A Brownian motion approach to log-concave measures and an application to the option on the minimum of several assets (Christer Borell)

Abstract. In this talk we will discuss a certain Brownian motion approach to log-concave measures and, moreover, we will give an application of log-concavity to the option on the minimum of several assets.

A positive measure μ in $\mathbf{R}^n$ is said to be log-concave if

$$\mu(\lambda A + (1-\lambda)B) \geq \mu^{\lambda}(A)\mu^{1-\lambda}(B)$$

for all $0 < \lambda < 1$ and all Borel sets A and B in $\mathbf{R}^n$. Motivated by some problems in stochastic programming Andras Prékopa proved in the early seventies that a measure μ in $\mathbf{R}^n$ is log-concave, if $\mu = fm_n$, where m_n is Lebesgue measure in $\mathbf{R}^n$ and is f log-concave. Recall that a non-negative function f in $\mathbf{R}^n$ is log-concave if $f(\lambda x + (1-\lambda)y) \geq f^{\lambda}(x)f^{1-\lambda}(y)$ for all $0 < \lambda < 1$ and all vectors x and y in $\mathbf{R}^n$. Clearly, the topological support of a log-concave measure μ is convex. Moreover, if the dimension of this set equals n and the measure is σ-finite, the author proved in the middle of the seventies that μ is absolutely continuous with respect to m_n and that the Radon-Nikodym derivative $d\mu/dm_n$ is log-concave. Together with Prékopa's result above this gives a complete description of the class of all σ-finite log-concave measures in $\mathbf{R}^n$.

Now suppose f, g and h are non-negative continuous functions in $\mathbf{R}^n$ such that $f(\lambda x + (1-\lambda)y) \geq g^{\lambda}(x)h^{1-\lambda}(y)$ for a fixed $0 < \lambda < 1$ and all vectors x and y in $\mathbf{R}^n$. Then

$$\int_{\mathbf{R}^n} f(x)dx \geq (\int_{\mathbf{R}^n} g(x)dx)^{\lambda}(\int_{\mathbf{R}^n} h(x)dx)^{1-\lambda}.$$

This result, which is refered to as the Prékopa inequality, follows from the classical Brunn-Minkowski inequality stating that

$$m_n^{\frac{1}{n}}(A+B) \geq m_n^{\frac{1}{n}}(A) + m_n^{\frac{1}{n}}(B)$$

for all non-empty Borel sets A and B in $\mathbf{R}^n$ (see e.g Prékopa, A.: Stochastic Programming, Kluwer Academic Publishers, Dordrecht, Boston, London, 1995). In fact, many proofs of the classical Brunn-Minkowski inequality also give the Prékopa inequality (see e.g. Borell, Ch.: Convex set functions in

$d-$space, Period. Math. Hungar. 6 (1975), 111-136 and Borell, Ch.: Geometric properties of some familiar diffusions in $\mathbf{R}^n$, Ann. Probab. 21 (1993), 482-489). A recent proof of the Prékopa inequality based on the so-called Brenier map is given by McCann (Adv. Math. 128 (1997), 153-179). All these proofs are of geometric or analytic type. In this talk we will discuss a certain Brownian motion proof of the Prékopa inequality.

To this end, let Ω be the vector space of all continuous mappings of the unit interval $[0,1]$ into $\mathbf{R}^n$ and let P be Wiener measure on Ω. Thus if $B_t(\omega) = \omega(t)$, $0 \leq t \leq 1$, $\omega \in \Omega$, then the process $(B_t)_{0 \leq t \leq 1}$ is normalized n-dimensional Brownian motion in the time interval from 0 to 1. Set $\mathcal{F}_t = \sigma(B_s;\ 0 \leq s \leq t)$, $0 \leq t \leq 1$. A function $u : [0,1] \times \Omega \to \mathbf{R}^n$ is said to be progressively measurable if the map $(s,\omega) \to u(s,\omega)$, $0 \leq s \leq t$, $\omega \in \Omega$, is $(\mathcal{B}[0,t] \times \mathcal{F}_t)$-measurable for all $0 \leq t \leq 1$. Here $\mathcal{B}[0,t]$ denotes the Borel sets in the interval $[0,t]$. Let $\mathcal{U}$ denote the class of all bounded progressively measurable functions $u : [0,1] \times \Omega \to \mathbf{R}^n$ and set

$$h(u,\omega) = \int_0^1 u(s,\omega)ds$$

if $u \in \mathcal{U}$ and $\omega \in \Omega$.

Now suppose $f : \mathbf{R}^n \to [a,b]$ is a given continuous function, where $0 < a \leq b < \infty$, and set $f = \exp(-F)$. Moreover, we introduce

$$Y_F^u(\omega) = F(\omega(1) + h(u,\omega)) + \frac{1}{2}\int_0^1 \mid u(s,\omega) \mid^2 ds$$

for any $u \in \mathcal{U}$ and $\omega \in \Omega$ and set

$$J_F^u = E\left[Y_F^u\right],\ \ u \in \mathcal{U}.$$

Then

$$-\ln \int_{\mathbf{R}^n} f(x) e^{-\frac{1}{2}|x|^2} \frac{dx}{\sqrt{2\pi}^n} = \inf_{u \in \mathcal{U}} J_F^u.$$

Clearly, the Prékopa inequality is immediate from this representation formula of Gauss integrals, which is, at least implicitly, known from stochastic control theory (see Fleming W. H. and Soner, H. M.: Controlled Markov Processes and Viscosity Solutions, Springer-Verlag,Berlin, Heidelberg, New York, 1993). All the details are discussed in a forthcoming paper by the author (Diffusion equations and geometric inequalities, Pot. Anal.).

Next in this talk we will give some applications of log-concavity to the option on the minimum of several assets. Therefore consider a Black-Scholes model with n asset prices $S_1(t), ..., S_n(t)$, $t \geq 0$, governed by a non-degenerate geometric Brownian motion in n dimensions and denote by r the interest rate. To be more explicit, let C be a non-singular n by n matrix with unit rows in Euclidean $\mathbf{R}^n$ and for each $1 \leq k \leq n$, let C_k denote the $k:th$ row of C and suppose $S_k(t) = \alpha_k t + \sigma_k C_k B(t)$, where $\alpha_k \in \mathbf{R}$ and $\sigma_k > 0$ are given. Here $(B(t))_{t\geq 0}$ denotes normalized Brownian motion in $\mathbf{R}^n$. Now define $S(t) = (S_1(t), ..., S_n(t))$, $t \geq 0$, and consider a derivative security of European type, which pays $f(S(T))$ to its owner at the maturity date T. Here the payoff function f is non-negative, $P\left[f(S(T)) > 0\right] > 0$, and, moreover, we assume that $f(s) \leq C(1+ \mid s \mid)^\kappa$ for appropriate positive constants C and κ. The theoretic price of the derivative security at the time point $t \leq T$ equals $u_f(t, S(t))$, where

$$u_f(t,s) = e^{-r(T-t)} E\left[f((s_k e^{(r-\frac{\sigma_k^2}{2})(T-t)} e^{\sigma_k C_k B(T-t)})_{k=1}^n) \right].$$

If $k \in \{1, ..., m\}$ is fixed, the quantity

$$\varepsilon_k^f(t,s) = \frac{s_k}{u_f(t,s)} \frac{\partial u_f(t,s)}{\partial s_k} = \frac{\partial u_f(t,s)/u_f(t,s)}{\partial s_k / s_k}$$

is called the elasticity of the option price relative to $k:th$ asset price. As is well known, the elasticity functions ε_k^f, $k = 1, ..., n$, enter quite naturally in connection with the hedging against $f(S(T))$.

If, in addition to the above premises on f it is assumed that $f(s)$ is a log-concave function of the log-prices $\ln s_1, ..., \ln s_n$, then the Prékopa inequality shows that $\varepsilon_k^f(t,s)$ is a non-increasing function of $k:th$ asset price s_k if all other prices and time are held fixed (for details see Borell, Ch.: Geometric inequalities in option pricing. In: Convex Geometry, MSRI Publications 34 (23 pages), Cambridge Univ. Press 1998 (to appear)). In particular, this result applies to the special payoff function $f(s) = \min_{1\leq k\leq n} \max(0, s_k - a)$, where $a \geq 0$. Here, if the strike price a equals zero, we have the so called quality option, which appears in connection with a future contract, when the short has some flexibility with regard to what will be delivered (see e.g. Boyle, P. P.: The quality option and the timing option in future contracts, The Journal of Finance XLIV(1989), 101-113).