\zeta)\wedge\mu(\zeta)$ and $p(z,\zeta)\wedge\mu(\zeta)$ are in

$\mathcal{W}(V'\times V)$, where $V'=D'\cap V$. Let $\tilde{\pi}\colon V'_z\times V_\zeta\to V'_z$ be the natural projection onto V'_z. It follows that

$$\mathscr{K}\mu(z) := \tilde{\pi}_*\big(k(z,\zeta)\wedge\mu(\zeta)\big),$$

$$\mathscr{P}\mu(z) := \tilde{\pi}_*\big(p(z,\zeta)\wedge\mu(\zeta)\big),$$

are in $\mathcal{W}(V'_z)$. The sheaves $\mathscr{A}_V^{0,\bullet}$ are then morally defined to be the smallest sheaves that contain $\mathcal{E}_V^{0,\bullet}$ and are closed under operators $\mathscr{K}$ and under multiplication with $\mathcal{E}_V^{0,\bullet}$. More precisely, the stalk $\mathscr{A}_{V,x}^{0,q}$ consists of those germs of currents which can be written as a finite sum of of terms

$$\xi_m\wedge\mathscr{K}_m\big(\cdots\xi_1\wedge\mathscr{K}_1(\xi_0)\cdots\big),$$

where ξ_j are smooth $(0,*)$-forms and $\mathscr{K}_j$ are integral operators at x of the above form; cf. [6, Definition 7.1].

3. Koppelman formulas for (n,q)-forms

Let V be a pure n-dimensional analytic subset of a pseudoconvex domain $D\subset\mathbb{C}^N$ and let ω be a structure form on V. Let g be a weight in $D\times D'$, where $D'\subset D$ and let $k(z,\zeta)$ and $p(z,\zeta)$ be the kernels defined respectively in (2.20) and (2.21). Since k and p are almost semi-meromorphic it follows from Proposition 2.2 that if $\mu=\mu(z)\in\mathcal{W}^{n,q}(V)$, then $k(z,\zeta)\wedge\mu(z)$ and $p(z,\zeta)\wedge\mu(z)$ are well-defined currents in $\mathcal{W}(V\times V)$. Assume that $z\mapsto g(z,\zeta)$ has compact support in D or that μ has compact support in V. Let $\pi\colon V_z\times V'_\zeta\to V'_\zeta$ be the natural projection, where, as above, $V'=D'\cap V$, and define

$$\check{\mathscr{K}}\mu(\zeta) := \pi_*\big(k(z,\zeta)\wedge\mu(z)\big) \tag{3.1}$$

$$\check{\mathscr{P}}\mu(\zeta) := \pi_*\big(p(z,\zeta)\wedge\mu(z)\big). \tag{3.2}$$

It follows that $\check{\mathscr{K}}\mu$ and $\check{\mathscr{P}}\mu$ are well-defined currents in $\mathcal{W}(V'_\zeta)$. Notice that $\check{\mathscr{P}}\mu$ is of the form $\sum_r\omega_r\wedge\xi_r$, where ξ_r is a smooth $(0,*)$-form (with values in an appropriate bundle) in general, and holomorphic if the weight $g(z,\zeta)$ is chosen holomorphic in ζ; cf. (2.21). It is natural to write

$$\check{\mathscr{K}}\mu(\zeta)=\int_{V_z}k(z,\zeta)\wedge\mu(z),\quad \check{\mathscr{P}}\mu(\zeta)=\int_{V_z}p(z,\zeta)\wedge\mu(z).$$

We have the following analogue of Proposition 6.3 in [6].

Proposition 3.1. *Let $\mu(z)\in\mathcal{W}^{n,q}(V)$ and assume that $\bar\partial\mu\in\mathcal{W}^{n,q+1}(V)$. Let $\check{\mathscr{K}}$ and $\check{\mathscr{P}}$ be as above. Then*

$$\mu=\bar\partial\check{\mathscr{K}}\mu+\check{\mathscr{K}}(\bar\partial\mu)+\check{\mathscr{P}}\mu \tag{3.3}$$

in the sense of currents on V'_{reg}.

Proof. If $\varphi=\varphi(\zeta)$ is a $(0,n-q)$-test form on V'_{reg} it follows, cf. the beginning of Section 2.4, from Proposition 2.7 that

$$\varphi(z)=\bar\partial_z\int_{V'_\zeta}k(z,\zeta)\wedge\varphi(\zeta)+\int_{V'_\zeta}k(z,\zeta)\wedge\bar\partial\varphi(\zeta)+\int_{V'_\zeta}p(z,\zeta)\wedge\varphi(\zeta) \tag{3.4}$$

for $z \in V'_{reg}$. By [6, Lemma 6.1][4] the first two terms on the right hand side are smooth on V'. The last term is smooth V' since $z \mapsto p(z,\zeta)$ is smooth. Assume that $z \mapsto g(z,\zeta)$ has compact support in D. Then so have $z \mapsto k(z,\zeta)$ and $z \mapsto p(z,\zeta)$. Thus each term in the right hand side of (3.4) is a test form in z, and so μ acts on each term. Thus (3.3) follows in this case. If μ has compact support (3.3) holds without the assumption that $z \mapsto g(z,\zeta)$ has compact support.

For the general case, let $h = h(z)$ be a holomorphic tuple such that $\{h = 0\} = V_{sing}$ and let $\chi_\epsilon = \chi(|h|/\epsilon)$. Then the proposition holds for $\chi_\epsilon\mu$ (since k and p have compact support in z). Since $k(z,\zeta) \wedge \mu(z)$ and $p(z,\zeta) \wedge \mu(z)$ are in $\mathcal{W}(V' \times V)$ it follows that $\check{\mathscr{K}}(\chi_\epsilon\mu) \to \check{\mathscr{K}}\mu$ and that $\check{\mathscr{P}}(\chi_\epsilon\mu) \to \check{\mathscr{P}}\mu$ in the sense of currents, and consequently $\bar\partial\check{\mathscr{K}}(\chi_\epsilon\mu) \to \bar\partial\check{\mathscr{K}}\mu$ in the current sense. It remains to see that $\lim_{\epsilon\to 0}\check{\mathscr{K}}(\bar\partial(\chi_\epsilon\mu)) = \check{\mathscr{K}}(\bar\partial\mu)$. In fact, since by assumption $\bar\partial\mu \in \mathcal{W}(V)$ it follows that $\check{\mathscr{K}}(\chi_\epsilon\bar\partial\mu) \to \check{\mathscr{K}}(\bar\partial\mu)$ and so

$$\lim_{\epsilon\to 0}\check{\mathscr{K}}(\bar\partial(\chi_\epsilon\mu)) = \check{\mathscr{K}}(\bar\partial\mu) + \lim_{\epsilon\to 0}\check{\mathscr{K}}(\bar\partial\chi_\epsilon \wedge \mu); \tag{3.5}$$

it also follows that

$$\bar\partial\chi_\epsilon \wedge \mu = \bar\partial(\chi_\epsilon\mu) - \chi_\epsilon\bar\partial\mu \to \bar\partial\mu - \bar\partial\mu = 0. \tag{3.6}$$

Now, if ζ is in a compact subset of V'_{reg} and ϵ is sufficiently small, then $k(z,\zeta)\wedge\bar\partial\chi_\epsilon(z)$ is a smooth form times $\omega = \omega(\zeta)$. Since $\mu(z)\wedge\omega(\zeta)$ is just a tensor product it follows from (3.6) that $\bar\partial\chi_\epsilon(z) \wedge \mu(z) \wedge \omega(\zeta) \to 0$. Hence, $\check{\mathscr{K}}(\bar\partial\chi_\epsilon \wedge \mu) \to 0$ as a current on V'_{reg} and so by (3.5) we have $\lim_{\epsilon\to 0}\check{\mathscr{K}}(\bar\partial(\chi_\epsilon\mu)) = \check{\mathscr{K}}(\bar\partial\mu)$. □

4. The dualizing Dolbeault complex of $\mathscr{B}_X^{n,q}$-currents

Let X be a reduced complex space of pure dimension n. We define our sheaves $\mathscr{B}_X^{n,\bullet}$ in a way similar to the definition of $\mathscr{A}_X^{0,\bullet}$; see the end of Section 2.5. In a moral sense $\oplus_q\mathscr{B}_X^{n,q}$ then becomes the smallest sheaf that contains $\oplus_q\mathcal{E}_X^{n,q}$ and that is closed under integral operators $\check{\mathscr{K}}$ and exterior products with elements of $\oplus_q\mathcal{E}_X^{0,q}$.

Definition 4.1. We say that an (n,q)-current ψ on an open set $V \subset X$ is a section of $\mathscr{B}_X^{n,q}$, $\psi \in \mathscr{B}^{n,q}(V)$, if, for every $x \in V$, the germ ψ_x can be written as a finite sum of terms

$$\xi_m \wedge \check{\mathscr{K}}_m\left(\cdots \xi_1 \wedge \check{\mathscr{K}}_1(\omega \wedge \xi_0)\cdots\right), \tag{4.1}$$

where ξ_j are smooth $(0,*)$-forms, $\check{\mathscr{K}}_j$ are integral operators at x given by (3.1) with kernels of the form (2.20), and ω is a structure form at x.

Notice that ω takes values in some bundle $\oplus_j E_j$ so we let ξ_0 take values in $\oplus_j E_j^*$ to make $\omega \wedge \xi_0$ scalar valued.

It is clear that $\check{\mathscr{K}}$ preserves $\oplus_q\mathscr{B}_X^{n,q}$. Notice that we allow $m = 0$ in the definition above so that $\mathscr{B}_X^{n,\bullet}$ contains all currents of the form $\omega \wedge \xi_0$, where ξ_0 is smooth with values in $\oplus_j E_j^*$. Since $\check{\mathscr{P}}\mu$ is of the form $\omega \wedge \xi$ for a smooth ξ, also $\check{\mathscr{P}}$ preserves $\oplus_q\mathscr{B}_X^{n,q}$.

Recall that if $\mu \in \mathcal{W}^{n,*}(V)$, then $\check{\mathscr{K}}\mu \in \mathcal{W}^{n,*}(V')$, where V' is a relatively compact subset of V. Since $\omega \wedge \xi_0 \in \mathcal{W}_X^{n,*}$ it follows that $\mathscr{B}_X^{n,q}$ is a subsheaf of $\mathcal{W}_X^{n,q}$. In fact, by Proposition 4.3 below we can say more.

[4]The proof goes through also in our setting, i.e., when g not necessarily has compact support in D_ζ but $\varphi(\zeta)$ has.

Definition 4.2. A current $\mu \in \oplus_q \mathcal{W}_X^{n,q}$ is said to be in *the domain of* $\bar\partial$, $\mu \in \operatorname{Dom}\bar\partial$, if $\bar\partial\mu \in \oplus_q \mathcal{W}_X^{n,q}$.

Assume that $\mu \in \mathcal{W}_X^{n,q}$ is smooth on X_{reg}, let h be a holomorphic tuple such that $\{h = 0\} = X_{sing}$, and, as above, let $\chi_\epsilon = \chi(|h|/\epsilon)$. Then $\bar\partial(\chi_\epsilon\mu) \to \bar\partial\mu$ since μ has the SEP. In view of the first equality in (3.6) it follows that $\bar\partial\mu$ has the SEP if and only if $\bar\partial\chi_\epsilon \wedge \mu \to 0$ as $\epsilon \to 0$; this last condition can be interpreted as a "boundary condition" on μ at X_{sing}.

Proposition 4.3. *Let X be a reduced complex space of pure dimension n. Then*
(i) $\mathscr{B}_X^{n,q}\big|_{X_{reg}} = \mathcal{E}_X^{n,q}\big|_{X_{reg}}$,
(ii) $\mathcal{E}_X^{n,q} \subset \mathscr{B}_X^{n,q} \subset Dom\,\bar\partial$.

To prove (i) we need to prove that if $\mu \in \mathcal{W}(V)$ is smooth in a neighborhood of a given point $x \in V'_{reg}$, then $\check{\mathscr{K}}\mu(z)$ is smooth in a neighborhood of x. This is proved in the same way as part (i) of Lemma 6.1 in [6]. The proof (of the second inclusion) of (ii) is similar to the proof that $\mathscr{A}_X^{0,q} \subset \operatorname{Dom}\bar\partial$ in [6], see Section 7 and Lemmas 6.4 and 4.1 in [6]. We include a proof for the reader's convenience.

Proof of (ii). Let ψ be a smooth (n,q)-form on X and let $\omega = \sum_r \omega_r$ be a structure form. Then, by Proposition 2.6, there is smooth $(0,q)$-form ξ (with values in the appropriate bundle) such that $\psi = \omega_0 \wedge \xi$ and so $\mathcal{E}_X^{n,q} \subset \mathscr{B}_X^{n,q}$.

To prove the second inclusion of (ii) we may assume that μ is of the form (4.1). Let $k_j(w^{j-1}, w^j)$, $j = 1, \dots, m$, be the integral kernel corresponding to $\check{\mathscr{K}}_j$; w^j are coordinates on V for each j. We define an almost semi-meromorphic current T on V^{m+1} (the $m+1$-fold Cartesian product) by

$$T := \bigwedge_{j=1}^{m} k_j(w^{j-1}, w^j) \wedge \omega(w^0), \tag{4.2}$$

and we let T_r be the term of T corresponding to ω_r. Notice that $\pi_*(\xi \wedge T) = \mu$ for a suitable smooth $(0,*)$-form ξ on V^{m+1}, where $\pi\colon V^{m+1} \to V_{w^m}$ is the natural projection. We claim that

$$\lim_{\epsilon\to 0} \bar\partial\chi(|h(w^m)|/\epsilon) \wedge T_r = 0 \tag{4.3}$$

for all r, where h is a holomorphic tuple such that $\{h = 0\} = V_{sing}$. Taking this for granted,

$$\lim_{\epsilon\to 0} \bar\partial\chi_\epsilon \wedge \mu = \pi_*\big(\lim_{\epsilon\to 0} \bar\partial\chi(|h(w^m)|/\epsilon) \wedge \xi \wedge T\big) = 0,$$

and thus $\mu \in \operatorname{Dom}\bar\partial$, cf. the discussion after Definition 4.2.

We will prove that (4.3) holds for all r by double induction over m and r. If $m = 0$ then $T = \omega(w^0)$ and, since $\bar\partial\omega_r = f_{r+1}|_V\omega_{r+1}$ by (2.5), it follows that $\bar\partial T$ has the SEP, i.e., $\lim_{\epsilon\to 0}\bar\partial\chi(|h|/\epsilon) \wedge T = 0$.

Assume that (4.3) holds for $m \le k-1$ and all r. The left hand side of (4.3), with $m = k$, defines a pseudomeromorphic current τ_r of bidegree $(*, kn-k+r+1)$ since each k_j has bidegree $(*, n-1)$ and clearly $\operatorname{supp}\tau_r \subset Sing(V_{w^m}) \times V^m$. If $w^j \neq w^{j-1}$, then $k_j(w^{j-1}, w^j)$ is a smooth form times some structure form $\tilde\omega(w^j)$. Thus T, with $m = k$, is a smooth form times the *tensor product* of two currents, each of which is of the form (4.2) with $m < k$. By the induction hypothesis, it follows that (4.3), with $m = k$, holds outside $\{w^j = w^{j-1}\}$ for all j. Hence, τ_r has support in $\{w^1 = \cdots = w^k\} \cap (Sing(V_{w^m}) \times V^m)$, which has codimension at least $kn+1$ in

V^{k+1}. Since τ_0 has bidegree $(*, kn-k+1)$, $k \geq 1$, it follows from the dimension principle that $\tau_0 = 0$.

By Proposition 2.5, there is a $(0,1)$-form α_1 such that $\omega_1 = \alpha_1 \omega_0$ and α_1 is smooth outside V^1 (cf. (2.4)) which has codimension at least 2 in V. Since $\tau_1 = \alpha_1(w^0)\tau_0$ outside $V^1_{w^0}$ and $\tau_0 = 0$ it follows that τ_1 has support in $\{w^1 = \cdots = w^k\} \cap (V^1_{w^0} \times V^m)$. This set has codimension at least $kn+2$ in V^{m+1} and τ_1 has bidegree $(*, kn-k+2)$ so the dimension principle shows that $\tau_1 = 0$. Continuing in this way we get that $\tau_r = 0$ for all r and hence, (4.3) holds with $m = k$. $\square$

Theorem 4.4. *Let X be a reduced complex space of pure dimension n. Then* $\bar\partial\colon \mathscr{B}^{n,q}_X \to \mathscr{B}^{n,q+1}_X$.

Proof. Let ψ be a germ of a current in $\mathscr{B}^{n,q}_X$ at some point x; we may assume that

$$\psi = \xi_m \wedge \check{\mathscr{K}}_m \left(\cdots \xi_1 \wedge \check{\mathscr{K}}_1(\omega \wedge \xi_0) \cdots \right),$$

see Definition 4.1.

We will prove the theorem by induction over m. Assume first that $m = 0$ so that $\psi = \omega \wedge \xi_0$; recall that ξ_0 takes values in $\oplus_j E^*_j$ so that ψ is scalar valued. Then, by Proposition 2.5, we have that

$$\bar\partial\psi = \bar\partial\omega \wedge \xi_0 \pm \omega \wedge \bar\partial\xi_0 = f\omega \wedge \xi_0 \pm \omega \wedge \bar\partial\xi_0 = \omega \wedge f^*\xi_0 \pm \omega \wedge \bar\partial\xi_0,$$

where $f = \oplus_{r=0}^{n} f_{p+r}|_V$ and f^* is the transpose of f. Hence, $\bar\partial\psi$ is in $\mathscr{B}^{n,q+1}_X$. Assume now that $\bar\partial\psi' \in \oplus_q \mathscr{B}^{n,q}_X$, where

$$\psi' = \xi_{m-1} \wedge \check{\mathscr{K}}_{m-1} \left(\cdots \xi_1 \wedge \check{\mathscr{K}}_1(\omega \wedge \xi_0) \cdots \right).$$

Then $\psi' \in \mathrm{Dom}\,\bar\partial \subset \mathcal{W}_X$ and by Proposition 4.3 ψ' is smooth on X_{reg}. Thus, from Proposition 3.1 it follows that

$$\psi' = \bar\partial\check{\mathscr{K}}_m\psi' + \check{\mathscr{K}}_m(\bar\partial\psi') + \check{\mathscr{P}}_m\psi' \tag{4.4}$$

in the current sense on V_{reg}, where V is some neighborhood of x. By the induction hypothesis, $\bar\partial\psi' \in \oplus_q \mathscr{B}^{n,q}_X$ and since $\check{\mathscr{K}}_m$ and $\check{\mathscr{P}}_m$ preserve $\oplus_q \mathscr{B}^{n,q}_X$ and furthermore $\oplus_q \mathscr{B}^{n,q}_X \subset \mathrm{Dom}\,\bar\partial$ it follows that every term of (4.4) has the SEP. Thus, (4.4) holds in fact on V. Finally, notice that $\psi = \xi_m \wedge \check{\mathscr{K}}_m\psi'$ and so, since ψ', $\check{\mathscr{K}}_m(\bar\partial\psi')$, and $\check{\mathscr{P}}_m\psi'$ all are in $\oplus_q \mathscr{B}^{n,q}_X$, it follows that $\bar\partial\psi \in \mathscr{B}^{n,q+1}_X$. $\square$

Proof of Theorem 1.1. Choose a weight g in $D \times D'$, where $D' \Subset D$, such that $z \mapsto g(z,\zeta)$ has compact support in D, cf. Section 2.4. Let $k(z,\zeta)$ and $p(z,\zeta)$ be the kernels defined by (2.20) and (2.21), respectively, and let $\check{\mathscr{K}}$ and $\check{\mathscr{P}}$ be the associated integral operators.

Let $\psi \in \mathscr{B}^{n,q}(V)$. By Proposition 3.1,

$$\psi = \bar\partial\check{\mathscr{K}}\psi + \check{\mathscr{K}}(\bar\partial\psi) + \check{\mathscr{P}}\psi \tag{4.5}$$

holds on V'_{reg}. Since $\check{\mathscr{K}}$ and $\check{\mathscr{P}}$ map $\oplus_q \mathscr{B}^{n,q}(V)$ to $\oplus_q \mathscr{B}^{n,q}(V')$ it follows from Theorem 4.4 that every term of (4.5) has the SEP. Hence, (4.5) holds on V' and the theorem follows. $\square$

Proof of Theorem 1.2. Let V be a pure n-dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, let $\mathcal{J}_V$ be the sheaf in D defined by V, let $i\colon V \hookrightarrow D$ be the inclusion, and, as above, let $\kappa = N - n$ be the codimension of V in D. Let (2.3) be a free resolution of $\mathscr{O}_D/\mathcal{J}_V$ in (possibly a slightly smaller domain still denoted) D and let $\omega = \sum_r \omega_r$ be an associated structure form.

Dualizing the complex (2.3) and tensoring with the invertible sheaf Ω_D^N gives the complex

$$0 \to \mathscr{O}(E_0^*) \otimes_{\mathscr{O}_D} \Omega_D^N \xrightarrow{f_1^*} \cdots \xrightarrow{f_m^*} \mathscr{O}(E_m^*) \otimes_{\mathscr{O}_D} \Omega_D^N \to 0. \tag{4.6}$$

It is well-known that the cohomology sheaves of (4.6) are isomorphic to $\mathscr{E}xt^\bullet(\mathscr{O}_D/\mathcal{J}_V, \Omega_D^N)$ and that $\mathscr{E}xt^k(\mathscr{O}_D/\mathcal{J}_V, \Omega_D^N) = 0$ for $k < \kappa$. Notice that if V is Cohen-Macaulay, i.e., if we can take $m = \kappa = \operatorname{codim} V$ in (2.3), then $\mathscr{E}xt^k(\mathscr{O}_D/\mathcal{J}_V, \Omega_D^N) = 0$ for $k \neq \kappa$.

We define mappings $\varrho_k\colon \mathscr{O}(E_{\kappa+k}^*) \otimes \Omega_D^N \to \mathscr{B}_V^{n,k}$ by letting $\varrho_k(hdz) = 0$ for $k < 0$ and $\varrho_k(hdz) = \omega_k \cdot h$ for $k \geq 0$; here we let $\mathscr{B}_V^{n,k} := 0$ for $k < 0$ and $\mathscr{O}(E_k^*) \otimes \Omega_D^N := 0$ for $k > m$. We get a map

$$\varrho_\bullet\colon \left(\mathscr{O}(E_{\kappa+\bullet}^*) \otimes \Omega_D^N, f_{\kappa+\bullet}^*\right) \longrightarrow \left(\mathscr{B}_V^{n,\bullet}, \bar\partial\right) \tag{4.7}$$

which is a morphism of complexes since if $h \in \mathscr{O}(E_{\kappa+k}^*)$, then, by Proposition 2.5,

$$\bar\partial \varrho_k(hdz) = \bar\partial\omega_k \cdot h = f_{\kappa+k+1}\omega_{k+1} \cdot h = \omega_{k+1} \cdot f_{\kappa+k+1}^* h = \varrho_{k+1}(f_{\kappa+k+1}^* h).$$

Hence, (4.7) induces a map on cohomology. We claim that $\varrho_\bullet$ in fact is a quasi-isomorphism, i.e., that $\varrho_\bullet$ induces an isomorphism on cohomology level. Given the claim it follows that $\mathscr{H}^k(\mathscr{B}_V^{n,\bullet})$ is coherent since the corresponding cohomology sheaf of $(\mathscr{O}(E_{\kappa+\bullet}^*) \otimes \Omega_D^N, f_{\kappa+\bullet}^*)$ is $\mathscr{E}xt^{\kappa+k}(\mathscr{O}_D/\mathcal{J}_V, \Omega_D^N)$, which is coherent.

To prove the claim, recall first that $i_*\omega_k = R_k \wedge dz$. Thus, by [4, Theorem 7.1] the mapping on cohomology is injective. For the surjectivity, choose a weight g in $D \times D'$, where $D' \Subset D$, such that g is holomorphic in ζ and has compact support in D_z, cf. Section 2.4, let $k(z,\zeta)$ and $p(z,\zeta)$ be the integral kernels defined by (2.20) and (2.21), respectively, and let $\check{\mathscr{K}}$ and $\check{\mathscr{P}}$ be the corresponding integral operators. Let $\psi \in \mathscr{B}^{n,k}(V)$ be $\bar\partial$-closed. By Theorem 1.1 we get

$$\psi(\zeta) = \bar\partial \int_{V_z} k(z,\zeta) \wedge \psi(z) + \int_{V_z} p(z,\zeta) \wedge \psi(z)$$

in $V \cap D'$. Hence, the $\bar\partial$-cohomology class of ψ is represented by the last integral. Since g is holomorphic in ζ, the summand with index k in (2.21) has exactly $n-k$ differentials of the form $d\bar z_j$ (and k differentials of the form $d\bar\zeta_j$). It follows that

$$\int_{V_z} p(z,\zeta) \wedge \psi(z) = \int_{V_z} C_\eta(z,\zeta)\epsilon_N^* \wedge \cdots \wedge \epsilon_1^* \lrcorner \hat H_{p+k}^0 \omega_k(\zeta) \wedge \hat g_{n-k,n-k} \wedge \psi(z)$$
$$=: \omega_k(\zeta) \wedge \int_{V_z} G(z,\zeta) \wedge \psi(z),$$

where G takes values in E_{p+k}^*. Note that G is holomorphic in ζ since g is. We will show that

$$f_{p+k+1}^* \int_{V_z} G(z,\zeta) \wedge \psi(z) = 0. \tag{4.8}$$

Taking (4.8) for granted, it follows that the class of ψ is in the image of the map on cohomology induced by ϱ_k, which proves the claim.

To prove (4.8) first note that $d\eta \wedge G = H^0_{p+k} \wedge g_{n-k,n-k}$. By (2.15),

$$f^*_{p+k+1} H^0_{p+k} \wedge g_{n-k,n-k} = H^0_{p+k+1} f_{p+k+1} \wedge g_{n-k,n-k} = \delta_\eta H^0_{p+k+1} \wedge g_{n-k,n-k} + f_1(z) H^1_{p+k} \wedge g_{n-k,n-k}. \tag{4.9}$$

Since $H^0_{p+k+1} \wedge g_{n-k,n-k}$ takes values in Λ_η and is of degree $(N+1, n-k)$ it vanishes and thus the first term in the right-most expression in (4.9) equals

$$\pm H^0_{p+k+1} \wedge \delta_\eta g_{n-k,n-k} = \pm H^0_{p+k+1} \wedge \bar\partial g_{n-k-1,n-k-1} = \pm\bar\partial\big(H^0_{p+k+1} \wedge g_{n-k-1,n-k-1}\big),$$

where we have used that $\nabla_\eta g = 0$ and that H^0_{p+k+1} is holomorphic. Using that $H^1_{p+k} \wedge g_{n-k,n-k}$ and $H^0_{p+k+1} \wedge g_{n-k-1,n-k-1}$ take values in Λ_η and have degree $(N,*)$ we get that

$$f^*_{p+k+1} H^0_{p+k} \wedge g_{n-k,n-k} = d\eta \wedge \big(\bar\partial A + f_1(z) B\big)$$

for some smooth A and B. Hence

$$f^*_{p+k+1} \int_{V_z} G(z,\zeta) \wedge \psi(z) = \int_{V_z} \bar\partial A \wedge \psi(z) + \int_{V_z} f_1(z) B \wedge \psi(z) = 0. \tag{4.10}$$

The first integral vanishes by Stokes' theorem since ψ is $\bar\partial$-closed and G has compact support in z since g has. The second integral vanishes since $f_1(z) = 0$ on V_z.

If V is Cohen-Macaulay, then (4.6) is exact except for at level p and so $(\mathscr{B}^{n,\bullet}_V, \bar\partial)$ is exact except for at level 0 where the cohomology is $\omega^{n,0}_V = \ker(\bar\partial\colon \mathscr{B}^{n,0}_V \to \mathscr{B}^{n,1}_V)$. Thus, (1.4) is exact. □

5. The trace map

The basic result of this section is the following theorem. It is the key to define our trace map.

Theorem 5.1. *Let X be a reduced complex space of pure dimension n. There is a unique map*

$$\wedge\colon \mathscr{B}^{n,q}_X \times \mathscr{A}^{0,q'}_X \to \mathcal{W}^{n,q+q'}_X \cap Dom\,\bar\partial$$

extending the exterior product on X_{reg}.

The uniqueness is clear since two currents with the SEP that are equal on X_{reg} are equal on X. It is moreover clear that $\wedge$ is $\mathcal{E}^{0,0}_X$-bilinear. Indeed, if, e.g., φ_1 and φ_2 are sections of $\mathscr{A}^{0,q'}_X$, ψ is a section of $\mathscr{B}^{n,q}_X$, and ξ_1 and ξ_2 are sections of $\mathcal{E}^{0,0}_X$, then $\psi \wedge (\xi_1\varphi_1 + \xi_2\varphi_2)$, $\psi \wedge \xi_1\varphi$, and $\psi \wedge \xi_2\varphi_2$ have the SEP by Theorem 5.1 and $\psi \wedge (\xi_1\varphi_1 + \xi_2\varphi_2) = \psi \wedge \xi_1\varphi_1 + \psi \wedge \xi_2\varphi_2$ on X_{reg}. We get bilinear pairings of $\mathbb{C}$-vector spaces, $\mathscr{B}^{n,n-q}_c(X) \times \mathscr{A}^{0,q}(X) \to \mathbb{C}$ and $\mathscr{B}^{n,n-q}(X) \times \mathscr{A}^{0,q}_c(X) \to \mathbb{C}$, given by $(\psi,\varphi) \mapsto \int_X \psi \wedge \varphi := \psi \wedge \varphi.1$, where 1 here denotes the function constantly equal to 1; we will refer to these maps as *trace maps on the level of currents*. We also get *trace maps on the level of cohomology*:

Corollary 5.2. *Let φ and ψ be sections of $\mathscr{A}^{0,q'}_X$ and $\mathscr{B}^{n,q}_X$ respectively. Then $\bar\partial(\psi \wedge \varphi) = \bar\partial\psi \wedge \varphi \pm \psi \wedge \bar\partial\varphi$. Moreover, there are bilinear maps of $\mathbb{C}$-vector spaces*

$$H^q\big(\mathscr{A}^{0,\bullet}(X), \bar\partial\big) \times H^{n-q}\big(\mathscr{B}^{n,\bullet}_c(X), \bar\partial\big) \to \mathbb{C},$$

$$H^q\big(\mathscr{A}^{0,\bullet}_c(X), \bar\partial\big) \times H^{n-q}\big(\mathscr{B}^{n,\bullet}(X), \bar\partial\big) \to \mathbb{C},$$

given by $([\varphi]_{\bar\partial}, [\psi]_{\bar\partial}) \mapsto \int_X \psi \wedge \varphi$.

Proof. By Theorem 5.1, $\bar\partial(\psi\wedge\varphi)$ has the SEP; cf. Definition 4.2. By Theorem 4.4 and [6, Theorem 1.2], respectively, $\bar\partial\psi$ is a section of $\mathscr{B}_X^{n,q+1}$ and $\bar\partial\varphi$ is a section of $\mathscr{A}_X^{0,q'+1}$. Thus, $\bar\partial\psi\wedge\varphi$ and $\psi\wedge\bar\partial\varphi$ have the SEP by Theorem 5.1 and so $\bar\partial(\psi\wedge\varphi)=\bar\partial\psi\wedge\varphi\pm\psi\wedge\bar\partial\varphi$ since it holds on X_{reg}. The last part of the corollary immediately follows. □

Proof of Theorem 5.1. We have already noticed that if $\psi|_{X_{reg}}\wedge\varphi|_{X_{reg}}$ has an extension with the SEP, then it is unique. To see that such an extension exists, let V be a relatively compact open subset of a pure n-dimensional analytic subset of some pseudoconvex domain in some $\mathbb{C}^N$. Let $\phi=(\phi_1,\dots,\phi_s)$ be generators for the radical ideal sheaf over $V\times V$ associated to the diagonal $\Delta^V\subset V\times V$. Let

$$A_\epsilon := \chi(|\phi|/\epsilon)\frac{\partial\log|\phi|^2}{2\pi i}\wedge(dd^c\log|\phi|^2)^{n-1}.$$

Notice that if $p\colon W\to V\times V$ is a holomorphic map such that, locally on W, $p^*\phi=\phi_0\phi'$ for a holomorphic function ϕ_0 and a non-vanishing holomorphic tuple ϕ', then

$$2\pi i p^*A_\epsilon = \chi(|\phi_0\phi'|/\epsilon)\big(d\phi_0/\phi_0+\partial|\phi'|^2/|\phi'|^2\big)\wedge(dd^c\log|\phi'|^2)^{n-1}. \tag{5.1}$$

Thus, in view of Section 2.1, $A:=\lim_{\epsilon\to0}A_\epsilon$ exists and defines an almost semi-meromorphic current on $V\times V$. Let

$$M_\epsilon := \bar\partial\chi(|\phi|/\epsilon)\wedge\frac{\partial\log|\phi|^2}{2\pi i}\wedge(dd^c\log|\phi|^2)^{n-1} = \bar\partial A_\epsilon-\chi(|\phi|/\epsilon)(dd^c\log|\phi|^2)^n. \tag{5.2}$$

Similarly to (5.1) one checks that the limit of the last term on the right-hand side defines an almost semi-meromorphic current. Thus, the limit $M:=\lim_{\epsilon\to0}M_\epsilon$ exists and defines a pseudomeromorphic (n,n)-current on $V\times V$ supported on Δ^V. Notice that M is the difference of an almost semi-meromorphic current and the $\bar\partial$-image of such a current. Hence, by Proposition 2.2, for any pseudomeromorphic current τ, $M\wedge\tau$ is a well-defined pseudomeromorphic current. It is well-known that $M=[\Delta^V]$ on $V_{reg}\times V_{reg}$ and so, in view of the dimension principle, $M=[\Delta^V]$ on $V\times V$; cf. [7, Corollary 1.3].

Let $\psi\in\mathscr{B}^{n,q}(V)$ and $\varphi\in\mathscr{A}^{0,q'}(V)$. The tensor product $\psi(w)\wedge\varphi(z)$ is a pseudomeromorphic current on $V\times V$ by Section 2.1, and so $M\wedge\psi(w)\wedge\varphi(z)=\lim_{\epsilon\to0}M_\epsilon\wedge\psi(w)\wedge\varphi(z)$ is a pseudomeromorphic currents on $V\times V$ with support on Δ^V. Notice also that since ψ and φ are smooth on V_{reg}, we have

$$M\wedge\psi(w)\wedge\varphi(z)=[\Delta^V]\wedge\psi(w)\wedge\varphi(z)=i_*(\psi|_{V_{reg}}\wedge\varphi|_{V_{reg}}) \tag{5.3}$$

on $V_{reg}\times V_{reg}$, where $i\colon\Delta^V\to V\times V$ is the inclusion and where we have made the identification $\Delta^V\simeq V$.

Lemma 5.3. *The pseudomeromorphic currents $M\wedge\psi(w)\wedge\varphi(z)$ and $\bar\partial\big(M\wedge\psi(w)\wedge\varphi(z)\big)$ have the SEP with respect to Δ^V.*

Let g be a holomorphic function such that $g|_{\Delta^V}=0$. Then $g[\Delta^V]=0=dg\wedge[\Delta^V]$ and so, since $\psi(w)\wedge\varphi(z)$ is smooth on $V_{reg}\times V_{reg}$ and $M=[\Delta^V]$, we have

$$gM\wedge\psi(w)\wedge\varphi(z)=dg\wedge M\wedge\psi(w)\wedge\varphi(z)=0 \tag{5.4}$$

on $V_{reg}\times V_{reg}$. In fact, by Lemma 5.3, (5.4) holds on $V\times V$ and so, by Proposition 2.3 and Lemma 5.3 again, there is a $\mu\in\mathcal{W}(V)$ such that $M\wedge\psi(w)\wedge\varphi(z)=i_*\mu$. Hence, in view of (5.3), μ is an extension of $\psi|_{V_{reg}}\wedge\varphi|_{V_{reg}}$ to V with the SEP. We will denote the extension by $\psi\wedge\varphi$.

It remains to see that $\psi\wedge\varphi$ is in $\operatorname{Dom}\bar\partial$. However, $\bar\partial\big(M\wedge\psi(w)\wedge\varphi(z)\big)=i_*\bar\partial(\psi\wedge\varphi)$ and $\bar\partial\big(M\wedge\psi(w)\wedge\varphi(z)\big)$ has the SEP with respect to Δ^V by Lemma 5.3. It follows that $\bar\partial(\psi\wedge\varphi)$ has the SEP on V, i.e., $\psi\wedge\varphi$ is in $\operatorname{Dom}\bar\partial$. □

Proof of Lemma 5.3. We may assume, cf. Definition 4.1 and the end of Section 2.5, that

$$\psi=\xi_m\wedge\check{\mathscr{K}}_m\left(\cdots\xi_1\wedge\check{\mathscr{K}}_1(\omega\wedge\xi_0)\cdots\right),\quad \varphi=\tilde\xi_\ell\wedge\mathscr{K}_\ell\left(\cdots\tilde\xi_1\wedge\mathscr{K}_1(\tilde\xi_0)\cdots\right),$$

where ξ_i and $\tilde\xi_j$ are smooth $(0,*)$-forms, $\omega=\sum_k\omega_k$ is a structure form associated with a free resolution (2.3), and $\check{\mathscr{K}}_i$ and $\mathscr{K}_j$ are integral operators for $(n,*)$-forms and $(0,*)$-forms respectively. Let $\check k_j(w^{j-1},w^j)$ be the integral kernel corresponding to $\check{\mathscr{K}}_j$ and let $k_j(z^j,z^{j-1})$ be the integral kernel corresponding to $\mathscr{K}_j$; w^j and z^j are coordinates on V. We will assume that for each j, $z^j\mapsto k_{j+1}(z^{j+1},z^j)$ has compact support where $z^j\mapsto k_j(z^j,z^{j-1})$ is defined and similarly for $\check k_j$; possibly we will have to multiply by a smooth cut-off function that we however will suppress. Now, consider

$$T:=\lim_{\epsilon\to0}M_\epsilon(z^\ell,w^m)\wedge\bigwedge_{j=1}^m\check k_j(w^{j-1},w^j)\wedge\omega(w^0)\wedge\bigwedge_{j=1}^\ell k_j(z^j,z^{j-1}),\tag{5.5}$$

which is a pseudomeromorphic current on $V^{\ell+m+2}$ supported on $\{z^\ell=w^m\}$; cf. Proposition 2.2.[5] Notice that $M(z^\ell,w^m)\wedge\psi(w^m)\wedge\varphi(z^\ell)=\pi_*(T\wedge\xi)$, where $\pi\colon V^{\ell+m+2}\to V_{z^\ell}\times V_{w^m}$ is the natural projection and ξ is a suitable smooth form on $V^{\ell+m+2}$. In view of the paragraph following the dimension principle in Section 2.1, it suffices to show that T and $\bar\partial T$ have the SEP with respect to $\{z^\ell=w^m\}$. Let $h=h(z^\ell,w^m)$ be a germ of a holomorphic tuple in $V\times V$ that is generically non-vanishing on the diagonal; we will consider h also as a germ of a tuple on $V^{\ell+m+2}$ and we denote its zero-set there by H. In view of Section 2.1, what we are to show is that $\mathbf{1}_HT=\mathbf{1}_H\bar\partial T=0$.

Let T_k be the part of T corresponding to $\omega_k(w^0)$ and notice that T_k is a pseudomeromorphic current of bidegree $(*,n(\ell+m+1)-m-\ell+k)$. We will show that T and $\bar\partial T$ have the SEP by double induction over $\ell+m$ and k.

Assume first that $\ell=m=0$. Then $T_k=M(z^0,w^0)\wedge\omega_k(w^0)$ and we know that $T_k=[\Delta^V]\wedge\omega_k(w^0)$ for $w^0\in V_{reg}$ since $\omega_k(w^0)$ is smooth there. Hence, since $[\Delta^V]$ has the SEP with respect to Δ^V, $\mathbf{1}_HT_k=0$ outside of $\{w^0\in V_{sing}\}$ and it follows that $\operatorname{supp}(\mathbf{1}_HT_k)\subset\{z^0=w^0\in V_{sing}\}$, which has codimension $\geq n+1$ in $V\times V$. Since $\mathbf{1}_HT_0$ has bidegree $(*,n)$, the dimension principle implies that $\mathbf{1}_HT_0=0$. By Proposition 2.5, $\omega_k=\alpha_k\omega_{k-1}$, where α_k is smooth outside of V^k, which has codimension $\geq k+1$ in V. Hence, $\operatorname{supp}\mathbf{1}_HT_1\subset\{w^0\in V^1\}$, which has codimension $\geq n+2$ in $V\times V$. Since $\mathbf{1}_HT_1$ has bidegree $(*,n+1)$, the dimension principle implies that also $\mathbf{1}_HT_1=0$. Continuing in this way, we get that $\mathbf{1}_HT_k=0$. Hence, $T=M(z^0,w^0)\wedge\omega(w^0)$ has the SEP with respect to Δ^V and arguing as in the paragraph following Lemma 5.3 we see that $T=i_*\omega$. Since $\bar\partial\omega=f\omega$ by Proposition 2.5, it follows that $\bar\partial T=i_*\bar\partial\omega=i_*f\omega$ and thus, $\bar\partial T$ has the SEP with respect to Δ^V.

[5]In this proof V^j will mean either the Cartesian product of j copies of V or the j^{th} set in (2.4). We hope that it will be clear from the context what we are aiming at.

Let now $\ell + m = s \geq 1$ in (5.5) and assume that T and $\bar\partial T$ have the SEP with respect to $\{z^\ell = w^m\}$ for $\ell + m \leq s-1$. Let $1 \leq r \leq \ell$; if $z^{r-1} \neq z^r$ then $k_r(z^r, z^{r-1})$ is a smooth form times some structure form $\tilde\omega(z^{r-1})$. Hence, outside of $\{z^r = z^{r-1}\}$, T is a smooth form times the *tensor product* of

$$\tilde\omega(z^{r-1}) \bigwedge_{j=1}^{r-1} k_j(z^j, z^{j-1})$$

and some current $\tilde T$, where $\tilde T$ is of the form (5.5) with $\ell + m = s - r$ depending on the variables $z^r, \ldots, z^\ell$ and $w^0, \ldots, w^m$. From the induction hypothesis it thus follows that $\mathbf{1}_H T$ and $\mathbf{1}_H \bar\partial T$ have supports contained in $\{z^0 = \ldots = z^\ell\}$. Similarly, let $1 \leq r \leq m$. If $w^{r-1} \neq w^r$ then $\check k_r(w^{r-1}, w^r)$ is a smooth form times some structure form $\tilde\omega(w^r)$ and so, outside of $\{w^{r-1} = w^r\}$, T is a smooth form times the tensor product of

$$\bigwedge_{j=1}^{r-1} \check k_j(w^{j-1}, w^j) \wedge \omega(w^0)$$

and a current of the form (5.5) with $\ell + m = s - r$ depending on the variables $z^0, \ldots, z^\ell$ and $w^r, \ldots, w^m$. Thus, again from the induction hypothesis, it follows that $\mathbf{1}_H T$ and $\mathbf{1}_H \bar\partial T$ have supports contained in $\{w^0 = \ldots = w^m\}$. In addition, since T vanishes outside of $\{z^\ell = w^m\}$, we have that the supports of $\mathbf{1}_H T$ and $\mathbf{1}_H \bar\partial T$ must be contained in the diagonal $\Delta^V = \{z^0 = \cdots = z^\ell = w^m = \cdots = w^0\} \subset V^{\ell+m+2}$. Hence, we see that $\mathbf{1}_H T$ and $\mathbf{1}_H \bar\partial T$ have supports contained in $\Delta^V \cap H$, which has codimension $\geq n(s+1)+1$. Since $\mathbf{1}_H T_0$ has bidegree $(*, n(s+1)-s)$ and $\mathbf{1}_H \bar\partial T_0$ has bidegree $(*, n(s+1)-s+1)$ we have $\mathbf{1}_H T_0 = \mathbf{1}_H \bar\partial T_0 = 0$ by the dimension principle. Since $T_1 = \pm\alpha_1(w^0) T_0$ and α_1 is smooth outside of V^1, which has codimension ≥ 2 in V, it follows that $\mathbf{1}_H T_1$ and $\mathbf{1}_H \bar\partial T_1$ have supports in $\Delta^V \cap \{w^0 \in V^1\}$, which then has codimension $\geq n(s+1)+2$. The dimension principle then shows that $\mathbf{1}_H T_1 = \mathbf{1}_H \bar\partial T_1 = 0$. By induction over k, using that $T_k = \pm\alpha_k(w^0) T_{k-1}$ with α_k smooth outside of V^k, that $\operatorname{codim}_V V^k \geq k+1$, and the dimension principle, we obtain $\mathbf{1}_H T_k = \mathbf{1}_H \bar\partial T_k = 0$ for all k. $\square$

6. Serre duality

6.1. **Local duality.** Let V be a pure n-dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, let $D' \Subset D$ be a strictly pseudoconvex subdomain, and let $V' = V \cap D'$. Consider the complexes

$$0 \to \mathscr{A}^{0,0}(V') \xrightarrow{\bar\partial} \mathscr{A}^{0,1}(V') \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{A}^{0,n}(V') \to 0 \tag{6.1}$$

$$0 \to \mathscr{B}_c^{n,0}(V') \xrightarrow{\bar\partial} \mathscr{B}_c^{n,1}(V') \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{B}_c^{n,n}(V') \to 0. \tag{6.2}$$

From Corollary 5.2 we have the trace map

$$Tr\colon H^0\left(\mathscr{A}^{0,\bullet}(V'), \bar\partial\right) \times H^n\left(\mathscr{B}_c^{n,\bullet}(V'), \bar\partial\right) \to \mathbb{C}, \quad Tr([\varphi],[\psi]) = \int_{V'} \varphi\psi. \tag{6.3}$$

By [6, Theorem 1.2] the complex (6.1) is exact except for at the level 0 where the cohomology is $\mathscr{O}(V')$, cf. the introduction.

Theorem 6.1. *The complex* (6.2) *is exact except for at the top level and the pairing* (6.3) *makes $H^n(\mathscr{B}_c^{n,\bullet}(V'))$ the topological dual of the Frechét space $H^0(\mathscr{A}^{0,\bullet}(V')) = \mathscr{O}(V')$; in particular* (6.3) *is non-degenerate.*

Proof. Let $\psi \in \mathscr{B}_c^{n,q}(V')$ be $\bar\partial$-closed. Moreover, let g be a weight in $D'' \times D'$, where $D'' \subset D'$ is a neighborhood of $\operatorname{supp}\psi$, such that g is holomorphic in z and has compact support in D'_ζ, cf. Section 2.4, and let $k(z,\zeta)$ and $p(z,\zeta)$ be the integral kernels defined by (2.20) and (2.21), respectively. Since ψ has compact support in D'', Theorem 1.1 shows that

$$\psi(\zeta) = \bar\partial_\zeta \int_{V'_z} k(z,\zeta)\wedge\psi(z) + \int_{V'_z} k(z,\zeta)\wedge\bar\partial\psi(z) + \int_{V'_z} p(z,\zeta)\wedge\psi(z), \tag{6.4}$$

holds on V'. The second term on the right hand side vanishes since $\bar\partial\psi = 0$. Since g is holomorphic in z the kernel p has degree 0 in $d\bar z_j$ and hence, also the last term vanishes if $q \neq n$. The first integral on the right hand side is in $\mathscr{B}_c^{n,q-1}(V')$ since g has compact support in D'_ζ and so (6.2) is exact except for at level n.

To see that $H^n(\mathscr{B}_c^{n,\bullet}(V'))$ is the topological dual of $\mathscr{O}(V')$, recall that the topology on $\mathscr{O}(V') \cong \mathscr{O}(D')/\mathcal{J}(D')$ is the quotient topology, where $\mathcal{J}_V$ be the sheaf in D associated with $V \subset D$. It is clear that each $[\psi] \in H^n(\mathscr{B}_c^{n,\bullet}(V'))$ yields a continuous linear functional on $\mathscr{O}(V')$ via (6.3). Moreover, if $q = n$ and $\int_{V'}\varphi\psi = 0$ for all $\varphi \in \mathscr{O}(V')$ then, since $p(z,\zeta)$ is holomorphic in z by the choice of g, the last integral on the right hand side of (6.4) vanishes and thus $[\psi] = 0$. Hence, $H^n(\mathscr{B}_c^{n,\bullet}(V'))$ is a subset of the topological dual of $\mathscr{O}(V')$.

To see that there is equality, let λ be a continuous linear functional on $\mathscr{O}(V')$. By composing with the projection $\mathscr{O}(D') \to \mathscr{O}(D')/\mathcal{J}(D')$ we get a continuous functional $\tilde\lambda$ on $\mathscr{O}(D')$. By definition of the topology on $\mathscr{O}(D')$, $\tilde\lambda$ is carried by some compact subset $K \Subset D'$. By the Hahn-Banach theorem, $\tilde\lambda$ can be extended to a continuous linear functional on $C^0(D')$ and so it is given as integration against some measure μ on D' that has support in a neighborhood $U(K) \Subset D'$ of K. Let $\tilde g$ be a weight in $U(K)\times D'$ that depends holomorphically on $z \in U(K)$ and that has compact support in D'_ζ, and let $\tilde p(z,\zeta)$ be the integral kernel defined from $\tilde g$ as in (2.21), and let $\mathscr{P}$ be the corresponding integral operator. Let $f \in \mathscr{O}(V')$ and define the sequence $f_\epsilon(z) \in \mathscr{O}(K)$ by

$$f_\epsilon(z) = \int_{V'_\zeta} \chi_\epsilon(\zeta)\tilde p(z,\zeta) f(\zeta),$$

where, as above, $\chi_\epsilon = \chi(|h|/\epsilon)$ and $h = h(\zeta)$ is a holomorphic tuple such that $\{h = 0\} = V_{sing}$. For each z in a neighborhood in V' of $K \cap V'$ we have that $\lim f_\epsilon(z) = \mathscr{P}f(z) = f(z)$ by [6, Theorem 1.4]. We claim that f_ϵ in fact converges uniformly in a neighborhood of K in D' to some $\tilde f \in \mathscr{O}(K)$, which then is an extension of f to a neighborhood in D' of K. To see this, first notice by (2.21) that $\tilde p(z,\zeta)$ is a sum of terms $\omega_k(\zeta)\wedge p_k(z,\zeta)$ where $p_k(z,\zeta)$ is smooth in both variables and holomorphic for $z \in U(K)$. By Proposition 2.5, the ω_k are almost semi-meromorphic.

The claim then follows from a simple instance of [18, Theorem 1][6]. We now get

$$\begin{aligned}\lambda(f) &= \lim_{\epsilon\to 0}\int_z f_\epsilon(z)d\mu(z) = \lim_{\epsilon\to 0}\int_z\int_{V'_\zeta}\chi_\epsilon(\zeta)\tilde{p}(z,\zeta)f(\zeta)d\mu(z)\\ &= \lim_{\epsilon\to 0}\int_{V'_\zeta} f(\zeta)\chi_\epsilon(\zeta)\int_z \tilde{p}(z,\zeta)d\mu(z)\\ &= \lim_{\epsilon\to 0}\int_{V'_\zeta} f(\zeta)\chi_\epsilon(\zeta)\sum_k \omega_k(\zeta)\wedge\int_z p_k(z,\zeta)d\mu(z)\\ &= \int_{V'_\zeta} f(\zeta)\sum_k \omega_k(\zeta)\wedge\int_z p_k(z,\zeta)d\mu(z).\end{aligned}$$

But $\zeta \mapsto \int_{V_z} p_k(z,\zeta)d\mu(z)$ is smooth and compactly supported in D' and so λ is given as integration against some element $\psi \in \mathscr{B}_c^{n,n}(V')$; hence λ is realized by the cohomology class $[\psi]$ and the theorem follows. $\square$

Corollary 6.2. *Let $F \to V$ be a vector bundle, $\mathscr{F} = \mathscr{O}(F)$ the associated locally free $\mathscr{O}$-module, and $\mathscr{F}^* = \mathscr{O}(F^*)$. Then the following pairing is non-degenerate*

$$Tr\colon H^0(V',\mathscr{F}) \times H^n\big(\mathscr{F}^*\otimes\mathscr{B}_c^{n,\bullet}(V')\big) \to \mathbb{C}, \quad ([\varphi],[\psi]) \mapsto \int_{V'}\varphi\psi.$$

By Theorem 1.2, if X is Cohen-Macaulay, then the complex $(\mathscr{F}^*\otimes\mathscr{B}_V^{n,\bullet},\bar\partial)$ is a resolution of $\mathscr{F}^*\otimes\omega_V^{n,0}$ and so we get a non-degenerate pairing

$$H^0(V',\mathscr{F}) \times H_c^n(V',\mathscr{F}^*\otimes\omega_V^{n,0}) \to \mathbb{C}.$$

6.2. **Global duality.** From the local duality an abstract global duality follows by a patching argument using Čech cohomology, see [27], cf. also [11, Theorem (I)]. To see that this abstract global duality is realized by Theorem 1.3 we will make this patching argument explicit using a perhaps non-standard formalism for Čech cohomology; cf. [23, Section 7.3]

Let $\mathscr{F}$ be a sheaf on X and let $\mathcal{V} = \{V_j\}$ be a locally finite covering of X. We let $C^k(\mathcal{V},\mathscr{F})$ be the group of formal sums

$$\sum_{i_0\cdots i_k} f_{i_0\cdots i_k}V_{i_0}\wedge\cdots\wedge V_{i_k}, \quad f_{i_0\cdots i_k}\in\mathscr{F}(V_{i_0}\cap\cdots\cap V_{i_k})$$

with the suggestive computation rules, e.g., $f_{12}V_1\wedge V_2 + f_{21}V_2\wedge V_1 = (f_{12}-f_{21})V_1\wedge V_2$. Each element of $C^k(\mathcal{V},\mathscr{F})$ thus has a unique representation of the form

$$\sum_{i_0<\cdots<i_k} f_{i_0\cdots i_k}V_{i_0}\wedge\cdots\wedge V_{i_k}$$

that we will abbreviate as $\sum'_{|I|=k+1} f_I V_I$. The coboundary operator $\delta\colon C^k(\mathcal{V},\mathscr{F}) \to C^{k+1}(\mathcal{V},\mathscr{F})$ can in this formalism be taken to be the formal wedge product

$$\delta(\sideset{}{'}\sum_{|I|=k+1} f_I V_I) = (\sideset{}{'}\sum_{|I|=k+1} f_I V_I)\wedge(\sum_j V_j).$$

[6] Take $p=0$, $q=1$, and $\mu=1$ in this theorem.

If $\mathcal{V}$ is a Leray covering for $\mathscr{F}$, then $H^k(C^\bullet(\mathcal{V},\mathscr{F}),\delta)\cong H^k(X,\mathscr{F})$. Indeed, let $(\mathscr{F}^\bullet,d)$ be a flabby resolution of $\mathscr{F}$. Then $H^k(X,\mathscr{F})=H^k(\mathscr{F}^\bullet(X),d)$ and applying standard homological algebra to the double complex $C^\bullet(\mathcal{V},\mathscr{F}^\bullet)$ one shows that $H^k(C^\bullet(\mathcal{V},\mathscr{F}),\delta)\simeq H^k(\mathscr{F}^\bullet(X),d)$. If $\mathscr{F}$ is fine, i.e., a $\mathcal{E}_X^{0,0}$-module, then the complex $(C^\bullet(\mathcal{V},\mathscr{F}),\delta)$ is exact except for at level 0 where $H^0(C^\bullet(\mathcal{V},\mathscr{F}),\delta)\cong H^0(X,\mathscr{F})$.

Let $\mathscr{G}'$ be a precosheaf on X. Recall, see, e.g., [11, Section 3], that a precosheaf of abelian groups is an assignment that to each open set V associates an abelian group $\mathscr{G}'(V)$, together with inclusion maps $i_W^V\colon\mathscr{G}'(V)\to\mathscr{G}'(W)$ for $V\subset W$ such that $i_W^{V'}=i_W^V i_V^{V'}$ if $V'\subset V\subset W$. We define $C_c^{-k}(\mathcal{V},\mathscr{G}')$ to be the group of formal sums

$$\sum_{i_0\cdots i_k} g_{i_0\cdots i_k}V_{i_0}^*\wedge\cdots\wedge V_{i_k}^*,$$

where $g_{i_0\cdots i_k}\in\mathscr{G}'(V_{i_0}\cap\cdots\cap V_{i_k})$ and only finitely many $g_{i_0\cdots i_k}$ are non-zero; for $k<0$ we let $C_c^{-k}(\mathcal{V},\mathscr{G}')=0$. We define a coboundary operator $\delta^*\colon C_c^{-k}(\mathcal{V},\mathscr{G}')\to C_c^{-k+1}(\mathcal{V},\mathscr{G}')$ by formal contraction

$$\delta^*(\sideset{}{'}\sum_{|I|=k+1} g_I V_I^*)=\sum_j V_j\lrcorner\sideset{}{'}\sum_{|I|=k+1} g_I V_I^*,$$

see (6.5) and (6.6) below. If $\mathscr{G}$ is a sheaf (of abelian groups), then $V\to\mathscr{G}_c(V)$ is a precosheaf $\mathscr{G}'$ by extending sections by 0. We will write $C_c^{-k}(\mathcal{V},\mathscr{G})$ in place of $C_c^{-k}(\mathcal{V},\mathscr{G}')$.

Assume now that there, for every open $V\subset X$, is a map $\mathscr{F}(V)\otimes\mathscr{G}'(V)\to\mathscr{F}'(V)$ where $\mathscr{F}'$ and $\mathscr{G}'$ are precosheaves on X. We then define a contraction map $\lrcorner\colon C^k(\mathcal{V},\mathscr{F})\times C_c^{-\ell}(\mathcal{V},\mathscr{G}')\to C_c^{k-\ell}(\mathcal{V},\mathscr{F}')$ by using the following computation rules.

$$V_i\lrcorner V_j^*=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}, \tag{6.5}$$

$$V_i\lrcorner(V_{j_0}^*\wedge\cdots\wedge V_{j_\ell}^*)=\sum_{m=0}^{\ell}(-1)^m V_{j_0}^*\wedge\cdots(V_i\lrcorner V_{j_m}^*)\cdots\wedge V_{j_\ell}^*, \tag{6.6}$$

$$(V_{i_0}\wedge\cdots\wedge V_{i_k})\lrcorner V_J^*=\begin{cases}0, & k>|J|\\ ((V_{i_0}\wedge\cdots\wedge V_{i_{k-1}}))\lrcorner(V_{i_k}\lrcorner V_J^*), & k\leq|J|\end{cases}.$$

If $\mathscr{F}'$ and $\mathscr{G}'$ are sheaves we define in a similar way also the contraction $\lrcorner\colon C_c^{-k}(\mathcal{V},\mathscr{G}')\times C^\ell(\mathcal{V},\mathscr{F})\to C^{\ell-k}(\mathcal{V},\mathscr{F}')$. If $g=g_IV_I^*$ and $f=f_JV_J$, then $g\lrcorner f=g_If_JV_I^*\lrcorner V_J$, where g_If_J is the extension to $\bigcap_{i\in J\setminus I}V_i$ by 0; this is well-defined since g_If_J is 0 in a neighborhood of the boundary of $\bigcap_{j\in J}V_j$ in $\bigcap_{i\in J\setminus I}V_i$.

Lemma 6.3. *If $\mathscr{G}$ is a fine sheaf, then*

$$H^{-k}(C_c^\bullet(\mathcal{V},\mathscr{G}),\delta^*)=\begin{cases}0, & k\neq 0\\ H_c^0(X,\mathscr{G}), & k=0\end{cases}.$$

Proof. Let $\{\chi_j\}$ be a smooth partition of unity subordinate to $\mathcal{V}$ and let $\chi=\sum_j\chi_jV_j^*$. Since $\delta^*\chi=\sum\chi_j=1$ we have

$$\delta^*(\chi\wedge g)=\delta^*(\chi)\cdot g-\chi\wedge\delta^*(g)=g-\chi\wedge\delta^*(g)$$

for $g \in C_c^{-k}(\mathcal{V}, \mathscr{G})$. Hence, if g is δ^*-closed, then g is δ^*-exact. It follows that the complex

$$\cdots \xrightarrow{\delta^*} C_c^{-1}(\mathcal{V}, \mathscr{G}) \xrightarrow{\delta^*} C_c^0(\mathcal{V}, \mathscr{G}) \xrightarrow{\delta^*} H_c^0(X, \mathscr{G}) \to 0$$

is exact and so the lemma follows. □

—

Let X be a paracompact reduced complex space of pure dimension n. Let $\aleph$ be the precosheaf on X defined by

$$\aleph(V) = H^n(\mathscr{B}_c^{n,\bullet}(V), \bar\partial),$$
$$i_W^V \colon \aleph(V) \to \aleph(W), \quad i_W^V([\psi]) = [\tilde\psi],$$

where $\psi \in \mathscr{B}_c^{n,n}(V)$ and $\tilde\psi$ is the extension of ψ by 0.[7] Let $\mathcal{V} = \{V_j\}$ be a suitable locally finite Leray covering of X and consider the complexes

$$0 \to C^0(\mathcal{V}, \mathscr{O}_X) \xrightarrow{\delta} C^1(\mathcal{V}, \mathscr{O}_X) \xrightarrow{\delta} \cdots \tag{6.7}$$

$$\cdots \xrightarrow{\delta^*} C_c^{-1}(\mathcal{V}, \aleph) \xrightarrow{\delta^*} C_c^0(\mathcal{V}, \aleph) \to 0. \tag{6.8}$$

By Theorem 6.1 we have non-degenerate pairings

$$Tr \colon C^k(\mathcal{V}, \mathscr{O}_X) \times C_c^{-k}(\mathcal{V}, \aleph) \to \mathbb{C}, \quad Tr(f, g) = \int_X f \lrcorner g,$$

induced by the trace map (6.3); in fact, Theorem 6.1 shows that these pairings make the complex (6.8) the topological dual of the complex of Frechét spaces (6.7). Moreover, if $f \in C^{k-1}(\mathcal{V}, \mathscr{O}_X)$ and $g \in C_c^{-k}(\mathcal{V}, \aleph)$ we have

$$\begin{aligned} Tr(\delta f, g) &= \int_X (\delta f) \lrcorner g = \int_X \big(f \wedge \sum_j V_j\big) \lrcorner g = \int_X f \lrcorner \big((\sum_j V_j) \lrcorner g\big) \\ &= \int_X f \lrcorner (\delta^* g) = Tr(f, \delta^* g). \end{aligned} \tag{6.9}$$

Hence, we get a well-defined pairing on cohomology level

$$Tr \colon H^k\left(C^\bullet(\mathcal{V}, \mathscr{O}_X)\right) \times H^{-k}\left(C_c^\bullet(\mathcal{V}, \aleph)\right) \to \mathbb{C}, \quad Tr([f], [g]) = \int_X f \lrcorner g. \tag{6.10}$$

Since $\mathcal{V}$ is a Leray covering we have

$$H^k\left(C^\bullet(\mathcal{V}, \mathscr{O}_X)\right) \cong H^k(X, \mathscr{O}_X) \cong H^k\left(\mathscr{A}^{0,\bullet}(X)\right), \tag{6.11}$$

and these isomorphisms induce canonical topologies on $H^k(X, \mathscr{O}_X)$ and $H^k\left(\mathscr{A}^{0,\bullet}(X)\right)$; cf. [27, Lemma 1]. To understand $H^{-k}\left(C_c^\bullet(\mathcal{V}, \aleph)\right)$, consider the double complex

$$K^{-i,j} := C_c^{-i}(\mathcal{V}, \mathscr{B}_X^{n,j}),$$

where the map $K^{-i,j} \to K^{-i+1,j}$ is the coboundary operator δ^* and the map $K^{-i,j} \to K^{-i,j+1}$ is $\bar\partial$. We have that $K^{-i,j} = 0$ if $i < 0$ or $j < 0$ or $j > n$. Moreover, the "rows" $K^{-i,\bullet}$ are, by Theorem 6.1, exact except for at the n^{th} level where the cohomology is $C_c^{-i}(\mathcal{V}, \aleph)$; the "columns" $K^{\bullet,j}$ are exact except for at level 0 where the cohomology is $\mathscr{B}_c^{n,j}(X)$ by Lemma 6.3 since the sheaf $\mathscr{B}_X^{n,j}$ is fine. By standard homological algebra (e.g., a spectral sequence argument) it follows that

[7] In view of Theorem 6.1 and [11, Proposition 8 (a)], $\aleph$ is in fact a cosheaf.

$$H^{-k}\left(C_c^{\bullet}(\mathcal{V},\aleph)\right)\cong H^{n-k}\left(\mathscr{B}_c^{n,\bullet}(X),\bar\partial\right), \tag{6.12}$$

cf. also the proof of Theorem 1.3 below. The vector space $C_c^{-k}(\mathcal{V},\aleph)$ has a natural topology since it is the topological dual of the Frechét space $C^k(\mathcal{V},\mathscr{O}_X)$; therefore (6.12) gives a natural topology on $H^{n-k}(\mathscr{B}_c^{n,\bullet}(X))$.

Lemma 6.4. *Assume that $H^k(X,\mathscr{O}_X)$ and $H^{k+1}(X,\mathscr{O}_X)$, considered as topological vector spaces, are Hausdorff. Then the pairing* (6.10) *is non-degenerate.*

Proof. Since (6.8) is the topological dual of (6.7) it follows (see, e.g., [27, Lemma 2]) that the topological dual of

$$\mathrm{Ker}\big(\delta\colon C^k(\mathcal{V},\mathscr{O}_X)\to C^{k+1}(\mathcal{V},\mathscr{O}_X)\big)/\overline{\mathrm{Im}\big(\delta\colon C^{k-1}(\mathcal{V},\mathscr{O}_X)\to C^k(\mathcal{V},\mathscr{O}_X)\big)} \tag{6.13}$$

equals

$$\mathrm{Ker}\big(\delta^*\colon C_c^{-k}(\mathcal{V},\omega_X^{n,n})\to C_c^{-k+1}(\mathcal{V},\omega_X^{n,n})\big)/\overline{\mathrm{Im}\big(\delta^*\colon C_c^{-k-1}(\mathcal{V},\omega_X^{n,n})\to C_c^{-k}(\mathcal{V},\omega_X^{n,n})\big)}. \tag{6.14}$$

Since $H^k(X,\mathscr{O}_X)$ and $H^{k+1}(X,\mathscr{O}_X)$ are Hausdorff it follows that the images of $\delta\colon C^{k-1}\to C^k$ and $\delta\colon C^k\to C^{k+1}$ are closed. Since the image of the latter map is closed it follows from the open mapping theorem and the Hahn-Banach theorem that also the image of $\delta^*\colon C_c^{-k-1}\to C_c^{-k}$ is closed. The images of δ and δ^* in (6.13) and (6.14) are thus closed and so the closure signs may be removed. Hence, (6.10) makes $H^{-k}(C_c^{\bullet}(\mathcal{V},\omega_X^{n,n}))$ the topological dual of $H^k(X,\mathscr{O}_X)$. □

Remark 6.5. If X is compact the Cartan-Serre theorem says that the cohomology of coherent sheaves on X is finite dimensional, in particular Hausdorff. In the compact case the pairing (6.10) is thus always non-degenerate. The pairing (6.10) is also always non-degenerate if X is holomorphically convex since then, by [26, Lemma II.1], $H^k(X,\mathscr{S})$ is Hausdorff for any coherent sheaf $\mathscr{S}$.

If X is q-convex it follows from the Andreotti-Grauert theorem that for any coherent sheaf $\mathscr{S}$, $H^k(X,\mathscr{S})$ is Hausdorff for $k\ge q$. Hence, in this case, (6.10) is non-degenerate for $k\ge q$.

Proof of Theorem 1.3. For notational convenience we assume that $\mathscr{F}=\mathscr{O}_X$. By Lemma 6.4 we know that (6.10) is non-degenerate. In view of the Dolbeault isomorphisms (6.11) and (6.12) we get an induced non-degenerate pairing

$$Tr\colon H^k\left(\mathscr{A}^{0,\bullet}(X)\right)\times H^{n-k}\left(\mathscr{B}_c^{n,\bullet}(X)\right)\to\mathbb{C}.$$

It remains to see that this induced trace map is realized by $([\varphi],[\psi])\mapsto\int_X\varphi\wedge\psi$; for this we will make (6.11) and (6.12) explicit.

Let $\{\chi_j\}$ be a partition of unity subordinate to $\mathcal{V}$, and let $\chi=\sum_j\chi_jV_j^*$. We will use the convention that forms *commute* with all V_i^* and V_j, i.e., if ξ is a differential form then

$$\xi V_I^*=V_I^*\xi,\quad V_I^*\lrcorner(\xi V_J)=\xi V_I^*\lrcorner V_J.$$

Moreover, we let $\bar\partial(\xi V_I^*)=\bar\partial\xi V_I^*$. We now let

$$T_{k,j}\colon C^k(\mathcal{V},\mathscr{O}_X)\to C^{k-j-1}(\mathcal{V},\mathscr{A}_X^{0,j}),\quad T_{k,j}(f)=(\chi\wedge(\bar\partial\chi)^j)\lrcorner f,$$

where we put $C^{-1}(\mathcal{V}, \mathscr{A}_X^{0,k}) = \mathscr{A}^{0,k}(X)$ and $C^{\ell}(\mathcal{V}, \mathscr{A}_X^{0,k}) = 0$ for $\ell < -1$.[8] Using that $\chi \lrcorner V = 1$ it is straightforward to verify that

$$T_{k,j}(\delta \tilde{f}) = \delta T_{k-1,j}(\tilde{f}) + (-1)^{k-j} \bar{\partial} T_{k-1,j-1}(\tilde{f}), \quad \tilde{f} \in C^{k-1}(\mathcal{V}, \mathscr{O}_X). \tag{6.15}$$

It follows that if $f \in C^k(\mathcal{V}, \mathscr{O}_X)$ is δ-closed then $T_{k,k}(f)$ is $\bar{\partial}$-closed and if f is δ-exact then $T_{k,k}(f)$ is $\bar{\partial}$-exact. Thus $T_{k,k}$ induces a map

$$\mathrm{Dol}\colon H^k(C^\bullet(\mathcal{V}, \mathscr{O}_X)) \to H^k(\mathscr{A}^{0,\bullet}(X)), \quad \mathrm{Dol}([f]_\delta) = [T_{k,k}(f)]_{\bar{\partial}};$$

this is a realization of the composed isomorphism (6.11).

To make (6.12) explicit, let $[g] \in C_c^{-k}(\mathcal{V}, \aleph)$, where $g \in C_c^{-k}(\mathcal{V}, \mathscr{B}_X^{n,n})$, be δ^*-closed. This means that there is a $\tau^{n-1} \in C_c^{-k+1}(\mathcal{V}, \mathscr{B}_X^{n,n-1})$ such that $\delta^* g = \bar{\partial}\tau^{n-1}$. Hence, $\bar{\partial}\delta^*\tau^{n-1} = \delta^*\bar{\partial}\tau^{n-1} = \delta^*\delta^* g = 0$ and so by Theorem 6.1 there is a $\tau^{n-2} \in C_c^{-k+2}(\mathcal{V}, \mathscr{B}_X^{n,n-2})$ such that $\delta^*\tau^{n-1} = \bar{\partial}\tau^{n-2}$. Continuing in this way we obtain, for all j, $\tau^{n-j} \in C_c^{-k+j}(\mathcal{V}, \mathscr{B}_X^{n,n-j})$ such that $\delta^*\tau^{n-j} = \bar{\partial}\tau^{n-j-1}$. It follows that $\delta^*\tau^{n-k} \in \mathscr{B}_c^{n,n-k}(X)$, cf. the proof of Lemma 6.3, and that it is $\bar{\partial}$-closed. One can verify that if $[g] \in C_c^{-k}(\mathcal{V}, \aleph)$ is δ^*-exact then $\delta^*\tau^{n-k}$ is $\bar{\partial}$-exact and so we get a well-defined map

$$\mathrm{Dol}^*\colon H^{-k}(C_c^\bullet(\mathcal{V}, \aleph)) \to H^{n-k}(\mathscr{B}_c^{n,\bullet}(X)), \quad \mathrm{Dol}^*([g]_{\bar{\partial}}) = [\delta^*\tau^{n-k}]_{\bar{\partial}};$$

this is a realization of the isomorphism (6.12).

Let now $f \in C^k(\mathcal{V}, \mathscr{O}_X)$ be δ-closed and let $[g] \in C_c^{-k}(\mathcal{V}, \aleph)$ be δ^*-closed. One checks that $\delta T_{k,0}(f) = (-1)^k f$ and thus, by (6.15), we have

$$\delta T_{k,j}(f) = \begin{cases} (-1)^{k-j}\bar{\partial} T_{k,j-1}(f), & 1 \le j \le k \\ (-1)^k f, & j = 0 \end{cases}.$$

Using this and the computation in (6.9) we get

$$\begin{aligned}
\int_X f \lrcorner g &= (-1)^k \int_X \delta T_{k,0}(f) \lrcorner g = (-1)^k \int_X T_{k,0}(f) \lrcorner \delta^* g = (-1)^k \int_X T_{k,0}(f) \lrcorner \bar{\partial}\tau^{n-1} \\
&= (-1)^{k+1} \int_X \bar{\partial} T_{k,0}(f) \lrcorner \tau^{n-1} = (-1)^{2k} \int_X \delta T_{k,1}(f) \lrcorner \tau^{n-1} \\
&= (-1)^{2k} \int_X T_{k,1}(f) \lrcorner \delta^*\tau^{n-1} = \cdots = (-1)^{k(k+1)} \int_X T_{k,k}(f) \lrcorner \delta^*\tau^{n-k} \\
&= \int_X \mathrm{Dol}([f]) \wedge \mathrm{Dol}^*([g]).
\end{aligned}$$

□

7. Compatibility with the cup product

Assume that X is compact and Cohen-Macaulay. In view of [6, Theorem 1.2] and Theorem 1.2 we have that

$$H^k(X, \mathscr{O}_X) \cong H^k\left(\mathscr{A}^{0,\bullet}(X), \bar{\partial}\right) \quad \text{and} \quad H^k(X, \omega_X^{n,0}) \cong H^k\left(\mathscr{B}^{n,\bullet}(X), \bar{\partial}\right), \tag{7.1}$$

cf. the introduction. Now we make these Dolbeault isomorphisms explicit in a slightly different way than in the previous section: We adopt in this section the standard

[8] In fact, the image of $T_{k,j}$ is contained in $C^{k-j-1}(\mathcal{V}, \mathcal{E}_X^{0,j})$.

definition of Čech cochain groups so that now

$$C^p(\mathcal{V},\mathscr{F}) := \prod_{\alpha_0\neq\alpha_1\neq\cdots\neq\alpha_p} \mathscr{F}(V_{\alpha_0}\cap\cdots\cap V_{\alpha_p})$$

for a sheaf $\mathscr{F}$ on X and a locally finite open cover $\mathcal{V}=\{V_\alpha\}$.

Let $\mathcal{V}$ be a Leray covering and let $\{\chi_\alpha\}$ be a smooth partition of unity subordinate to $\mathcal{V}$. Following [16, Chapter IV, §6], given Čech cocycles $c\in C^p(\mathcal{V},\mathscr{O}_X)$ and $c'\in C^q(\mathcal{V},\omega_X^{n,0})$ we define Čech cochains $f\in C^0(\mathcal{V},\mathscr{A}_X^{0,p})$ and $f'\in C^0(\mathcal{V},\mathscr{B}_X^{n,q})$ by

$$f_\alpha = \sum_{\nu_0,\dots,\nu_{p-1}} \bar\partial\chi_{\nu_0}\wedge\cdots\wedge\bar\partial\chi_{\nu_{p-1}}\cdot c_{\nu_0\cdots\nu_{p-1}\alpha} \quad\text{in}\quad V_\alpha,$$

$$f'_\alpha = \sum_{\nu_0,\dots,\nu_{q-1}} \bar\partial\chi_{\nu_0}\wedge\cdots\wedge\bar\partial\chi_{\nu_{q-1}}\wedge c'_{\nu_0\cdots\nu_{q-1}\alpha} \quad\text{in}\quad V_\alpha.$$

In fact, f and f' are cocycles and define $\bar\partial$-closed global sections

$$\varphi = \sum_{\nu_p}\chi_{\nu_p} f_{\nu_p} = \sum_{\nu_0,\dots,\nu_p}\chi_{\nu_p}\bar\partial\chi_{\nu_0}\wedge\cdots\wedge\bar\partial\chi_{\nu_{p-1}}\cdot c_{\nu_0\cdots\nu_p}\in\mathscr{A}^{0,p}(X), \tag{7.2}$$

$$\varphi' = \sum_{\nu_q}\chi_{\nu_q} f'_{\nu_q} = \sum_{\nu_0,\dots,\nu_q}\chi_{\nu_q}\bar\partial\chi_{\nu_0}\wedge\cdots\wedge\bar\partial\chi_{\nu_{q-1}}\wedge c'_{\nu_0\cdots\nu_q}\in\mathscr{B}^{n,q}(X). \tag{7.3}$$

The Dolbeault isomorphisms (7.1) are then realized by

$$H^p(X,\mathscr{O}_X)\xrightarrow{\simeq} H^p(\mathscr{A}^{0,\bullet}(X)),\quad [c]\mapsto[\varphi],\quad\text{and}$$
$$H^q(X,\omega_X^{n,0})\xrightarrow{\simeq} H^q(\mathscr{B}^{n,\bullet}(X)),\quad [c']\mapsto[\varphi'],$$

respectively.

We can now show that the cup product is compatible with our trace map on the level of cohomology.

Proposition 7.1. *The following diagram commutes.*

$$\begin{array}{ccc} H^p(X,\mathscr{O}_X)\times H^q(X,\omega_X^{n,0}) & \xrightarrow{\cup} & H^{p+q}(X,\omega_X^{n,0}) \\ \downarrow & & \downarrow \\ H^p(\mathscr{A}^{0,\bullet}(X))\times H^q(\mathscr{B}^{n,\bullet}(X)) & \xrightarrow{\wedge} & H^{p+q}(\mathscr{B}^{n,\bullet}(X)), \end{array}$$

where the vertical mappings are the Dolbeault isomorphisms.

Proof. Let $\mathcal{V}=\{V_\alpha\}$ be a Leray covering of X. Let $[c]\in H^p(X,\mathscr{O}_X)$ and $[c']\in H^q(X,\omega_X^{n,0})$, where $c\in C^p(\mathcal{V},\mathscr{O}_X)$ and $c'\in C^q(\mathcal{V},\omega_X^{n,0})$ are cocycles. Then $c\cup c'\in C^{p+q}(\mathcal{V},\omega_X^{n,0})$, defined by

$$(c\cup c')_{\alpha_0\cdots\alpha_{p+q}} = c_{\alpha_0\cdots\alpha_p}\cdot c'_{\alpha_p\cdots\alpha_{p+q}}\quad\text{in } V_{\alpha_0}\cap\cdots\cap V_{\alpha_{p+q}},$$

is a cocycle representing $[c]\cup[c']\in\check{H}^{p+q}(X,\omega_X^{n,0})$. The image of $[c]\cup[c']$ in $H^{p+q}(\mathscr{B}^{n,\bullet}(X))$ is the cohomology class defined by the $\bar\partial$-closed current

$$\sum_{\nu_0,\dots,\nu_{p+q}}\chi_{\nu_{p+q}}\bar\partial\chi_{\nu_0}\wedge\cdots\wedge\bar\partial\chi_{\nu_{p+q-1}}\wedge c_{\nu_0\cdots\nu_p}\cdot c'_{\nu_p\cdots\nu_{p+q}}\in\mathscr{B}^{n,p+q}(X). \tag{7.4}$$

The images of $[c]$ and $[c']$ in Dolbeault cohomology are, respectively, the cohomology classes of the $\bar\partial$-closed currents φ and φ' defined by (7.2) and (7.3). Notice that

$$\varphi|_{V_{\nu_p}} = \sum_{\nu_0,\dots,\nu_{p-1}}\bar\partial\chi_{\nu_0}\wedge\cdots\wedge\bar\partial\chi_{\nu_{p-1}}\cdot c_{\nu_0\cdots\nu_{p-1}\nu_p}.$$

Therefore, $\varphi \wedge \varphi'$ is given by (7.4) as well. □

Notice that $H^n(X, \omega_X^{n,0}) \simeq \mathbb{C}$ (e.g. as it is the dual of $H^0(X, \mathscr{O}_X)$) and any two realizations of this isomorphism are the same up to a multiplicative constant. In the compact Cohen-Macaulay case it thus follows from Proposition 7.1 that the duality of this paper, up to a multiplicative constant, is the same as the abstractly defined duality in complex and algebraic geometry.

References

[1] M. Andersson: Integral representation with weights I. *Math. Ann.*, **326** (2003), 1–18.

[2] M. Andersson: Residue currents and ideals of holomorphic functions. *Bull. Sci. Math.* 128 (2004), no. 6, 481–512.

[3] M. Andersson: Integral representation with weights II, division and interpolation formulas. *Math. Z.*, **254** (2006), 315–332.

[4] M. Andersson: Coleff-Herrera currents, duality, and Noetherian operators. *Bull. Soc. Math. France*, **139** (2011), no. 4, 535–554.

[5] M. Andersson: Pseudomeromorphic currents on subvarieties. *Complex Var. Elliptic Equ.* 61 (2016), no. 11, 1533–1540.

[6] M. Andersson, H. Samuelsson: A Dolbeault-Grothendieck lemma on complex spaces via Koppelman formulas. *Invent. Math.*, **190** (2012), 261–297.

[7] M. Andersson, H. Samuelsson, E. Wulcan, A. Yger: Segre numbers, a generalized King formula, and local intersections. *J. reine angew. Math.*, to appear. DOI 10.1515/crelle-2014-0109.

[8] M. Andersson, E. Wulcan: Residue currents with prescribed annihilator ideals. *Ann. Sci. Éc. Norm. Super.*, **40** (2007), 985–1007.

[9] M. Andersson, E. Wulcan: Decomposition of residue currents. *J. reine angew. Math.*, **638** (2010), 103–118.

[10] M. Andersson, E. Wulcan: Direct images of semi-meromorphic currents. *arXiv:1411.4832*

[11] A. Andreotti, A. Kas: Duality on complex spaces. *Ann. Scuola Norm. Sup. Pisa (3)* **27** (1973), 187–263.

[12] D. Barlet: Le faisceau $\omega_X^\bullet$ sur un espace analytique X de dimension pure. *Lecture Notes in Math., 670*, Springer, Berlin, 1978.

[13] B. Berndtsson: A formula for interpolation and division in $\mathbb{C}^n$. *Math. Ann.* **263** (1983) no. 4, 399–418.

[14] N. Coleff, M. Herrera: Les courants résiduels associés à une forme méromorphe. *Lecture Notes in Mathematics, 633.* Springer, Berlin, 1978.

[15] B. Conrad: Grothendieck duality and base change. *Lecture Notes in Mathematics, 1750*, Springer-Verlag, Berlin, 2000.

[16] J.-P. Demailly: Complex Analytic and Differential Geometry. *Online book, available at* `http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`.

[17] D. Eisenbud: Commutative Algebra. With a View Toward Algebraic Geometry. *Graduate Texts in Mathematics, vol. 150*, Springer, New York, 1995.

[18] J.-E. Björk, H. Samuelsson: Regularizations of residue currents. *J. reine angew. Math.*, **649** (2010), 33–54.

[19] R. Hartshorne: Algebraic geometry. *Graduate Texts in Mathematics, No. 52.* Springer-Verlag, New York-Heidelberg, 1977.

[20] R. Hartshorne: Residues and duality. *Lecture Notes in Mathematics, No. 20*, Springer-Verlag, Berlin-New York, 1966.

[21] G. Henkin, M. Passare: Abelian differentials on singular varieties and variations on a theorem of Lie-Griffith. *Invent. Math.*, **135** (1999), 297–328.

[22] M. Herrera, D. Liebermann: Residues and principal values on complex spaces. *Math. Ann.* **194** (1971), 259–294.

[23] L. Hörmander: An introduction to complex analysis in several variables. Third edition. North-Holland Mathematical Library, 7. *North-Holland Publishing Co., Amsterdam*, 1990.

[24] R. Lärkäng, H. Samuelsson: Various approaches to products of residue currents. *J. Funct. Anal.*, **264** (2013), 118–138.

[25] M. Passare: Residues, currents, and their relation to ideals of holomorphic functions. *Math. Scand.*, **62** (1988), 75–152.

[26] D. Prill: The divisor class groups of some rings of holomorphic functions. *Math. Z.* **121** (1971), 58–80.

[27] J.-P. Ramis, G. Ruget: Complexe dualisant et théorèmes de dualité en géométrie analytique complexe. *Inst. Hautes Études Sci. Publ. Math.*, No. 38 1970, 77–91.

[28] G. Ruget: Complexe dualisant et residues. *Journées Géo. analyt. (Poitiers, 1972), Bull. Soc. math. France, Mémoire 38*, 1974, 31–34.

[29] J.-P. Serre: Un théorème de dualité. *Comm. Math. Helv.*, **29** (1955), 9–26.

H. Samuelsson Kalm, E. Wulcan, Department of Mathematical Sciences, Division of Mathematics, University of Gothenburg and Chalmers University of Technology, SE-412 96 Göteborg, Sweden

E-mail address: `hasam@chalmers.se, wulcan@chalmers.se`

J. Ruppenthal, Department of Mathematics, University of Wuppertal, Gaussstr. 20, 42119 Wuppertal, Germany

E-mail address: `ruppenthal@uni-wuppertal.de`