CANONICAL HEIGHTS FOR PLANE POLYNOMIAL MAPS OF SMALL TOPOLOGICAL DEGREE

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ABSTRACT. We study canonical heights for plane polynomial mappings of small topological degree. In particular, we prove that for points of canonical height zero, the arithmetic degree is bounded by the topological degree and hence strictly smaller than the first dynamical degree. The proof uses the existence, proved by Favre and the first author, of certain compactifications of the plane adapted to the dynamics.

1. INTRODUCTION

J. Silverman [Sil11] recently proposed a number of conjectures on the growth of heights and degrees under iterates of rational selfmaps of projective space, and proved them for monomial maps. Here we study the growth of heights for a large class of plane polynomial maps.

Consider a polynomial mapping $f : \mathbf{A}^2 \to \mathbf{A}^2$ defined over the field $\overline{\mathbf{Q}}$ of algebraic numbers. The *first dynamical degree* λ_1 is defined by

$$\lambda_1 := \lim_{n \to \infty} (\deg f^n)^{1/n}.$$

The *second dynamical degree* λ_2 is the number of preimages under f of a general closed point in $\mathbf{A}^2$. It follows from Bézout's Theorem that $\lambda_2 \leq \lambda_1^2$. Following Guedj [Gue02] we say that f has *small topological degree* if $\lambda_2 < \lambda_1$.

Let h be the standard logarithmic height on $\mathbf{P}^2(\overline{\mathbf{Q}}) \supseteq \mathbf{A}^2(\overline{\mathbf{Q}})$.

Main Theorem. *Let $f : \mathbf{A}^2 \to \mathbf{A}^2$ be a polynomial mapping of small topological degree, $\lambda_2 < \lambda_1$. Then the limit*

$$\hat{h} := \lim_{n \to \infty} \lambda_1^{-n} h \circ f^n$$

exists, and is finite, pointwise on $\mathbf{A}^2(\overline{\mathbf{Q}})$. We have $\hat{h} \not\equiv 0$ and $\hat{h} \circ f = \lambda_1 \hat{h}$. Further, if $P \in \mathbf{A}^2(\overline{\mathbf{Q}})$ and $\hat{h}(P) = 0$, then

$$\limsup_{n \to \infty} h(f^n(P))^{1/n} \leq \lambda_2 < \lambda_1. \qquad (*)$$

If f is moreover a polynomial automorphism, then $\hat{h}(P) = 0$ iff P is periodic.

As in [Sil11] we call $\hat{h}$ the *canonical height* associated to f, whereas the left hand side of (*) is the *arithmetic degree* of the point P. Our Main Theorem says that for points of canonical height zero, the height along the orbit grows relatively slowly.

A related Conjecture 3 in [Sil11] states that $\hat{h}(P) > 0$ whenever P has Zariski dense orbit in $\mathbf{A}^2$. By the Main Theorem, this holds for polynomial automorphisms, but we have not been able to establish it for noninvertible polynomial mappings of small topological degree. For such maps, the locus $\hat{h} = 0$ may contain points for which $h \circ f^n$ grows exponentially, see §3.4.

The basic arithmetic dynamics of plane polynomial automorphisms is well understood, thanks to work of Silverman [Sil94], Denis [Den95], Marcello [Mar00, Mar03], Kawaguchi [Kaw06, Kaw09], Lee [Lee09], Ingram [Ing11] and others. In particular, the Main Theorem above and Conjecture 3 in [Sil11] were already known[1] for polynomial automorphisms f with $\lambda_1(f) > 1$. The existing proofs make use of the inverse map f^{-1} and also rely crucially on the Friedland-Milnor classification [FM89], which shows that, up to conjugation, f is a composition of generalized Hénon maps and in particular *regular* in the sense of [Sib99]. In fact, the two results above are true for regular polynomial automorphisms of any dimension, see [Sil11, Theorem 36].

For noninvertible maps, no algebraic classification is known. Instead we exploit a result by Favre and the first author [FJ11] which shows that maps of small topological degree always admit compactifications that are well adapted to the dynamics. We refer to §2.3 for a precise statement. Given such a compactification, we work locally with a given absolute value and estimate the growth of the local height under iteration. When the absolute value is Archimedean, this was essentially carried out in [FJ11, §7], adapting techniques from the early work on the complex Hénon map, see [Hub86, HO94, FM89, BS91, FS92].

The existence of a compactification adapted to the dynamics is proved in [FJ11] using the induced dynamics on a suitable space of valuations, see also [FJ04, FJ07, Jon12]. In order to address Silverman's Conjecture 3 in our setting, one would likely have to refine this valuative analysis, something which is beyond the scope of the present paper.

The ergodic theory of polynomial maps of small topological degree over the complex numbers is studied in detail in [DDG1, DDG2, DDG3]. It would be interesting to see if these results have non-Archimedean or arithmetic analogues.

The paper is organized as follows. In §2 we recall some facts concerning dynamical degrees and heights. We also state the key result from [FJ11] on the existence of a compactification adapted to the dynamics. The Main Theorem is proved in §3.

2. Background

Unless indicated otherwise, we work over the field $\overline{\mathbf{Q}}$ of algebraic numbers.

2.1. **Admissible compactifications.** We use the standard embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ given in coordinates by $(x_1, x_2) \mapsto [1 : x_1 : x_2]$. Let $\mathcal{L}$ be the set of affine functions on $\mathbf{A}^2$.

Definition 2.1. [FJ11] An *admissible compactification* of $\mathbf{A}^2$ is a smooth projective surface X together with a birational morphism $\pi : X \to \mathbf{P}^2$ that is an isomorphism above $\mathbf{A}^2$.

We can thus view $\mathbf{A}^2$ as a Zariski open subset of X. By the structure theorem for birational morphisms of surfaces, π is a finite composition of point blowups, so the divisor $X \setminus \mathbf{A}^2$ has normal crossing singularities.

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[1]The existence of $\hat{h}$ as a limit rather than a lim sup seems to be new, however.

Let $\xi \in X \setminus \mathbf{A}^2$ be a closed point. The closure in X of at least one of the curves $\{x_i = 0\} \subseteq \mathbf{A}^2$, $i = 1, 2$, does not contain ξ: this is true already when $X = \mathbf{P}^2$. Pick local coordinates $z_1, z_2 \in \mathcal{O}_{X,\xi}$ such that $X \setminus \mathbf{A}^2 \subseteq E_1 \cup E_2$, locally at ξ, where $E_i = \{z_i = 0\}$. Write $b_i = -\max_{\ell \in \mathcal{L}} \operatorname{ord}_{E_i}(\ell) \in \mathbf{Z}_{\geq 0}$, where $\operatorname{ord}_{E_i}(\ell)$ is the order of vanishing of ℓ along E_i. Note that $b_i = -\operatorname{ord}_{E_i}(\ell)$ for a general affine function $\ell \in \mathcal{L}$. Thus $b_i = 0$ iff $E_i \cap \mathbf{A}^2 \neq \emptyset$. We can write

$$x_i = z_1^{-b_1} z_2^{-b_2} \psi_i, \tag{2.1}$$

where $\psi_i \in \mathcal{O}_{X,\xi}$ and where $\psi_i(\xi) \neq 0$ for at least one i.

Similarly, if $f : \mathbf{A}^2 \to \mathbf{A}^2$ is any dominant polynomial mapping, then

$$f^* x_i = z_1^{\operatorname{ord}_{E_1}(f^*\ell)} z_2^{\operatorname{ord}_{E_2}(f^*\ell)} \chi_i, \tag{2.2}$$

for a general affine function $\ell \in \mathcal{L}$, where $\chi_i \in \mathcal{O}_{X,\xi}$. In this case, we do not claim that $\chi_i(\xi) \neq 0$ for some i.

2.2. **Degree growth.** See [Gue02, FJ07, FJ11] for more details on what follows. Let $f : \mathbf{A}^2 \to \mathbf{A}^2$ be a dominant polynomial mapping defined over $\overline{\mathbf{Q}}$. The *degree* $\deg f$ of f is the degree of $f^*\ell$ for a general affine function $\ell \in \mathcal{L}$. Thus $\deg f = \max_i \deg f^* x_i$.

It is easy to see that $\deg f^{n+m} \leq (\deg f^n)(\deg f^m)$, hence the limit $\lambda_1 := \lambda_1(f) := \lim_{n \to \infty} (\deg f^n)^{1/n}$ exists. It is called the *first dynamical degree* of f. The *second dynamical degree* $\lambda_2 := \lambda_2(f)$ is the topological degree of f, i.e. the number of preimages of a general closed point in $\mathbf{A}^2$. Note that $\lambda_i(f^n) = \lambda_i(f)^n$. It follows from Bézout's Theorem that $\lambda_2 \leq \lambda_1^2$. While the degree $\deg f$ depends on the choice of embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$, the dynamical degrees $\lambda_i(f)$ do not.

The degree growth sequence $(\deg f^n)_{n=0}^\infty$ of plane polynomial maps was studied in detail in [FJ07, FJ11]; see also [BFJ08]. In particular, we have

Theorem 2.2. [FJ07, Theorem A'.] *If $\lambda_1 > 1$, then $\deg f^n \sim n^l \lambda_1^n$ as $n \to \infty$, where $l \in \{0, 1\}$. Further, $l = 0$ unless $\lambda_2 = \lambda_1^2$ and f is conjugate to a skew product.*

2.3. **Dynamical compactifications.** The following result from [FJ11] plays a key role in the proof of the Main Theorem.

Theorem 2.3. *Let $f : \mathbf{A}^2 \to \mathbf{A}^2$ be a polynomial map of small topological degree: $\lambda_2 < \lambda_1$. Then, for every $\varepsilon > 0$ there exist an integer $n_0 \geq 1$, an admissible compactification X of $\mathbf{A}^2$ and a decomposition $X \setminus \mathbf{A}^2 = Z^+ \cup Z^-$ into (possibly reducible) curves Z^+, Z^- without common components, such that the following properties hold.*

(1) *If F is any irreducible component of Z^- and $\ell \in \mathcal{L}$ is a general affine function, then*

$$\operatorname{ord}_F(f^{n_0*}\ell) \geq (\lambda_2 + \varepsilon)^{n_0} \operatorname{ord}_F(\ell). \tag{2.3}$$

(2) *The extension $f^{n_0} : X \dashrightarrow X$ of f^{n_0} as a rational map is regular at any point on Z^+.*

(3) *There exists a closed point $\xi_+ \in Z^+ \setminus Z^-$ such that $f^{n_0}(Z^+) = \{\xi_+\}$ and $f(\xi_+) = \xi_+$. Further, we are in one of the following two cases:*

(a) *$\lambda_1 \notin \mathbf{Q}$, there are two irreducible components E_1, E_2 of Z^+ containing ξ_+ and, locally at ξ_+, we have $f^* E_i = a_{i1} E_1 + a_{i2} E_2$, where $a_{ij} \in \mathbf{N}$, and the 2×2 matrix (a_{ij}) has spectral radius λ_1;*

(b) *$\lambda_1 \in \mathbf{N}$, there is a unique irreducible component E of Z^+ containing ξ_+, $f(E) = \{\xi_+\}$ and, locally at ξ_+, we have $f^* E = \lambda_1 E$.*

The theorem above corresponds to [FJ11, Lemma 7.3].[2] The fact that we may assume $\lambda_1 \notin \mathbf{Q}$ in case (a) follows from Proposition 2.5 and the proof of Theorem 3.1 in [FJ11].

Note that everything in [FJ11] is stated over the complex numbers, and the assertions in (a) and (b) are slightly more precise than what is written here: they give normal forms in (possibly transcendental) local coordinates, based on the work of Favre [Fav00]. However, the main analysis in [FJ11] is purely algebraic. When working over a general algebraically closed field of characteristic zero one obtains Theorem 2.3.

2.4. **Absolute values.** Consider an admissible compactification X of $\mathbf{A}^2$. Let K be a number field such that X and the map $\pi : X \to \mathbf{P}^2$ are defined over K.

Let M_K be the set of (normalized) absolute values on K. For each $v \in M_K$, let K_v be the completion. Write $X(K)$ for the set of K-rational points of X. Similarly, $X(K_v)$ is the set of K_v-rational points of $X \otimes_K K_v$. Define $\mathbf{A}^2(K)$ and $\mathbf{A}^2(K_v)$ in the same way and equip all these spaces with the topology induced by v. Then $X(K_v)$ is compact and contains $X(K)$ as a dense subset. Further, $\mathbf{A}^2(K)$ is dense in $X(K)$, hence also in $X(K_v)$. This follows since every point in X admits a Zariski open neighborhood U isomorphic to affine 2-space such that $U \setminus \mathbf{A}^2$ is a proper Zariski closed subset.

2.5. **Heights.** See [Sil07, §3.1] or [Lan83, §3] for an introduction to heights. Let K be a number field. For any absolute value $v \in M_K$ define a *local height* $\tau_v : \mathbf{A}^2(K) \to \mathbf{R}_+$ by

$$\tau_v = \log^+ \max\{|x_1|_v, |x_2|_v\} = \log \max\{1, |x_1|_v, |x_2|_v\}$$

and define the (absolute, global) *logarithmic height* $h = h_K : \mathbf{A}^2(K) \to \mathbf{R}_+$ by

$$h = \sum_{v \in M_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \tau_v.$$

If L/K is a finite extension, then $h_L = h_K$ on $\mathbf{A}^2(K)$, see [Sil07, Proposition 3.4].

Let us express the local height τ_v in local coordinates at infinity. Consider an admissible compactification X of $\mathbf{A}^2$ and a closed point $\xi \in X \setminus \mathbf{A}^2$. Let $L \supseteq K$ be a number field over which X and ξ are defined and extend v as an absolute value on L. Pick local coordinates $z_i \in \mathcal{O}_{X,\xi}$, $i = 1, 2$, defined over L such that $X \setminus \mathbf{A}^2 \subseteq E_1 \cup E_2$, locally at ξ, where $E_i = \{z_i = 0\}$. Given $\delta_i > 0$ define

$$\Omega_{v,\delta} := \{P \in \mathbf{A}^2(L_v) \mid |z_i(P)|_v < \delta_i, i = 1, 2\}. \tag{2.4}$$

By §2.4 we have $\Omega_{v,\delta} \cap \mathbf{A}^2(L) \neq \emptyset$. It follows from (2.1) that if $0 < \delta_i \ll 1$, then

$$\tau_v = -\sum_{i=1}^{2} b_i \log |z_i|_v + O(1) \tag{2.5}$$

on $\Omega_{v,\delta}$. Similarly, it follows from (2.2) that if $f : \mathbf{A}^2 \to \mathbf{A}^2$ is a dominant polynomial mapping defined over L, then

$$\tau_v \circ f \leq \sum_{i=1}^{2} \mathrm{ord}_{E_i}(f^*\ell) \log |z_i|_v + O(1) \tag{2.6}$$

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[2] The inequality (2.3) is reversed in [FJ11] but this is a typo. The right hand side of (2.3) is negative.

on $\Omega_{v,\delta}$, where $\ell \in \mathcal{L}$ is a general affine function.

2.6. **Canonical height and arithmetic degree.** Consider a dominant polynomial mapping $f : \mathbf{A}^2 \to \mathbf{A}^2$ defined over $\overline{\mathbf{Q}}$. For simplicity assume $\lambda_1 = \lambda_1(f) > 1$. Silverman defines the *arithmetic degree* on $\mathbf{A}^2(\overline{\mathbf{Q}})$ by

$$\alpha := \alpha_f := \limsup_{n\to\infty}(h \circ f^n)^{1/n}$$

and shows that $\alpha \le \lambda_1$, see [Sil11, Proposition 12].

By Theorem 2.2 we have $\deg(f^n) \sim n^l \lambda_1^n$, where $l \in \{0, 1\}$. Further, $l = 0$ when $\lambda_2 < \lambda_1^2$. Silverman defines the *canonical height* on $\mathbf{A}^2(\overline{\mathbf{Q}})$ by

$$\hat{h} := \hat{h}_f := \limsup \frac{1}{n^l \lambda_1^n} h \circ f^n$$

and shows in [Sil11, Proposition 19 (d)] that $\alpha(P) < \lambda_1$ implies $\hat{h}(P) = 0$. Inspired by Silverman's work, we state

Conjecture 2.4. We have $\hat{h}(P) < \infty$ for all $P \in \mathbf{A}^2(\overline{\mathbf{Q}})$. Moreover, if $\lambda_2 < \lambda_1^2$, then $\hat{h}(P) = 0$ implies $\alpha(P) < \lambda_1$.

The second part of this conjecture would provide a converse to Proposition 19 (d) in [Sil11], whereas the first part would give an affirmative answer to a special case of Question 18 in *loc. cit.*

Our Main Theorem gives a positive answer to Conjecture 2.4 when $\lambda_2 < \lambda_1$ and shows that, under this assumption, the $\limsup$ in the definition of $\hat{h}$ is in fact a limit. On the other hand, the following example shows that the last statement in Conjecture 2.4 may fail when $\lambda_2 = \lambda_1^2$.

Example 2.5. [Sil11, Example 16]. If $f(x, y) = (x^2, xy^2)$, then $\lambda_1 = 2$, $\lambda_2 = 4$ and $\deg f^n = (n + 2) \cdot 2^{n-1}$, so $l = 1$. For $P = (2, 0)$ we have $h(f^n(P)) = 2^n \log 2$. In particular, $\hat{h}(P) = 0$ but $\alpha(P) = \lambda_1 = 2$.

See also [KS12] for definitions and results of canonical heights and arithmetic degrees for rational selfmaps of more general algebraic varieties.

3. Proof of the Main Theorem

Fix a polynomial mapping $f : \mathbf{A}^2 \to \mathbf{A}^2$ defined over $\overline{\mathbf{Q}}$ and of small topological degree: $\lambda_2 < \lambda_1$. Let $K_0 = K_0(f)$ be a number field over which f is defined.

3.1. **Growth of local heights.**

Proposition 3.1. *We may choose K_0 above such that if $K \supseteq K_0$ and $v \in M_K$ is any normalized absolute value, then:*
   (i) *the limit $\hat{\tau}_v := \lim_{n\to\infty} \lambda_1^{-n} \tau_v \circ f^n$ exists, and is finite, pointwise on $\mathbf{A}^2(K)$;*
   (ii) *the set $U_v^+ := \{\hat{\tau}_v > 0\} \subseteq \mathbf{A}^2(K)$ is open and nonempty;*
   (iii) *$\limsup_{n\to\infty}(\tau_v \circ f^n)^{1/n} \le \lambda_2$ on $\mathbf{A}^2(K) \setminus U_v^+$.*

To prove this, pick $0 < \varepsilon < \lambda_1 - \lambda_2$. Given this ε, let X, ξ_+ and n_0 be as in Theorem 2.3 and choose a number field $L = L(\varepsilon) \supseteq K$ such that f and X are defined over L and the closed point ξ_+ is L-rational. Extend v as an absolute value on L.

Lemma 3.2. *There exists an open subset $V_v^+ \subseteq \mathbf{A}^2(L_v)$ such that $f(V_v^+) \subseteq V_v^+$ and*
   (i) *the limit $\hat{\tau}_v := \lim_{n\to\infty} \lambda_1^{-n} \tau_v \circ f^n$ exists pointwise on V_v^+ and $0 < \hat{\tau}_v < \infty$;*

(ii) $\mathbf{A}^2(L) \cap V_v^+ \neq \emptyset$;

(iii) $\tau_v \circ f^{n_0} \leq (\lambda_2 + \varepsilon)^{n_0} \tau_v + O(1)$ on $\mathbf{A}^2(L_v) \setminus V_v^+$, with n_0 as in Theorem 2.3.

This result is an analogue of [FJ11, Lemma 7.2], which in turn is modeled on results for the complex Hénon map. See also [Ing11, Lemma 2.1].

Granted Lemma 3.2 we conclude the proof of Proposition 3.1 as follows. Set

$$U_v^+ := \mathbf{A}^2(K) \cap \bigcup_{n \geq 0} f^{-n}(V_v^+). \tag{3.1}$$

By Lemma 3.2 (i), the limit defining $\hat{\tau}_v$ exists on the open set U_v^+ and $0 < \hat{\tau}_v < \infty$ there. Now suppose $P \in \mathbf{A}^2(K) \setminus U_v^+$. Since $f^{jn_0}(P) \notin V_v^+$ for all $j \geq 0$, Lemma 3.2 (iii) yields $\limsup_{j \to \infty} (\tau_v(f^{jn_0}(P)))^{1/jn_0} \leq \lambda_2 + \varepsilon$. Using the bound $\tau_v \circ f \leq (\deg f)\tau_v + O(1)$ on $\mathbf{A}^2(K)$ (see [Sil07, (3.6) p.92]) we obtain

$$\limsup_{n \to \infty} \tau_v(f^n(P))^{1/n} \leq \lambda_2 + \varepsilon < \lambda_1.$$

Thus $\hat{\tau}_v$ is well defined and finite everywhere on $\mathbf{A}^2(K)$ and the set U_v^+ is characterized, independently of L, as $U_v^+ = \{\hat{\tau}_v > 0\}$. Letting $\varepsilon \to 0$ we obtain (iii).

It only remains to prove that U_v^+ is nonempty, assuming that K_0 is well chosen and $K \supseteq K_0$. For this, it suffices to consider the case $K = K_0$. Fix ε_0 with $0 < \varepsilon_0 < \lambda_1 - \lambda_2$, apply Theorem 2.3 with $\varepsilon = \varepsilon_0$ and pick K_0 so that all the relevant data in that theorem (as well as f) are defined over K_0. Given $v \in M_{K_0}$, we now apply Lemma 3.2 with $L = K = K_0$. We obtain $U_v^+ \supseteq \mathbf{A}^2(K_0) \cap V_v^+ \neq \emptyset$ as desired.

Proof of Lemma 3.2. We first analyze the situation locally at infinity near the fixed point ξ_+. Pick local coordinates (z_1, z_2) at ξ_+ such that $X \setminus \mathbf{A}^2 \subseteq E_1 \cup E_2$, locally at ξ, where $E_i = \{z_i = 0\}$. Given $0 < \delta_i \ll 1$ define $\Omega_{v,\delta}$ as in (2.4). We claim that for suitable choices of δ_i we have $f(\Omega_{v,\delta}) \subseteq \Omega_{v,\delta}$ and that $\lambda_1^{-n} \tau_v \circ f^n$ converges pointwise on $\Omega_{v,\delta}$ to a strictly positive function $\hat{\tau}_v$. To see this, we consider the two cases (a) and (b) in Theorem 2.3 (3) separately.

In case (b) we have $E_1 \cap \mathbf{A}^2 = \emptyset \neq E_2 \cap \mathbf{A}^2$ and

$$f^* z_1 = z_1^{\lambda_1} \phi_1 \quad \text{and} \quad f^* z_2 = z_1 \phi_2, \tag{3.2}$$

where $\phi_i \in \mathcal{O}_{X,\xi_+}$ and $\phi_1(\xi_+) \neq 0$. For $0 < \delta_1 \ll \delta_2 \ll 1$ it follows, using $\lambda_1 \geq 2$, that $f(\Omega_{v,\delta}) \subseteq \Omega_{v,\delta}$. We claim that as $n \to \infty$, the functions $s_n := \lambda_1^{-n} \log|f^{n*}z_1|_v$ converge pointwise on $\Omega_{v,\delta}$ to a function $s_\infty < 0$. Indeed, (3.2) yields $s_{n+1} = s_n + \lambda_1^{-(n+1)} \log|\phi_1 \circ f^n|_v$ and hence

$$s_n \to s_\infty := \log|z_1|_v + \sum_0^\infty \lambda_1^{-(j+1)} \log|\phi_1 \circ f^j|_v \quad \text{as } n \to \infty.$$

Here the series is uniformly bounded on $\Omega_{v,\delta}$, so $s_\infty < 0$. Clearly $s_\infty \circ f = \lambda_1 s_\infty$. Using (2.5) we can rewrite this in terms of the local height. Note that $b_1 > 0 = b_2$ since $E_1 \cap \mathbf{A}^2 = \emptyset \neq E_2 \cap \mathbf{A}^2$. Thus $\tau_v = -b_1 \log|z_1|_v + O(1)$ on $\Omega_{v,\delta}$. It is then clear that $\lambda_1^{-n} \tau_v \circ f^n$ converges pointwise on $\Omega_{v,\delta}$ to the function $\hat{\tau}_v := -b_1 s_\infty > 0$.

In case (a) we instead have $E_i \cap \mathbf{A}^2 = \emptyset$ for $i = 1, 2$ and

$$f^* z_i = z_1^{a_{i1}} z_2^{a_{i2}} \phi_i \tag{3.3}$$

for $i = 1, 2$, where $a_{ij} \in \mathbf{N}$, $\phi_i \in \mathcal{O}_{X,\xi_+}$ and $\phi_i(\xi_+) \neq 0$. The matrix (a_{ij}) has irrational spectral radius λ_1 and hence admits an eigenvector $\zeta = (\zeta_1, \zeta_2)$ with $\zeta_i > 0$.

Set $\delta_i = \kappa^{\zeta_i}$ with $0 < \kappa \ll 1$. Since $\lambda_1 > 1$ it follows that $f(\Omega_{v,\delta}) \subseteq \Omega_{v,\delta}$. We have

$$f^{n*}z_i = z_1^{a_{i1}^{(n)}} z_2^{a_{i2}^{(n)}} \prod_{l=0}^{n-1} (\phi_1^{a_{i1}^{(l)}} \phi_2^{a_{i2}^{(l)}}) \circ f^{n-1-l}, \tag{3.4}$$

where, for $0 \leq l \leq n$, $a_{ij}^{(l)}$ are the entries in the matrix A^l. By the Perron-Frobenius Theorem, $\lambda_1^{-n}A^n$ converges as $n \to \infty$ to a matrix with strictly positive coefficients. Using (3.4) we see that for $i = 1, 2$, the sequence $(s_{i,n})_{n=1}^{\infty}$ of functions on $\Omega_{v,\delta}$, defined by $s_{i,n} = \lambda_1^{-n} \log |f^{n*}z_i|_v$, converges pointwise to a strictly negative function $s_{i,\infty}$. Clearly $s_{i,\infty} \circ f = \lambda_1 s_{i,\infty}$. Using (2.5), this implies that $\lambda_1^{-n}\tau_v \circ f^n$ converges pointwise on $\Omega_{v,\delta}$ to the function $\hat{\tau}_v := -\sum_{i=1}^{2} b_i s_{i,\infty}$. Since $b_i > 0$ we again have $\hat{\tau}_v > 0$.

Set $V_v^+ := f^{-n_0}(\Omega_{v,\delta})$. Clearly V_v^+ is open in $\mathbf{A}^2(L_v)$ and $f(V_v^+) \subseteq V_v^+$. Further, $\lambda_1^{-n}\tau_v \circ f^n$ converges pointwise on V_v^+ to a function $\hat{\tau}_v > 0$. By §2.4, we have $V_v^+ \cap \mathbf{A}^2(L) \supseteq \Omega_{v,\delta} \cap \mathbf{A}^2(L) \neq \emptyset$.

It remains to prove the estimate in (iii). Let $\tilde{V}_v^+ \subseteq X(L_v)$ be an open set such that $\mathbf{A}^2(L_v) \cap \tilde{V}_v^+ = V_v^+$ and $Z^+(L_v) \subseteq \tilde{V}_v^+$. Such a set can be constructed as follows. Following (2.4) set

$$\tilde{\Omega}_{v,\delta} := \{P \in X(L_v) \mid |z_i(P)|_v < \delta_i, i = 1, 2\},$$

so that $\tilde{\Omega}_{v,\delta} \cap \mathbf{A}^2(L_v) = \Omega_{v,\delta}$. Recall that the extension of f^{n_0} as a rational selfmap of X is regular at every point of Z^+ and that $f^{n_0}(Z^+) = \{\xi^+\}$. We can therefore set

$$\tilde{V}_v^+ := (f^{n_0})^{-1}(\tilde{\Omega}_{v,\delta}) \setminus I,$$

where I is the indeterminacy set of $f^{n_0} : X \dashrightarrow X$.

In order to prove (iii) it suffices, by compactness, to prove the estimate

$$\tau_v \circ f^{n_0} \leq (\lambda_2 + \varepsilon)^{n_0}\tau_v + O(1) \tag{3.5}$$

in a neighborhood of any point $\eta \in X(L_v) \setminus \tilde{V}_v^+$. This is clear if $\eta \in \mathbf{A}^2(L_v)$, so we may assume $\eta \notin \mathbf{A}^2(L_v)$. Then $\eta \in Z^-(L_v) \setminus Z^+(L_v)$. Pick local coordinates (w_1, w_2) at η, defined over L_v, such that $X(L_v) \setminus \mathbf{A}^2(L_v) \subseteq F_1(L_v) \cup F_2(L_v)$ locally at η, where $F_i = \{w_i = 0\}$. Let $\tilde{G}_\eta$ be a small neighborhood of $\eta \in X(L_v)$. On $\tilde{G}_\eta \cap \mathbf{A}^2(L_v)$ we then have

$$\tau_v \circ f^{n_0} \leq \sum_{i=1}^{2} \operatorname{ord}_{F_i}(f^{n_0*}\ell) \log |w_i|_v + O(1)$$

$$\leq (\lambda_2 + \varepsilon)^{n_0} \sum_{i=1}^{2} \operatorname{ord}_{F_i}(\ell) \log |w_i|_v + O(1)$$

$$\leq (\lambda_2 + \varepsilon)^{n_0}\tau_v + O(1),$$

where $\ell \in \mathcal{L}$ is a general affine function. Here the first and third inequality follow from (2.6) and (2.5), respectively. The second inequality results from (2.3) when $F_i \subseteq Z^-$ and is trivial otherwise, since $\operatorname{ord}_{F_i}(f^*\ell) \leq 0 = \operatorname{ord}_{F_i}(\ell)$ in this case.

Thus (3.5) holds, which completes the proof. $\square$

3.2. **Growth of heights.** Let K_0 be as in Proposition 3.1. Pick any point $P \in \mathbf{A}^2(\overline{\mathbf{Q}})$ and let $K \supseteq K_0$ be a number field such that P is K-rational. We claim that there exists a finite subset $M'_K = M'_K(P) \subseteq M_K$ such that

$$\tau_v(f^n(P)) = 0 \quad \text{for all } v \in M_K \setminus M'_K \text{ and all } n \geq 0. \tag{3.6}$$

To see this, note that $f^* x_i \in K[x_1, x_2]$, $i = 1, 2$, so for all but finitely many $v \in M_K$ we will have $f^* x_i \in \mathfrak{o}_v[x_i]$, where $\mathfrak{o}_v = \{a \in K \mid |a|_v \leq 1\}$. For these v we then also have $f^{n*} x_i \in \mathfrak{o}_v[x_i]$ for all $n \geq 1$. Further, for all but finitely many v we have $x_i(P) \in \mathfrak{o}_v$. Combining these statements and the definition of τ_v, we obtain (3.6).

It is now easy to conclude. We have

$$h(f^n(P)) = \sum_{v \in M_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \tau_v(f^n(P)) = \sum_{v \in M'_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \tau_v(f^n(P)),$$

so since M'_K is finite, all assertions in the Main Theorem follow from Proposition 3.1, except for the claim about polynomial automorphisms.

3.3. **Polynomial automorphisms.** Now assume that f is a polynomial automorphism with $\lambda_1 > 1$. We must prove that if $\hat{h}(P) = 0$, then P is periodic.

First suppose f is *regular* [Sib99] in the sense that the indeterminacy loci of the birational maps $f, f^{-1} : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ are disjoint.

From Propositions 2.2.2 and 2.3.2 in [Sib99] we deduce the following facts: $\deg f = \deg f^{-1} = \lambda_1$; the indeterminacy locus of f (resp. f^{-1}) is a single point ξ_- (resp. ξ_+) on the line at infinity $L_\infty = \mathbf{P}^2 \setminus \mathbf{A}^2$ with $\xi_+ \neq \xi_-$; $f(L_\infty \setminus \{\xi_-\}) = \xi_+$ and $f^{-1}(L_\infty \setminus \{\xi_+\}) = \xi_-$.

Let X be the minimal admissible compactification of $\mathbf{A}^2$ for which the induced birational map $f : X \dashrightarrow \mathbf{P}^2$ is a morphism (i.e. without indeterminacy points). Concretely, X is obtained from $\mathbf{P}^2$ by successively blowing up the indeterminacy locus of f. Since f is birational, $X \setminus \mathbf{A}^2$ has a unique irreducible component E_- that is mapped onto L_∞ by f. All other irreducible components are mapped to ξ_+.

Set $Z^- := E_-$ and $Z^+ := X \setminus (\mathbf{A}^2 \cup E_-)$. We claim that a strong version of Theorem 2.3 holds, with $n_0 = 1$. Indeed, note that statement (2) holds. The same is true of statement (3): we are in case (b), with $E = L_\infty$. Further, we claim that

$$\operatorname{ord}_{E_-}(f^* \ell) = -1 \quad \text{and} \quad \operatorname{ord}_{E_-}(\ell) = -\lambda_1, \tag{3.7}$$

for a general affine function ℓ. Since $\lambda_1 > \lambda_2 = 1$, this is stronger than (1).

To prove (3.7), recall that $f(E_-) = L_\infty$. Since f is an automorphism, this implies $\operatorname{ord}_{E_-}(f^* \ell) = \operatorname{ord}_{L_\infty}(\ell) = -1$. Similarly, $\operatorname{ord}_{E_-}(\ell) = \operatorname{ord}_{L_\infty}((f^{-1})^* \ell) = -\deg f^{-1} = -\lambda_1$.

Fix a number field K_0 over which f and X are defined. Suppose $K \supseteq K_0$ and consider an absolute value $v \in M_K$. Using (3.7) we prove Lemma 3.2 with $L = K$ and the following estimate, which is stronger than the one in (iii):

$$\tau_v \circ f \leq \lambda_1^{-1} \tau_v + O(1)$$

on $\mathbf{A}^2(K_v) \setminus V_v^+$. Since $\lambda_1 > 1$, it follows that the sequence $(\tau_v \circ f^n)_{n=0}^\infty$ must be pointwise bounded on $\mathbf{A}^2(K) \setminus U_v^+$, where U_v^+ is defined as in (3.1).

Now fix $P \in \mathbf{A}^2(\overline{\mathbf{Q}})$ with $\hat{h}(P) = 0$. Pick $K \supseteq K_0$ such that $P \in \mathbf{A}^2(K)$. From the preceding paragraph and the arguments in §3.2 it follows that the sequence

$(h(f^n(P)))_{n=0}^{\infty}$ is bounded, which by the Northcott Theorem implies that P is preperiodic, hence periodic.

Finally we treat the general case when f is not necessarily regular. It follows from [FM89] that there exists a polynomial automorphism $g : \mathbf{A}^2 \to \mathbf{A}^2$ such that $\tilde{f} := g^{-1}fg$ is regular. Set $D = (\deg g)(\deg g^{-1})$. Since $f^n = g\tilde{f}^n g^{-1}$ we have $D^{-1} \deg \tilde{f}^n \le \deg f^n \le D \deg \tilde{f}^n$ and hence $\lambda_1(\tilde{f}) = \lambda_1(f)$.

As for the growth of heights, we already know that the limits

$$\hat{h}_f := \lim_{n\to\infty} \lambda_1^{-n} h \circ f^n \quad \text{and} \quad \hat{h}_{\tilde{f}} := \lim_{n\to\infty} \lambda_1^{-n} h \circ \tilde{f}^n$$

exist, pointwise on $\mathbf{A}^2(\overline{\mathbf{Q}})$. Since $\deg g^{\pm 1} \le D$, we have by [Sil07, Theorem 3.11] that

$$D^{-1}(h \circ \tilde{f}^n) + O(1) \le h \circ f^n \circ g \le D(h \circ \tilde{f}^n) + O(1)$$

and hence $D^{-1}\hat{h}_{\tilde{f}} \le \hat{h}_f \circ g \le D\hat{h}_{\tilde{f}}$. In particular, if $\hat{h}_f(P) = 0$, then $\hat{h}_{\tilde{f}}(g^{-1}(P)) = 0$. By what precedes, $g^{-1}(P)$ is then periodic under $\tilde{f}$, so that P is periodic under f. This concludes the proof of the Main Theorem.

3.4. **A non-invertible example.** Let us consider the map

$$f(x,y) = (y^2(xy + 1), x(xy^3 + 1)).$$

The topological degree is $\lambda_2 = 4$. For example, the point $(0,0)$ has three preimages $(0,0)$, $(1,-1)$ and $(-1,1)$, with multiplicities 2, 1 and 1, respectively. It is easy to find a recursion relation for $(\deg f^n)_{n=0}^{\infty}$ and show that the first dynamical degree is $\lambda_1 = 2 + \sqrt{7}$. In particular, $\lambda_1 > \lambda_2$, so f is of small topological degree.

Note that $f^2(x,0) = (x^2, 0)$ and $f^2(0,y) = (0,y^2)$. Since $\lambda_1 > \sqrt{2}$, the coordinate axes are contained in the locus $\{\hat{h} = 0\}$, which therefore contains points for which $h \circ f^n$ grows exponentially.

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