ON WEIGHTED BOCHNER-MARTINELLI RESIDUE CURRENTS

ELIZABETH WULCAN

ABSTRACT. We study the weighted Bochner-Martinelli residue current $R^p(f)$ associated with a sequence $f = (f_1, \dots, f_m)$ of holomorphic germs at $0 \in \mathbf{C}^n$, whose common zero set equals the origin, and $p = (p_1, \dots, p_m) \in \mathbb{N}^m$. Our main results are a description of $R^p(f)$ in terms of the Rees valuations of the ideal generated by $(f_1^{p_1}, \dots, f_m^{p_m})$ and an explicit description of $R^p(f)$ when f is monomial. For a monomial sequence f we show that $R^p(f)$ is independent of p if and only if f is a regular sequence.

1. Introduction

Let $f = (f_1, \dots, f_m)$ be a sequence of germs of holomorphic functions at $0 \in \mathbb{C}^n$, such that $V(f) := \{f_1 = \dots = f_m = 0\} = \{0\}$. If f is a *regular sequence*, that is, $m = n$, then there is a canonical residue (current) associated with f - the Grothendieck residue $\text{Res}(\frac{\bullet}{f_1 \cdots f_m})$, see [12], and its current avatar the *Coleff-Herrera product* $R_{CH}(f) = \bar\partial[1/f_1] \wedge \cdots \wedge \bar\partial[1/f_m]$, introduced in [9]. In [19] Passare-Tsikh-Yger constructed residue currents based on the Bochner-Martinelli kernel as a natural generalization of the Coleff-Herrera product. This idea is further developed in [6], where Berenstein-Yger introduced *weighted Bochner-Martinelli residue currents*.

Let $p = (p_1, \dots, p_m) \in \mathbb{N}^m$ and let f^p denote the sequence $(f_1^{p_1}, \dots, f_m^{p_m})$; here $\mathbb{N}$ denotes the natural numbers $1, 2, \dots$. For each ordered multi-index $\mathcal{I} = \{i_1, \dots, i_n\} \subseteq \{1, \dots, m\}$ let

(1.1)
$$R^p_{\mathcal{I}}(f) = \bar\partial |f^p|^{2\lambda} \wedge c_n \sum_{\ell=1}^{n} (-1)^{\ell-1} \frac{\bar f_{i_\ell} |f_{i_\ell}|^{2(p_{i_\ell}-1)} \bigwedge'_{q\neq\ell} \bar\partial(\bar f_{i_q} |f_{i_q}|^{2(p_{i_q}-1)})}{|f^p|^{2n}} \Bigg|_{\lambda=0},$$

where $c_n = (-1)^{n(n-1)/2}(n-1)!$, $|f^p|^2 = |f_1^{p_1}|^2 + \dots + |f_m^{p_m}|^2$, $\bigwedge'$ denotes increasing order in q in the wedge product, and $\alpha|_{\lambda=0}$ denotes the analytic continuation of the form α to $\lambda = 0$. Moreover, let $R^p(f)$ denote the vector-valued current with entries $R^p_{\mathcal{I}}(f)$; we will refer to this as the *Bochner-Martinelli residue current of weight* p associated with f. Then $R^p(f)$ is a well-defined $(0, n)$-current with support at the origin

Date: August 1, 2011.
The author was partially supported by the Swedish Research Council and NSF grant DMS-0901073.

and $\bar{g}R^p_{\mathcal{I}}(f) = 0$ if g is a holomorphic function that vanishes at the origin. It follows that the coefficients of the $R^p_{\mathcal{I}}(f)$ are just finite sums of holomorphic derivatives at the origin. If $p = (1, \ldots, 1)$, then $R^p(f)$ is the *Bochner-Martinelli residue current* associated with f, introduced in [19]; we denote it by $R(f)$ and its entries by $R_{\mathcal{I}}(f)$. Note that, in fact,

$$R^p_{\mathcal{I}}(f) = f_{i_1}^{p_{i_1}-1} \cdots f_{i_n}^{p_{i_n}-1} R_{\mathcal{I}}(f^p). \tag{1.2}$$

Indeed, the sequence f^p in the factor $\bar{\partial}|f^p|^{2\lambda}$ in (1.1) can be replaced by any sequence of functions that vanish at the origin.

Let $\mathcal{O}^n_0$ be the local ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$. Given a germ of a current μ at $0 \in \mathbb{C}^n$, let $\operatorname{ann}\mu$ denote the (holomorphic) *annihilator ideal* of μ, that is, $\operatorname{ann}\mu = \{h \in \mathcal{O}^n_0, h\mu = 0\}$. Our first result concerns $\operatorname{ann} R^p(f)$. Let $\mathfrak{a}(f)$ denote the ideal generated by the f_i in $\mathcal{O}^n_0$. Recall that $h \in \mathcal{O}^n_0$ is in the *integral closure* of $\mathfrak{a}(f)$, denoted by $\overline{\mathfrak{a}(f)}$, if $|h| \leq C|f|$, for some constant C. Moreover, recall that $\mathfrak{a}(f)$ is a *complete intersection ideal* if it can be generated by $n = \operatorname{codim} V(f)$ functions. Note that this condition is slightly weaker than that f is a regular sequence. Also, recall that, given ideals $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}^n_0$, the *colon ideal* $\mathfrak{a} \colon \mathfrak{b}$ is the ideal $\mathfrak{a} \colon \mathfrak{b} = \{h \in \mathcal{O}^n_0 : h\mathfrak{b} \subseteq \mathfrak{a}\}$.

We also provide a characterization of the non-vanishing entries of $R^p(f)$. Let $\pi : X \to (\mathbb{C}^n, 0)$ be a log-resolution of $\mathfrak{a}(f)$, see [16, Def. 9.1.12]. Following [15] we say that a multi-index $\mathcal{I} = \{i_1, \ldots, i_n\}$ is *essential* with respect to f if there is an exceptional prime $E \subseteq \pi^{-1}(0)$ of X such that the mapping $[f_{i_1} \circ \pi : \ldots : f_{i_n} \circ \pi] : E \to \mathbb{CP}^{n-1}$ is surjective and moreover $\operatorname{ord}_E(f_{i_k}) \leq \operatorname{ord}_E(f_\ell)$ for $1 \leq k \leq n, 1 \leq \ell \leq m$, see Section 2 and also [15, Section 3] for details. The valuations ord_E that satisfy this are precisely the *Rees valuations* of $\mathfrak{a}(f)$. We say that $\mathcal{I}$ is *p-essential* if it is essential with respect to f^p. For $h \in \mathcal{O}^n_0$, let (h) denote the ideal generated by h.

Theorem A. *Suppose that f is a sequence of germs of holomorphic functions at $0 \in \mathbb{C}^n$, such that $V(f) = \{0\}$. Let $R^p(f)$ be the corresponding Bochner-Martinelli residue current of weight p. Then the entry $R^p_{\mathcal{I}}(f) \not\equiv 0$ if and only if $\mathcal{I}$ is p-essential. Moreover*

$$\bigcap_{\mathcal{I}\ p-essential} \overline{\mathfrak{a}(f^p)^n} \colon (f_{i_1}^{p_{i_1}-1} \cdots f_{i_n}^{p_{i_n}-1}) \subseteq ann\, R^p(f) \subseteq \mathfrak{a}(f). \tag{1.3}$$

The left inclusion in (1.3) *is strict whenever $n \geq 2$. If the right inclusion is an equality, then $\mathfrak{a}(f)$ is a complete intersection ideal.*

The new results in Theorem A are the characterization of the non-vanishing entries and the last two statements. Berenstein-Yger [6] showed that $\overline{\mathfrak{a}(f^p)^n} \colon (f_{i_1}^{p_{i_1}} \cdots f_{i_n}^{p_{i_n}}) \subseteq \operatorname{ann} R^p_{\mathcal{I}}(f)$, and it is easy to see from Andersson's construction of residue currents in [1] that the right inclusion in (1.3) holds. In fact, Berenstein-Yger defined currents $R^p_{\mathcal{I}}(f)$

also when $\dim V(f) > 0$. The inclusions in (1.3) hold true also in this case, and one can replace the leftmost ideal by $\bigcap_{\mathcal{I}=\{i_1,\ldots,i_\mu\}} \overline{\mathfrak{a}(f^p)^\mu} : (f_{i_1}^{p_{i_1}-1} \cdots f_{i_\mu}^{p_{i_\mu}-1})$, where $\mu = \min(m, n)$.

Also, for $R(f) = R^{(1,\ldots,1)}(f)$ Theorem A was proved in parts in [19], [1], and [15]. If f is a regular sequence, then the only entry $R_{\{1,\ldots,m\}}(f)$ of $R(f)$ coincides with the Coleff-Herrera product $R_{CH}(f)$, whose annihilator ideal is precisely $\mathfrak{a}(f)$, see [10, 18]. This should be compared to [12, Chapter 5.1] where $\operatorname{Res}(\frac{\bullet}{f_1\cdots f_m})$ is defined using the Bochner-Martinelli kernel. The idea of regarding (complete intersection) ideals of holomorphic functions as the annihilator ideals of certain residue currents is central for many applications, see, for example, [7]. For $p = (1, \ldots, 1)$, the inclusions (1.3) read $\overline{\mathfrak{a}(f)^n} \subseteq \operatorname{ann} R(f) \subseteq \mathfrak{a}(f)$, which gives a direct proof of the Briançon-Skoda Theorem [8]: $\overline{\mathfrak{a}(f)^n} \subseteq \mathfrak{a}(f)$. For other applications of Bochner-Martinelli residue currents, see for example [3], [4], and [22].

Weighted Bochner-Martinelli residue currents were introduced in [6] as a tool to construct Green currents but also as a natural extension of Bochner-Martinelli residue currents in the spirit of Lipman [17]; the currents have been further studied in [5] and [24]. In the monograph [17] not only the residue $\operatorname{Res}(\frac{\bullet}{f_1\cdots f_m})$ associated with a sequence f plays a role but also residues of the form $\operatorname{Res}(\frac{f_1^{p_1-1}\cdots f_m^{p_m-1}\bullet}{f_1^{p_1}\cdots f_m^{p_m}})$. The currents $R^p(f)$ can thus be seen as analogues of these residues. If f is a regular sequence, then $\operatorname{Res}(\frac{f_1^{p_1-1}\cdots f_m^{p_m-1}\bullet}{f_1^{p_1}\cdots f_m^{p_m}}) = \operatorname{Res}(\frac{\bullet}{f_1\cdots f_m})$, which in current language reads

$$f_1^{p_1-1} \cdots f_m^{p_m-1} R_{CH}(f^p) = R_{CH}(f). \tag{1.4}$$

It follows that $R^p(f)$ is independent of p if f is a regular sequence. In general, however, $R^p(f)$ depends in an essential way on p; the set of non-vanishing entries as well as $\operatorname{ann} R^p(f)$ depend on p, see Sections 4 and 5. Proposition 5.1 asserts that if f is monomial, then R^p is independent of p if and only if f is a regular sequence. This motivates the following question.

Question B. *Suppose that $f = (f_1, \ldots, f_m)$ is a sequence of germs of holomorphic functions at $0 \in \mathbb{C}^n$. Let $R^p(f)$ be the Bochner-Martinelli residue current of weight p. Is it true that $R^p(f)$ is independent of p if and only if f is a regular sequence?*

Question B could be asked also for $\operatorname{ann} R^p(f)$: is it true that $\operatorname{ann} R^p(f)$ is independent of p if and only if f is a regular sequence?

Lemma 1.2 in [6] asserts that

$$\sum_{\mathcal{I}=\{i_1,\ldots,i_n\}\subseteq\{1,\ldots,m\}} R^p_{\mathcal{I}}(f) \wedge df_{i_n} \wedge \cdots \wedge df_{i_1}/(2\pi i)^n = e^p(f)[0], \tag{1.5}$$

where $e^p(f)$ is a positive number; in fact each term in (1.5) is a positive current with support at the origin, see Lemma 3.1. Andersson [2] showed that $e^{(1,\dots,1)}(f)$ is the Hilbert-Samuel multiplicity of the ideal $\mathfrak{a}(f)$. In general $e^p(f)$ depends on p, see Example 4.4, but it can also happen that $e^p(f)$ is independent of p even if $\operatorname{ann} R^p(f)$ and $R^p(f)$ vary with p, as shown in Example 5.5.

In general it is hard to compute $R^p(f)$, as well as $\operatorname{ann} R^p(f)$ and $e^p(f)$. However, if the f_j are monomials we can give an explicit description of $R^p(f)$ based on [23, Thm. 3.1]. For $A = \{a^1, \dots, a^m\} \subseteq \mathbb{Z}^n$, let z^A denote the sequence of monomials $z^{a^1}, \dots, z^{a^m}$, where $z^{a^j} = z_1^{a_1^j} \cdots z_n^{a_n^j}$ if $a^j = (a_1^j, \dots, a_n^j)$. Moreover, for $p \in \mathbb{N}^m$, let pA denote the set $pA = \{p_1 a^1, \dots, p_m a^m\} \subseteq \mathbb{Z}^n$. Given a holomorphic function g we will use the notation $\bar\partial[1/g]$ for the value at $\lambda = 0$ of $\bar\partial|g|^{2\lambda}/g$, a priori defined for $\operatorname{Re}\lambda >> 0$, and analogously by $[1/g]$ we will mean $|g|^{2\lambda}/g|_{\lambda=0}$, that is, the principal value of $1/g$.

Theorem C. *Suppose that z^A is a sequence of germs of holomorphic monomials at $0 \in \mathbb{C}^n$, such that $V(z^A) = \{0\}$. Let $R^p(z^A)$ be the corresponding Bochner-Martinelli residue current of weight p. Then*

$$R^p_{\mathcal{I}}(z^A) = sgn\,(A_{\mathcal{I}}) C_{\mathcal{I}}\; \bar\partial \left[\frac{1}{z_1^{\alpha_1^{\mathcal{I}}}}\right] \wedge \cdots \wedge \bar\partial \left[\frac{1}{z_n^{\alpha_n^{\mathcal{I}}}}\right]; \tag{1.6}$$

here $sgn\,(A_{\mathcal{I}})$ is the sign of the determinant of the matrix with rows $a^{i_1}, \dots, a^{i_n}$, $C_{\mathcal{I}} > 0$ if $\mathcal{I}$ is p-essential and $C_{\mathcal{I}} = 0$ otherwise, and $(\alpha_1^{\mathcal{I}}, \dots, \alpha_n^{\mathcal{I}}) = \alpha^{\mathcal{I}} = \sum_{j\in\mathcal{I}} a^j$.

In particular, Theorem C implies that

$$\operatorname{ann} R^p(z^A) = \bigcap_{\mathcal{I}\ \text{p-essential}} (z_1^{\alpha_1^{\mathcal{I}}}, \dots, z_n^{\alpha_n^{\mathcal{I}}}).$$

In Section 2 we provide some background on Rees valuations, whereas the proof of Theorem A occupies Section 3. In Section 4 we focus on the case when f is monomial; we prove Theorem C and compute the coefficients $C_{\mathcal{I}}$ in some special cases. Finally, in Section 5 we discuss Question B and some related questions.

2. Rees valuations and essential multi-indices

Let $f = (f_1, \dots, f_m)$ be a sequence of germs of holomorphic functions at $0 \in \mathbb{C}^n$, such that $V(f) = \{0\}$. The *Rees valuations* of $\mathfrak{a}(f)$ are defined in terms of the normalized blowup $\nu : X^+ \to (\mathbb{C}^n, 0)$ of $\mathfrak{a}(f)$, see [13, Ch.II.7]. Since $V(\mathfrak{a}) = \{0\}$, ν is an isomorphism outside $0 \in \mathbb{C}^n$ and $\nu^{-1}(0)$ is the union of finitely many prime divisors $E \subseteq X^+$. The Rees valuations of $\mathfrak{a}(f)$ are then the associated divisorial valuations ord_E on $\mathcal{O}_0^n$: $\operatorname{ord}_E(g)$ is the order of vanishing of $\nu^* g$ along E.

Let $\pi : X \to (\mathbb{C}^n, 0)$ be a log-resolution of $\mathfrak{a}(f)$, see [16, Def. 9.1.12]. Then, in fact, a divisorial valuation ord_E is a Rees valuation of $\mathfrak{a}(f)$ if and only if the image of the prime divisor $E \subseteq \pi^{-1}(0)$ under the rational mapping $\Psi = [f_1 \circ \pi : \ldots : f_m \circ \pi] : X \dashrightarrow \mathbb{CP}^{m-1}$ is of (maximal) dimension $n-1$, see [20, p. 332].

Consider a multi-index $\mathcal{I} = \{i_1, \ldots, i_n\} \subseteq \{1, \ldots, m\}$. Let $\pi_{\mathcal{I}} : \mathbb{CP}^{m-1} \setminus W_{\mathcal{I}} \to \mathbb{CP}^{n-1}$, where $W_{\mathcal{I}} := \{w_{i_1} = \ldots = w_{i_n} = 0\} \subseteq \mathbb{CP}^n$, be the projection $[w_1 : \ldots : w_m] \mapsto [w_{i_1} : \ldots : w_{i_n}]$. Following [15] we say that $\mathcal{I}$ is *essential* with respect to E (and the sequence f) if $\Psi(E) \not\subseteq W_{\mathcal{I}}$ and the mapping $\pi_{\mathcal{I}} \circ \Psi : E \dashrightarrow \mathbb{CP}^{n-1}$ is surjective; in particular, $\mathrm{ord}_E(f_{i_1}) = \ldots = \mathrm{ord}_E(f_{i_n}) = \mathrm{ord}_E(\mathfrak{a})$. Moreover we say that $\mathcal{I}$ is *essential* (with respect to f) if $\mathcal{I}$ is essential with respect to at least one exceptional prime. Furthermore we say that $\mathcal{I}$ is p-*essential* with respect to E (and f) if $\mathcal{I}$ is essential with respect to the divisor E and the sequence f^p, and that $\mathcal{I}$ is p-*essential* (with respect to f) if $\mathcal{I}$ is essential with respect to the sequence f^p.

Observe that if $\mathcal{I}$ is p-essential with respect to E, then ord_E must be a Rees valuation of $\mathfrak{a}(f^p)$. Conversely, if ord_E is a Rees valuation of $\mathfrak{a}(f^p)$, then there exists at least one multi-index $\mathcal{I}$, which is p-essential with respect to E. However, note that $\mathcal{I}$ can be p-essential with respect to more than one divisor E, and conversely there can be several multi-indices that are p-essential with respect to a given E.

Recall that the integral closure of $\mathfrak{a} \subseteq \mathcal{O}_0^n$ can be defined in terms of the Rees valuations of $\mathfrak{a}$. Indeed, $h \in \mathcal{O}_0^n$ is in $\overline{\mathfrak{a}}$ if and only if $\mathrm{ord}_E(h) \geq \mathrm{ord}_E(\mathfrak{a})$ for all Rees valuations ord_E of $\mathfrak{a}$, see for example [16, Ex. 9.6.8].

Given a sequence f and a multi-index $\mathcal{I} = \{i_1, \ldots, i_n\}$, let $f_{\mathcal{I}}$ denote the sequence $(f_{i_1}, \ldots, f_{i_n})$.

3. Proof of Theorem A

The proof of Theorem A is very much inspired by and based on (the proofs of) Theorems A and B in [15] and it also uses Andersson's construction of residue currents in [1]. The following result is Theorem B and Lemma 4.3 in [15].

Lemma 3.1. *$R_{\mathcal{I}}(f) \not\equiv 0$ if and only if $\mathcal{I}$ is essential with respect to f. Moreover $R_{\mathcal{I}}(f) \wedge df_{i_n} \wedge \cdots \wedge df_{i_1}/(2\pi i)^n$ is a positive current and its mass is strictly positive if and only if $\mathcal{I}$ is essential.*

We first prove that $R^p_{\mathcal{I}}(f) \not\equiv 0$ precisely if $\mathcal{I}$ is p-essential. If $\mathcal{I}$ is not p-essential, then $R_{\mathcal{I}}(f^p) = 0$ by Lemma 3.1, and hence in light of (1.2) $R^p_{\mathcal{I}}(f) = 0$. For the converse, note that

$$R^p_{\mathcal{I}}(f) \wedge df_{i_n} \wedge \cdots \wedge df_{i_1} = \frac{1}{p_{i_1} \cdots p_{i_n}} R_{\mathcal{I}}(f^p) \wedge df^{p_{i_n}}_{i_n} \wedge \cdots \wedge df^{p_{i_1}}_{i_1} \tag{3.1}$$

by (1.2). Lemma 3.1 asserts that the right hand side of (3.1) is non-vanishing if $\mathcal{I}$ is essential with respect to f^p. Thus $R^p_{\mathcal{I}}(f) \not\equiv 0$ if $\mathcal{I}$ is p-essential.

The inclusion $\operatorname{ann} R^p(f) \subseteq \mathfrak{a}(f)$ follows from Andersson's construction of global Bochner-Martinelli residue currents based on the Koszul complex in [1]. We provide (a sketch of) a proof for completeness.

We identify the sequence $f = (f_1, \dots, f_m)$ with a holomorphic section of the dual bundle V^* of a trivial rank m vector bundle V over some neighborhood $\mathcal{U}$ of $0 \in \mathbb{C}^n$, endowed with the trivial metric. If $\{e_i\}_{i=1}^m$ is a global holomorphic frame for V and $\{e_i^*\}_{i=1}^m$ is the dual frame, we can write $f = \sum_{i=1}^m f_i e_i^*$. Let s^p be the section $s^p = \sum_{i=1}^m \bar{f}_i |f_i|^{2(p_i-1)} e_i$, and let

$$u^p = \sum_\ell \frac{s^p \wedge (\bar\partial s^p)^{\ell-1}}{|f^p|^{2\ell}}.$$

Then u^p is a section of $\Lambda(V \oplus T^*_{0,1}(\mathcal{U}))$ (where $e_j \wedge d\bar{z}_i = -d\bar{z}_i \wedge e_j$), that is clearly well-defined and smooth outside $V(f) = \{0\}$, and moreover $\bar\partial |f^p|^{2\lambda} \wedge u^p$, has an analytic continuation as a current to $\operatorname{Re}\lambda > -\epsilon$, see [1]. Note that the $e_{i_n} \wedge \dots \wedge e_{i_1}$-coefficient of $R(u^p) := \bar\partial |f|^{2\lambda} \wedge u^p|_{\lambda=0}$ is just the current $R^p_{\mathcal{I}}(f)$, and thus in particular, $\operatorname{ann} R(u^p) = \operatorname{ann} R^p(f)$. Let $\nabla = \delta_f - \bar\partial : \Lambda(V \oplus T^*_{0,1}(\mathcal{U})) \to \Lambda(V \oplus T^*_{0,1}(\mathcal{U}))$; here δ_f denotes interior multiplication by f. Observe that $\nabla u^p = 1$ outside $V(f)$. In [1] it was proved if u is any section of $\Lambda(V \oplus T^*_{0,1}(\mathcal{U}))$ that is smooth and satisfies $\nabla u = 1$ outside $V(f)$, then the corresponding current $R(u) := \bar\partial |f|^{2\lambda} \wedge u|_{\lambda=0}$ satisfies that $\operatorname{ann} R(u) \subseteq \mathfrak{a}(f)$. We conclude that $\operatorname{ann} R^p(f) \subseteq \mathfrak{a}(f)$.

Given a sequence of germs $g_1, \dots, g_n \in \mathcal{O}_0^n$, let $\operatorname{Jac}(g)$ denote the Jacobian determinant $\operatorname{Jac}(g) = \left|\frac{\partial g_i}{\partial z_j}\right|_{1 \le i,j \le n}$. Note that $df_{i_n} \wedge \dots \wedge df_{i_1} = \pm \operatorname{Jac}(f_I) dz_n \wedge \dots \wedge dz_1$. Thus in light of (3.1) and Lemma 3.1, $\operatorname{Jac}(f_{\mathcal{I}}) \in \operatorname{ann} R^p_{\mathcal{I}}(f)$ if and only if $R^p_{\mathcal{I}}(f) \equiv 0$. Given this we can show that $\operatorname{ann} R^p(f) = \mathfrak{a}(f)$ implies that $\mathfrak{a}(f)$ is a complete intersection ideal by following the proof of Theorem A in [15, Section 5].

It remains to prove that the right inclusion in (1.3) is strict when $n \ge 2$. Given a multi-index $\mathcal{I} = \{i_1, \dots, i_n\}$, let $P(\mathcal{I}) = \sum_{j=1}^n \frac{1}{p_{i_j}}$. Pick two multi-indices $\mathcal{I}$ and $\mathcal{J}$, such that $P(\mathcal{I}) \ge P(\mathcal{J})$. We claim that then $R^p_{\mathcal{J}}(f) \wedge df_{i_n} \wedge \dots \wedge df_{i_1}$ either vanishes or is a positive pointmass at the origin.

Let $\pi : X \to (\mathbb{C}^n, 0)$ be a log-resolution of $\mathfrak{a}(f^p)$. Then $R_{\mathcal{J}}(f^p)$ is the push-forward of a current $\tilde{R}$ on X, which has support on the exceptional primes with respect to whom $\mathcal{J}$ is essential. More precisely, $\tilde{R}$ can be decomposed as $\tilde{R} = \sum \tilde{R}_E$, where the sum is over the exceptional primes $E \subseteq X$, such that $\mathcal{J}$ is essential with respect to E, and $\tilde{R}_E$ has support on E, see [15, Section 6].

Let E_1 be an exceptional prime, such that $\mathcal{J}$ is essential with respect to E_1. Then we can choose local coordinates σ on X, so that $E_1 = \{\sigma_1 = 0\}$ and $\tilde{R}_{E_1}$ is of the form $\bar\partial[1/\sigma_1^{na_1}] \wedge [1/(\sigma_2^{na_2}\cdots\sigma_n^{na_n})] \wedge \beta$, where β is a smooth form and $a_j = \operatorname{ord}_{E_j}(f^p)$, where $E_j = \{\sigma_j = 0\}$. Observe that for $1 \le \ell \le m$, $\pi^* f_\ell^{p_\ell}$ is divisible by $\sigma_j^{a_j}$ and so $\pi^* f_\ell$ is divisible by $\sigma_j^{\lceil a_j/p_\ell \rceil}$. It follows that

$$\pi^*(f_{j_1}^{p_{j_1}-1}\cdots f_{j_n}^{p_{j_n}-1})\tilde{R}_{E_1} = \bar\partial[1/\sigma_1^{b_1}] \wedge [1/(\sigma_2^{b_2}\cdots\sigma_n^{b_n})] \wedge \beta,$$

where $b_j \le a_j P(\mathcal{J})$. A computation following [15, p. 2130] yields that

$$\pi^*(df_{i_n} \wedge \cdots \wedge df_{i_1}) = \sigma_1^{c_1-1}(\sigma_2^{c_2}\cdots\sigma_n^{c_n}\gamma + \sigma_1\delta)d\sigma_1 \wedge \cdots \wedge d\sigma_n,$$

where $c_j \ge a_j P(\mathcal{I})$ and γ and δ are holomorphic functions. Since, by assumption, $P(\mathcal{I}) \ge P(\mathcal{J})$, $\pi^*(f_{j_1}^{p_{j_1}-1}\cdots f_{j_n}^{p_{j_n}-1})\tilde{R}_{E_1} \wedge \pi^*(df_{i_n} \wedge \cdots \wedge df_{i_1})$ is of the form $\bar\partial[1/\sigma_1] \wedge d\sigma_1 \wedge \tilde\beta = 2\pi i[E_1] \wedge \tilde\beta$, where $\tilde\beta$ is a smooth form. Hence

$$R^p_{\mathcal{J}}(f) \wedge df_{i_n} \wedge \cdots \wedge df_{i_1} = \sum_E \pi_*\left(\pi^*(f_{j_1}^{p_{j_1}-1}\cdots f_{j_n}^{p_{j_n}-1})\tilde{R}_E \wedge \pi^*(df_{i_n} \wedge \cdots \wedge df_{i_1})\right)$$

is a non-negative point mass at 0 and the claim is proved.

Now pick a p-essential multi-index $\mathcal{I}$, for which $P(\mathcal{I}) = \max_{\mathcal{J}\ p\text{-essential}} P(\mathcal{J})$. Then the non-vanishing entries of $R^p(f) \wedge df_{i_n} \wedge \cdots \wedge df_{i_1}$ are just point-masses at the origin; in particular, $\operatorname{Jac}(f_{\mathcal{I}})\mathfrak{m} \subseteq \operatorname{ann} R^p(f)$, where $\mathfrak{m}$ denotes the maximal ideal in $\mathcal{O}_0^n$. Let E be an exceptional prime, such that $\mathcal{I}$ is p-essential with respect to E. A direct computation gives that $\operatorname{ord}_E(df_{i_1}^{p_{i_1}} \wedge \ldots \wedge df_{i_n}^{p_{i_n}}) = n \operatorname{ord}_E(f^p) - 1$ and $\operatorname{ord}_E(dz_1 \wedge \ldots \wedge dz_n) \ge \sum_{i=1}^n \operatorname{ord}_E(z_i) - 1$. Note that $\operatorname{ord}_E(z_k) \ge 1$ for $1 \le k \le n$. Since $df_{i_1}^{p_{i_1}} \wedge \cdots \wedge df_{i_n}^{p_{i_n}} = p_{i_1}\cdots p_{i_n} f_{i_1}^{p_{i_1}-1}\cdots f_{i_n}^{p_{i_n}-1}\operatorname{Jac}(f_{\mathcal{I}})dz_1 \wedge \cdots \wedge dz_n$ it follows that

$$\operatorname{ord}_E(z_k f_{i_1}^{p_{i_1}-1}\cdots f_{i_n}^{p_{i_n}-1}\operatorname{Jac}(f_{\mathcal{I}})) \le n \operatorname{ord}_E(f^p) - n + 1 = \operatorname{ord}_E(\overline{\mathfrak{a}(f^p)^n}) - n + 1$$

for $1 \le k \le n$. Here we have used that $\overline{\mathfrak{a}}$ is the set of all $h \in \mathcal{O}_0^n$, that satisfy $\operatorname{ord}_E(h) \ge \operatorname{ord}_E(\mathfrak{a})$ for all Rees valuations ord_E of $\mathfrak{a}$, see Section 2. Hence, if $n \ge 2$, there are elements, for example $z_k\operatorname{Jac}(f_{\mathcal{I}})$, in $\operatorname{ann} R^p(f)$ that are not in $\overline{\mathfrak{a}(f^p)^n} \colon (f_{i_1}^{p_{i_1}-1}\cdots f_{i_n}^{p_{i_n}-1})$. This proves that the first inclusion in (1.3) is strict for $n \ge 2$ and concludes the proof of Theorem A.

4. The monomial case

Let $z^A = (z^{a^1}, \ldots, z^{a^m})$ be a sequence of germs of monomials in $\mathcal{O}_0^n$. Recall that the *Newton polyhedron* $\operatorname{NP}(A)$ is defined as the convex hull in $\mathbb{R}^n$ of the set of exponents of monomials in $\mathfrak{a}(z^A)$. The Rees

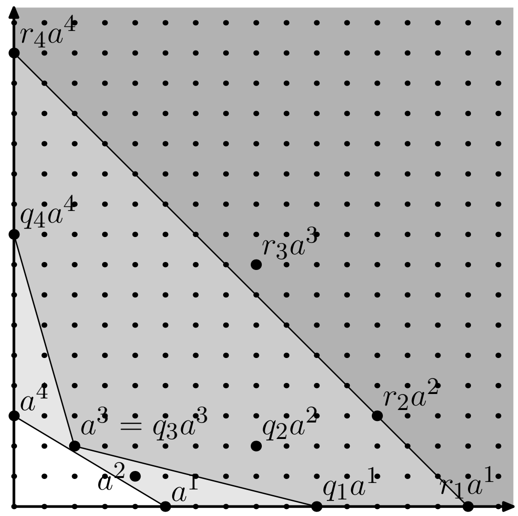

FIGURE 1. The Newton polytopes of the sequences z^A (light grey), z^{qA} (medium grey), and z^{rA} (dark grey) in Example 4.1.

valuations of $\mathfrak{a}(z^A)$ are monomial and in one-to-one correspondence with the compact facets (faces of dimension $n-1$) of $\mathrm{NP}(A)$. More precisely, the facet τ with normal vector $\rho = (\rho_1, \dots, \rho_n)$ corresponds to the monomial valuation $\mathrm{ord}_\tau(z_1^{a_1} \cdots z_n^{a_n}) = \rho_1 a_1 + \cdots + \rho_n a_n$, see for example [14, Thm. 10.3.5]. Given a multi-index $\mathcal{I} = \{i_1, \dots, i_n\}$, let $A_\mathcal{I}$ denote the set $\{a^{i_1}, \dots, a^{i_n}\} \subseteq A$ so that $z^{A_\mathcal{I}}$ is the sequence $z^{a^{i_1}}, \dots, z^{a^{i_n}}$. Moreover, let $\det(A_\mathcal{I})$ denote the determinant of the matrix with rows $a^{i_1}, \dots, a^{i_n}$. It follows that a multi-index $\mathcal{I}$ is essential with respect to E_τ precisely if $A_\mathcal{I}$ is contained in τ and $\det(A_\mathcal{I}) \neq 0$; here E_τ denotes the exceptional prime associated with τ. This means that $\mathcal{I}$ is p-essential if and only if $pA_\mathcal{I} := \{p_{i_1} a^{i_1}, \dots, p_{i_n} a^{i_n}\}$ is contained in a facet of $\mathrm{NP}(pA)$ and $\det(A_\mathcal{I}) \neq 0$.

Observe that if $V(z^A) = \{0\}$, then z^A is regular precisely if $m = n$ and z^{a^j} is of the form $z_j^{b_j}$ (possibly after reordering the variables). Moreover, recall that the integral closure of $\mathfrak{a}(z^A)$ is the monomial ideal generated by monomials with exponents in $\mathrm{NP}(A)$, see for example [20].

Let us illustrate Theorem C with some examples.

Example 4.1. Let z^A be the sequence of monomials $z^A = (z^{a^1}, \dots, z^{a^4}) = (z_1^5, z_1^4 z_2, z_1^2 z_2^2, z_2^3)$. Then $\mathrm{NP}(A)$ has just one compact facet and so $\mathfrak{a}(z^A)$ has exactly one Rees valuation, which is the monomial valuation ord_E given by $\mathrm{ord}_E(z_1^{b_1} z_2^{b_2}) = 3b_1 + 5b_2$. Moreover the only essential multi-index with respect to z^A is $\{1,4\}$ and so Theorem C asserts that $R(z^A) = R^p(z^A)$, where $p = (1,1,1,1)$, has one non-vanishing entry $R_{\{1,4\}}(z^A) = C_{\{1,4\}} \bar\partial \left[\frac{1}{z_1^5}\right] \wedge \bar\partial \left[\frac{1}{z_2^3}\right]$ and $\mathrm{ann}\, R(z^A) = (z_1^5, z_2^3)$.

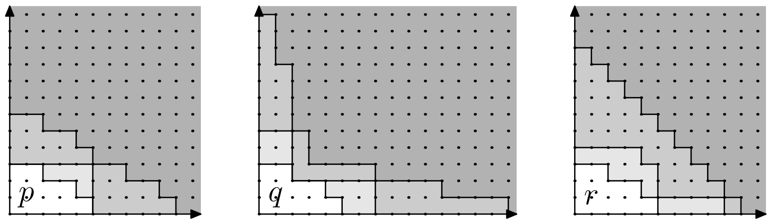

FIGURE 2. The exponent sets of the ideals $\overline{\mathfrak{a}(z^A)}$ (light grey), $\operatorname{ann} R^p(z^A)$ (medium grey) and $\bigcap \overline{(z^A)^2} : (z^{(p_{i_1}-1)a^{i_1}} z^{(p_{i_2}-1)a^{i_2}})$ (dark grey) for weights p, q, and r in Example 4.1.

Let $q = (2,2,1,3)$. Then $\operatorname{NP}(qA)$ has two compact facets, so that $\overline{\mathfrak{a}(z^{qA})} = (z_1^{10}, z_1^8 z_2^2, z_1^2 z_2^2, z_2^9)$ has two Rees valuations: $\operatorname{ord}_{E_1}(z_1^{b_1} z_2^{b_2}) = b_1 + 4b_2$ and $\operatorname{ord}_{E_2}(z_1^{b_1} z_2^{b_2}) = 7b_1 + 2b_2$. Moreover there are two q-essential multi-indices, $\{1,3\}$ and $\{3,4\}$, corresponding to E_1 and E_2, respectively. It follows from Theorem C that $\operatorname{ann} R^q(z^A) = (z_1^7, z_2^2) \cap (z_1^2, z_2^5) = (z_1^7, z_1^2 z_2^2, z_2^5)$. Note that $\operatorname{ann} R^p \not\subseteq \operatorname{ann} R^q$ and $\operatorname{ann} R^q \not\subseteq \operatorname{ann} R^p$, which illustrates that in general no relation between the weights p and q is reflected in the relation between $\operatorname{ann} R^p(z^A)$ and $\operatorname{ann} R^q(z^A)$. One can check that by varying the weight p one gets all together 9 different annihilator ideals. Let us consider one more example. Let $r = (3,3,4,5)$. Then $\operatorname{NP}(rA)$ has one compact facet, so that $\mathfrak{a}(z^{rA})$ has one Rees valuation. However, there are three r-essential multi-indices, $\{1,2\}$, $\{1,4\}$, and $\{2,4\}$, and $\operatorname{ann} R^r(z^A) = (z_1^9, z_2) \cap (z_1^5, z_2^3) \cap (z_1^4, z_2^4)$. In Figure 1 we have drawn $\operatorname{NP}(pA)$ and also marked the elements in pA, for the weights p, q, and r.

Note that $z^{(q_1-1)a^1} z^{(q_3-1)a^3} = z_1^5$ and $z^{(q_3-1)a^3} z^{(q_4-1)a^4} = z_2^6$. It follows that for the weight q the leftmost ideal in (1.3) is given by $(z_1^{15}, z_1^{11} z_2, z_1^7 z_2^2, z_1^3 z_2^3, z_1^2 z_2^5, z_1 z_2^9, z_2^{12})$ and so ones sees directly that the left inclusion in (1.3) is strict in this case. In Figure 2 the three ideals in (1.3) are depicted for weights p, q, and r. Note that $\operatorname{ann} R^p(z^A)$ is strictly included in $\overline{\mathfrak{a}(z^A)}$ is all three cases. Also note that $\overline{\mathfrak{a}(z^A)^2} \not\subseteq \operatorname{ann} R^r(z^A)$, which shows that it is not true in general that $\overline{\mathfrak{a}(f)^n} \subseteq \operatorname{ann} R^p(f)$.

□

Example 4.2. Let $z^A = (z, z^2)$. Then $\mathfrak{a}(z^A)$ is just the maximal ideal $\mathfrak{m} \subseteq \mathcal{O}_0^1$. Note that since $n = 1$ there is a unique Rees valuation associated with $\mathfrak{a}(z^A)$, namely the order of vanishing at the origin. For $j \in \mathbb{N}$, let $p^j = (j, 1)$. Then $R(z^A) = R^{p^1}(z^A) = \big(\bar\partial\,[1/z]\,, 0\big)$, $R^{p^2}(z^A) = \big(\bar\partial\,[1/z]\,, \bar\partial\,[1/z^2]\big)$, and $R^{p^j}(z^A) = \big(0, \bar\partial\,[1/z^2]\big)$ for $j \geq 3$. It follows that $\operatorname{ann} R = \mathfrak{m}$, whereas $\operatorname{ann} R^{p^j} = \mathfrak{m}^2$ for $j \geq 2$. □

Example 4.2 shows that in general $R^p(f)$, as well as $\operatorname{ann} R^p(f)$, depends in an essential way on the particular sequence f and not only on the ideal $\mathfrak{a}(f)$. Theorem A in [15] asserts that $\operatorname{ann} R(f) = \mathfrak{a}(f)$ if and only if $\mathfrak{a}(f)$ is a complete intersection ideal. Theorem A says that the only if-direction of this statement holds for any p, whereas Example 4.2 shows that the if-direction fails in general. Moreover, in the monomial case $R(f)$ only depends on $\mathfrak{a}(f)$ and not on the particular sequence f. Question D in [15] asks whether it is always true (as long as $V(f) = \{0\}$) that $\operatorname{ann} R(f)$ only depends on $\mathfrak{a}(f)$.

4.1. **Proof of Theorem C.** Theorem 3.1 in [23] states that if $\mathcal{I}$ is essential with respect to z^A, then $R_{\mathcal{I}}(f)$ is of the form (1.6), where $C_{\mathcal{I}}$ is a nonzero constant. Thus, using (1.2) and (1.4), we conclude that the entries of $R^p(f)$ are of the form (1.6).

Assume that $\mathcal{I}$ is p-essential. Then by Lemma 3.1, (3.1) times $1/(2\pi i)^n$ has strictly positive mass. Note that $dz^{a^{i_n}} \wedge \cdots \wedge dz^{a^{i_1}} = \det(A_{\mathcal{I}}) z_1^{\alpha_1^{\mathcal{I}}-1} \cdots z_n^{\alpha_n^{\mathcal{I}}-1} dz_n \wedge \cdots \wedge dz_1$. Since $\bar\partial \left[\frac{1}{z}\right] \wedge dz = 2\pi i[0]$, it follows that the left hand side of (3.1) is equal to $(2\pi i)^n C_{\mathcal{I}} |\det(A_{\mathcal{I}})| [0]$, and so $C_{\mathcal{I}} \geq 0$.

4.2. **The coefficients $C_{\mathcal{I}}$.** Given a sequence of monomials z^A one can find a log-resolution $X_A \to (\mathbb{C}^n, 0)$ of $\mathfrak{a}(z^A)$, where X_A is a toric variety constructed from (the normal fan of) $\mathrm{NP}(A)$, see [7, p. 82]. In [23] we computed $R(z^A)$ as the push-forward of a certain current on X_A. Assume that $\mathcal{I}$ is essential with respect to E_τ, where τ is a facet of $\mathrm{NP}(A)$. According to [23, p. 381], the coefficient $C_{\mathcal{I}}$ is of the form $C_{\mathcal{I}} = \pm \frac{1}{(2\pi i)^{n-1}} (n-1)! D I$, where I is an integral of the form

$$I = \int \frac{\prod_{j=1}^{n-1} |t_j|^{2(c_{j1}+\ldots+c_{jn}-1)}}{\sum_{k=1}^{\ell} \prod_{j=1}^{n-1} |t_j|^{2c_{jk}}}\, d\bar t_1 \wedge \cdots \wedge d\bar t_{n-1} \wedge dt_{n-1} \wedge \cdots \wedge dt_1,$$

for some $n \leq \ell \leq m$ and $\{c_{jk}\}_{1\leq j\leq n-1, 1\leq k\leq \ell}$, and D is the determinant of the matrix with entries $\{d_{jk}\}_{1\leq j,k\leq n}$, where $d_{jk} = c_{jk}$ if $j \leq n-1$ and $d_{nk} = 1$. The terms in the denominator correspond to the $a^j \in A$ that lie in τ; in particular, $C_{\mathcal{I}}$ depends only on $\tau \cap A$. (Assuming that $\mathcal{I} = \{1, \ldots, n\}$ and that $\{a^1, \ldots, a^\ell\}$ are the exponents in τ, then, in the terminology of [23], $c_{jk} = \rho_j \cdot (b_k - a_0)$.) In general the integral I is hard to compute; compare to (5.1).

Assume that $\ell = n$ and that $c_{jk} = 0$ unless $j = k$, possibly after rearranging the variables t_j. Then

$$I = \int \frac{\prod_{j=1}^{n-1} |t_j|^{2(c_j-1)}}{(1+\sum_{j=1}^{n-1} |t_j|^{2c_j})^n}\, d\bar t_1 \wedge \cdots \wedge d\bar t_{n-1} \wedge dt_{n-1} \wedge \cdots \wedge dt_1,$$

where c_j just denotes c_{jj}. A direct computation gives that

$$\int \frac{|s|^{2(N-1)}}{(1+|s|^{2N})^p} d\bar{s} \wedge ds = 2\pi i \frac{1}{p-1} \frac{1}{N},$$

which implies that $I = \frac{(2\pi i)^{n-1}}{(n-1)!} \frac{1}{c_1 \cdots c_{n-1}}$. Moreover $D = c_1 \cdots c_{n-1}$, and since $C_{\mathcal{I}} \geq 0$, we conclude that $C_{\mathcal{I}} = 1$.

The assumption that $\ell = n$ is satisfied precisely if $\mathcal{I}$ is the unique multi-index that is essential with respect to a certain Rees valuation. The assumption that $c_{jk} = 0$ for $j \neq k$ is for example satisfied if the normal fan of $\mathrm{NP}(A)$ is regular, see [11]. It is also satisfied if $n = 2$.

Given a facet τ of $\mathrm{NP}(A)$, let $\det(\tau)$ be the normalized volume, that is, $n!$ times the Euclidean volume, of the convex hull of τ and the origin in $\mathbb{R}^n$. If τ is simplicial with vertices $b^1, \ldots, b^n$, then $\det(\tau)$ is just (the absolute value of) the determinant of the matrix with rows $b^1, \ldots, b^n$. For $n = 2$ we have the following description of the coefficients $C_{\mathcal{I}}$:

$$\sum_{A_{\mathcal{I}} \subseteq \tau} |\det(A_{\mathcal{I}})| C_{\mathcal{I}} = \det(\tau). \tag{4.1}$$

To prove this, recall that if $V(z^A) = \{0\}$, then the Hilbert-Samuel multiplicity $e(z^A)$ of $\mathfrak{a}(z^A)$ equals the normalized volume $\mathrm{Vol}(\mathbb{R}^n_+ \setminus \mathrm{NP}(A))$ of the complement in $\mathbb{R}^n_+$ of $\mathrm{NP}(A)$, see for example [21]. Observe that $\mathrm{Vol}(\mathbb{R}^n_+ \setminus \mathrm{NP}(A)) = \sum \det(\tau)$, where the sum runs over the facets τ of $\mathrm{NP}(A)$. Now (4.1) follows in light of (1.5) and the fact that if $\mathcal{I}$ is essential with respect to E_τ, then $C_{\mathcal{I}}$ depends only on $a^j \in A \cap \tau$.

Question 4.3. *Does* (4.1) *hold also when* $n > 2$*?*

Example 4.4. Let z^A and p, q, and r be as in Example 4.1, and let $s = (2, 1, 1, 2)$. From [2] we know that $e^p(z^A)$ is the Hilbert-Samuel multiplicity of $\mathfrak{a}(z^A)$. Since there is only one essential multi-index with respect to z^A we can also compute this directly from (4.1). Indeed $C_{\{1,4\}} = 1$ and so $e^p(z^A) = |\det(A_{\{1,4\}})| = 15$.

Moreover, recall that $\mathfrak{a}(z^{qA})$ has two Rees valuations and that there is one q-essential multi-index associated with each divisor: $\{1, 3\}$ and $\{3, 4\}$. Hence $C_{\{1,3\}} = C_{\{3,4\}} = 1$ and so $e^q(z^A) = |\det(A_{\{1,3\}})| + |\det(A_{\{3,4\}})| = 10 + 6 = 16$, that is, the normalized area of the convex hull of $a^1 = (5, 0)$, $a^3 = (2, 2)$, and $a^4 = (0, 3)$. Similarly $\mathfrak{a}(z^{sA})$ has three Rees valuations and there is one s-essential multi-index for each valuation; it follows that $e^s(z^A) = 17$, see Figure 3.

Finally $\mathfrak{a}(z^{rA})$ has one Rees valuation, but there are three r-essential multi-indices. From (4.1) we know that $C_{\{1,2\}} |\det(A_{\{1,2\}})| + C_{\{1,4\}} |\det(A_{\{1,4\}})| + C_{\{2,4\}} |\det(A_{\{2,4\}})| = |\det(A_{\{1,4\}})|$, which means $5C_{\{1,2\}} + 15C_{\{1,4\}} + 12C_{\{2,4\}} = 15$. However, we cannot say more; in particular, we cannot determine $e^r(z^A)$.

□

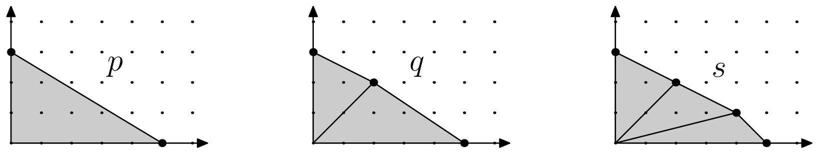

FIGURE 3. The multiplicities $e^p(z^A)$, $e^q(z^A)$, and $e^s(z^A)$ in Example 4.4.

5. Discussion of Question B

Theorem C allows us to give an affirmative answer to Question B in the monomial case. Recall that if $\mathfrak{a}(f)$ is a complete intersection ideal, then $\mathfrak{a}(f)$ is, in fact, generated by n of the f_j. This follows for example from Nakayama's Lemma.

Proposition 5.1. *Suppose that $z^A = (z^{a^j})_{j=1}^m$ is a sequence of germs of holomorphic monomials at $0 \in \mathbb{C}^n$, such that $V(z^A) = \{0\}$. Then $R^p(z^A)$ is independent of p if and only if z^A is a regular sequence.*

Moreover, ann $R^p(z^A)$ is independent of p if and only if for each $\mathcal{I} = \{i_1, \ldots, i_n\} \subseteq \{1, \ldots, m\}$, either $z^{A_\mathcal{I}}$ generates $\mathfrak{a}(z^A)$ or $\det(A_\mathcal{I}) = 0$.

Note that the condition that either $z^{A_\mathcal{I}}$ generates $\mathfrak{a}(z^A)$ or $\det(A_\mathcal{I}) = 0$ is equivalent to that $\mathfrak{a}(z^A)$ is a complete intersection ideal, generated by say $z_1^{b_1}, \ldots, z_n^{b_n}$, and that moreover, for $1 \leq j \leq m$, z^{a^j} is equal to $z_k^{b_k}$ for some $1 \leq k \leq n$.

Proof. First, note that the if-directions of the statements in Proposition 5.1 follow immediately from Theorem C. Thus we need to prove the only if-directions.

Let $\mathcal{I}$ be a multi-index defined by that $z^{a_{i_j}}$ is of the form $z_j^{b_j}$, where b_j is the smallest number such that $z_j^{b_j}$ is among the entries of z^A. Without loss of generality we may assume that $\mathcal{I} = \{1, \ldots, n\}$. Choose $p \in \mathbb{N}^m$, so that $p_i = 1$ if $i \leq n$ and $p_i >> 1$ otherwise. Then $\mathcal{I}$ is the unique p-essential multi-index.

Assume that $m > n$ and choose j, such that $n < j \leq m$. Moreover, choose $q \in \mathbb{N}^m$ such that a^j lies in the one of the compact facets of $\mathrm{NP}(qA)$. For example, let q be defined by $q_i = |a^1| + \ldots + |a^{i-1}| + |a^{i+1}| + \ldots + |a^m|$, where $|a^\ell| = a_1^\ell + \ldots + a_n^\ell$. Then j is contained in a q-essential multi-index, say $\mathcal{J}$. It follows that $R^q_\mathcal{J}(z^A) \neq 0$, whereas $R^p_\mathcal{J}(z^A) = 0$. Hence $R^p(z^A) \neq R^q(z^A)$ and we have proved the first part of Proposition 5.1.

Next, assume that there is an $a^j \in A$ such that z^A is not equal to any of $z_1^{b_1}, \ldots, z_n^{b_n}$. Since $V(z^A) = \{0\}$, at least one of the entries of a^j is positive, say $a_k^j > 0$. Let $\mathcal{J} = \{1, \ldots, k-1, k+1, \ldots, n, j\}$. Then $\det(A_\mathcal{J}) \neq 0$, which means that we can find a weight q such that $\mathcal{J}$ is q-essential; for instance we can take q as above. By assumption,

$a^j_k > b_k$ or $a^j_i > 0$ for some $i \neq k$. In both cases, for some ℓ, the ℓth entry of $\sum_{j\in\mathcal{J}} a^j$ is strictly larger than the ℓth entry of $\sum_{j\in\mathcal{I}} a^j$ and thus $\operatorname{ann} R^q_{\mathcal{J}}(z^A) \not\supseteq \operatorname{ann} R^p_{\mathcal{I}}(z^A)$. This proves the second part of Proposition 5.1. □

Observe that a necessary condition for Question B to be true would be that the set of p-essential multi-indices is independent of p if and only if f is a regular sequence. As we saw in the above proof this is true if f is monomial, but we do not know if it holds in general. When f is monomial, the essential multi-indices are rather special. For example, a multi-index can be essential with respect to at most one Rees valuation, which is not the case in general. Indeed, if $m = n$, then $\mathcal{I} = \{1, \dots, n\}$ is essential with respect to all Rees valuations (and there can be more than one Rees valuation). The following example illustrates another phenomenon, which does not occur in the monomial case.

Example 5.2. Let $f = (z_1^4 - z_2^4, z_1^2 z_2, z_1 z_2^2)$. Then $\mathfrak{a}(f)$ has three Rees valuations, namely the monomial valuations $\operatorname{ord}_{E_1}(z_1^{b_1} z_2^{b_2}) = b_1 + b_2$, $\operatorname{ord}_{E_2}(z_1^{b_1} z_2^{b_2}) = 2b_1 + b_2$, $\operatorname{ord}_{E_3}(z_1^{b_1} z_2^{b_2}) = b_1 + 2b_2$, and $\{2,3\}$, $\{1,3\}$ and $\{1,2\}$ are the unique essential multi-indices with respect to ord_{E_1}, ord_{E_2}, and ord_{E_3}, respectively. Note that this situation cannot happen if f_j are all monomials.

Let $q = (1, 2, 2)$. Then $\mathfrak{a}(f^q) = (z_1^4 - z_2^4, z_1^4 z_2^2, z_1^2 z_2^4)$ has four Rees valuations, $\operatorname{ord}_{E_1}, \dots, \operatorname{ord}_{E_4}$. To see this, note that after blowing up the origin once, the strict transform of $\mathfrak{a}(f^q)$ has support at four points $x_1, \dots, x_4$. The divisor E_j is obtained by further blowing up x_j twice. A computation yields that $\{1,2\}$ and $\{1,3\}$ are both q-essential with respect to E_j for $1 \leq j \leq 4$, whereas $\{2,3\}$ is not q-essential. Hence $R(f) \neq R^q(f)$. □

Note that $\det(A_{\mathcal{I}}) = 0$ is equivalent to that $dz^{a^{i_1}} \wedge \cdots \wedge dz^{a^{i_n}}$ vanishes identically, which in turn implies that $\mathcal{I}$ is not p-essential for any $p \in \mathbb{N}^m$. This motivates the following version of Question B.

Question B'. *Is it true that* $\operatorname{ann} R^p(f)$ *is independent of* p *if and only if for any* $\mathcal{I} = \{i_1, \dots, i_n\}$, *either* $f_{\mathcal{I}}$ *generates* $\mathfrak{a}(f)$ *or the form* $df_{i_1} \wedge \cdots \wedge df_{i_n}$ *vanishes identically.*

Let us mention some partial answers to Question B'. Theorem C in [15] asserts that if $\mathfrak{a}(f)$ is a complete intersection ideal, then $R_{\mathcal{I}}(f)$ is a constant times the Coleff-Herrera product $R_{CH}(f_{\mathcal{I}})$ if $f_{\mathcal{I}}$ generates $\mathfrak{a}(f)$ and 0 otherwise. Using this and (1.4) one can check that $\operatorname{ann} R^p(f)$ is independent of p if $\mathfrak{a}(f)$ is a complete intersection ideal, generated by say $f_1, \dots, f_n$, and moreover for $j > n$, f_j is equal to (a constant times) one of the f_k for $1 \leq k \leq n$; compare this to (the discussion right after) Proposition 5.1.

Example 5.3. Let $f=(z_1,z_2,z_1+z_2)$. Then $\mathfrak{a}(f)$ is just the maximal ideal in $\mathcal{O}_0^2$, which is clearly a complete intersection ideal, and thus by Theorem C in [15], $\operatorname{ann} R(f)=\mathfrak{a}(f)$. Note that any choice of f_i and f_j generate $\mathfrak{a}(f)$, so f satisfies the condition in Question B'.

Let $p=(3,3,3)$. Observe that $\mathfrak{a}(f^p)=(z_1^3,z_2^3,z_1^2z_2+z_1z_2^2)$ is not a complete intersection ideal. A computation yields that

$$R_{\{1,3\}}(f^p)=A_1\bar\partial[1/z_1^5]\wedge\bar\partial[1/z_2]+A_2\bar\partial[1/z_1^4]\wedge\bar\partial[1/z_2^2]+A_3\bar\partial[1/z_1^3]\wedge\bar\partial[1/z_2^3],$$

for some constants A_1, A_2, and A_3. It follows that $R^p_{\{1,3\}}(f)=(A_1+2A_2+A_3)\bar\partial[1/z_1]\wedge\bar\partial[1/z_2]$. In fact, also the other entries of R^p are of this form and so $\operatorname{ann} R^p=\mathfrak{a}(f)$. □

Note that if there is a subsequence $f_{\mathcal{I}}=(f_{i_1},\dots,f_{i_n})$ of f such that $V(f_{\mathcal{I}})=\{0\}$, then by choosing $p_j=1$ if $j\in\mathcal{I}$ and $p_j>>1$ for $j\notin\mathcal{I}$, the only non-vanishing entry of $R^p(f)$ is $R^p_{\mathcal{I}}(f)$, which is a constant times $R_{CH}(f_{\mathcal{I}})$. Thus, given that there exists such an $f_{\mathcal{I}}$, $\operatorname{ann} R^p(f)$ is not independent of p as soon as, for example, there is another multi-index $\mathcal{J}$, such that $V(f_{\mathcal{J}})=\{0\}$, or as soon as $\operatorname{ann} R(f)$ is not a complete intersection ideal. One can, however, not always find such an $f_{\mathcal{I}}$, as the following example shows.

Example 5.4. Let $f=(f_1,f_2,f_3)=(z_1z_2,z_1(z_1+z_2),z_2(z_1+z_2))$. Then $V(f_{\mathcal{I}})$ is a line through the origin for all $\mathcal{I}=\{i_1,i_2\}$; in particular, $V(f_{\mathcal{I}})\neq\{0\}$. Moreover, $\mathfrak{a}(f)$ is the (monomial) ideal $\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}_0^2$. Thus the only Rees valuation of $\mathfrak{a}(f)$ is the order of vanishing at the origin and so $R(f)$ can be computed by blowing up the origin once. Note that all multi-indices $\mathcal{I}=\{i_1,i_2\}$ are essential. Let $R^{\ell,k}$ denote the current $\bar\partial[1/z_1^\ell]\wedge\bar\partial[1/z_2^k]$, and let

$$C_j=\frac{1}{2\pi i}\int\frac{|t|^{2j}d\bar t\wedge dt}{(|t|^2+|1+t|^2+|t(1+t)|^2)^2}. \tag{5.1}$$

Then, a computation yields that $R_{\{1,2\}}(f)=-C_0R^{3,1}$, $R_{\{1,3\}}(f)=2C_2R^{1,3}$, and $R_{\{2,3\}}(f)=C_0R^{3,1}+2C_1R^{2,2}+C_2R^{1,3}$. It follows that $\operatorname{ann} R(f)=\mathfrak{m}^3$.

Let $p=(2,1,1)$. Then $\mathfrak{a}(f^p)$ has two Rees valuations, ord_{E_1} and ord_{E_2}, where E_1 is the exceptional divisor obtained by blowing up the origin once, whereas E_2 is obtained by further blowing up a point on E_1 twice. Moreover, $\{2,3\}$ is essential with respect to E_1 and $\{1,2\}$ and $\{1,3\}$ are essential with respect to E_2. A computation gives that $R^p_{\{1,2\}}(f)=R^p_{\{1,3\}}(f)=-1/2(R^{3,1}-R^{2,2}+R^{1,3})$ and $R^p_{\{2,3\}}(f)=A^{3,1}R^{3,1}+A^{2,2}R^{2,2}+A^{1,3}R^{1,3}$, where $A^{i,j}>0$.

Note that $R^p_{\mathcal{I}}(f)\neq R_{\mathcal{I}}(f)$, as well as $\operatorname{ann} R^p_{\mathcal{I}}(f)\neq\operatorname{ann} R_{\mathcal{I}}(f)$, for, at least, $\mathcal{I}=\{1,2\},\{1,3\}$. Moreover, note that $\operatorname{ann} R(f)$ is strictly included in $\operatorname{ann} R^p(f)$. Indeed, $(A^{2,2}+A^{1,3})z_1^2+(A^{1,3}-A^{3,1})z_1z_2-(A^{3,1}+A^{2,2})z_2^2\in\operatorname{ann} R^p(f)\setminus\operatorname{ann} R(f)$. □

5.1. **Related questions.** Question B could be posed also for the currents (1.5). The following example shows that $e^p(f)$ does not necessarily vary with p even if $R^p(f)$ and ann $R^p(f)$ do.

Example 5.5. Let $z^A = (z_1^2, z_1z_2, z_2^2)$. Then by varying p there are three different possibilities of p-essential multi-indices. First, all three multi-indices $\mathcal{I} = \{i_1, i_2\}$ could be p-essential, which for example is the case for $p = (1,1,1)$. Next, for $p = (1,2,1)$, $\{1,3\}$ is the only p-essential multi-index, and for $p = (2,1,1)$, the p-essential multi-indices are $\{1,2\}$ and $\{2,3\}$. In the first situation, by [2], $e^p(z^A)$ is the Hilbert-Samuel multiplicity of $\mathfrak{a}(z^A)$, which is equal to $\mathrm{Vol}(\mathbb{R}^n_+ \setminus \mathrm{NP}(A)) = |\det(A_{\{1,3\}})| = 4$. In light of (4.1) it is not hard to check that this is holds true also if p is another weight such that all $\mathcal{I}$ are p-essential. In the latter two cases, by Section 4.2, the coefficients $C_\mathcal{I}$ are all 1, when $\mathcal{I}$ is p-essential. It follows that $e^p = |\det(A_{\{1,3\}})| = 4$ and $e^p = |\det(A_{\{1,3\}})| + |\det(A_{\{2,3\}})| = 2 + 2$, respectively, so in fact $e^p(z_A)$ is independent of p. □

One can also ask in what sense $R^p_\mathcal{I}(f)$ and ann $R^p_\mathcal{I}(f)$ depend on p, once $\mathcal{I}$ is p-essential. In the monomial case ann $R^p_\mathcal{I}(f)$ is fix as long as $\mathcal{I}$ is essential but the coefficient $C_\mathcal{I}$ in (1.6) vary in general. Indeed, in Example 5.5 above, in the first case, for $p = (1,1,1)$, $C_\mathcal{I}$ are all strictly between 0 and 1, whereas in the latter cases they are either 0 or 1. In general, also ann $R^p_\mathcal{I}(f)$ varies with p, see Example 5.4 above. Computations, such as in Example 5.4, suggest that in general there may be infinitely many different annihilator ideals ann $R^p_\mathcal{I}(f)$ and ann $R^p(f)$ as p varies over $\mathbb{N}^m$. This contrasts the monomial case, where there are always finitely many different ideals ann $R^p(f)$.

References

[1] M. Andersson: *Residue currents and ideals of holomorphic functions.*, Bull. Sci. Math. **128** (2004), no. 6 481–512.

[2] M. Andersson: *Residues of holomorphic sections and Lelong currents*, Ark. Mat. **43** (2005), no. 2, 201–219.

[3] M. Andersson & E. Götmark: *Explicit representation of membership in polynomial ideals*, Math. Ann. **349** (2011), no. 2, 345–365.

[4] M. Andersson & H. Samuelsson & J. Sznajdman: *On the Briançon-Skoda theorem on a singular variety*, Ann. Inst. Fourier **60** (2010), no. 2, 417–432.

[5] C. Berenstein & A. Vidras & A. Yger: *Analytic residues along algebraic cycles*, J. Complexity **21** (2005), no. 1, 5–42.

[6] C. Berenstein & A. Yger: *Analytic residue theory in the non-complete intersection case*, J. Reine Angew. Math. **527** (2000), 203–235.

[7] C. A. Berenstein & R. Gay & A. Vidras & A. Yger: *Residue currents and Bezout identities*, Progress in Mathematics **114** Birkhäuser Yerlag (1993).

[8] J. Briançon, H. Skoda : *Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de* $\mathbb{C}^n$, C. R. Acad. Sci. Paris Sér. A **278** (1974) 949–951.

[9] N. Coleff & M. Herrera: *Les courants résiduels associcés à une forme mŕomorphe*, Lecture Notes in Mathematics **633** Springer Verlag, Berlin, 1978.

[10] A. Dickenstein & C. Sessa: *Canonical representatives in moderate cohomology*, Invent. Math. **80** (1985), 417–434.

[11] W. Fulton: *Introduction to toric varieties*, Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[12] P. Griffiths & J. Harris: *Principles of algebraic geometry*, Pure and Applied Mathematics. Wiley-Interscience New York, 1978.

[13] R. Hartshorne: *Algebraic Geometry*, Graduate Texts in Mathematics, 52. Springer, New York, 1977

[14] C. Huneke & I. Swanson: *Integral closure of ideals, rings, and modules, London Mathematical Society Lecture Note Series, 336*, Cambridge University Press, Cambridge, 2006.

[15] M. Jonsson & E. Wulcan: *On Bochner-Martinelli residue currents and their annihilator ideals*, Ann. Inst. Fourier **59** (2009), no. 6, 2119–2142.

[16] R. Lazarsfeld: *Positivity in algebraic geometry. I & II, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Volumes 48 & 49*, Springer-Verlag, Berlin, **2004**.

[17] J. Lipman: *Residues and traces of differential forms via Hochschild homology*, Contemporary Mathematics, 61. American Mathematical Society, Providence, RI, 1987.

[18] M. Passare: *Residues, currents, and their relation to ideals of holomorphic functions*, Math. Scand. **62** (1988), no. 1, 75–152.

[19] M. Passare & A. Tsikh & A. Yger: *Residue currents of the Bochner-Martinelli type*, Publ. Mat. **44** (2000), 85–117.

[20] B. Teissier: *Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney* Algebraic geometry (La Rábida, 1981), Lecture Notes in Mathematics **961** Springer Verlag, Berlin, 1982, pp 314–491.

[21] B. Teissier: *Monômes, volumes et multiplicités*, Introduction à la thórie des singularités, II, 127–141, Travaux en Cours, 37, Hermann, Paris, 1988.

[22] A. Vidras & A. Yger: *On some generalizations of Jacobi's residue formula*, Ann. Sci. École Norm. Sup. **34** (2001), no. 1, 131–157.

[23] E. Wulcan: *Residue currents of monomial ideals*, Indiana Univ. Math. J. **56** (2007), no. 1, 365–388.

[24] A. Yger: *Aspects opérationnels de la théorie des résidus hors du cadre intersection complète*, Géométrie complexe. II. Aspects contemporains dans les mathématiques et la physique, 182–204, Hermann Éd. Sci. Arts, Paris, 2004.

Department of Mathematics, Chalmers University of Technology and the University of Gothenburg, SE-412 96 Göteborg, SWEDEN

E-mail address: wulcan@chalmers.se