ON THE INFLUENCE OF WEIGHTS ON EXTREMES

by

Charles EL-NOUTY and Armelle GUILLOU *

Extended abstract. Let $U_1, ..., U_n$ be a sample of n independent Uniform $(0, \theta)$, $\theta > 0$, random variables, where θ is an unknown parameter, and $U_1^W, ..., U_n^W$ be a sample from $\mathbb{F}_{W,n}$, the weighted empirical distribution function of $U_1, ..., U_n$. We define the order statistics $U_{1,n}^W \leq ... \leq U_{n,n}^W$ of $U_1^W, ..., U_n^W$. The motivation of this article is that "the weights choice" corresponds to a practical situation. For example, ratio are commonly used in poll or insurance to associate a weight to the data.

To this purpose, consider a vector of random weights $\mathcal{W}_n = (W_{n,1}, ..., W_{n,n})$ independent of $U_1, ..., U_n$. Assume that, for each integer $n \geq 1$, the components of $\mathcal{W}_n$ are exchangeable. Typically, the weights $\mathcal{W}_n$ will also satisfy the conditions below:

$$W_{n,i} \geq 0, \; i \in \{1, ..., n\}, n \geq 1, \tag{1}$$

$$\textstyle\sum_{i=1}^n W_{n,i} = n, \tag{2}$$

$$\mathbb{E}W_{n,i}^2 = O(1), \; i \in \{1, ..., n\}, \tag{3}$$

$$\text{for some } c > 0, \; \frac{1}{n}\sum_{i=1}^n \Big(W_{n,i} - 1\Big)^2 \longrightarrow_P c, \text{ as } n \to \infty, \tag{4}$$

$$\frac{1}{n} \max_{1 \leq i \leq n} W_{n,i}^2 \longrightarrow_P 0, \text{ as } n \to \infty. \tag{5}$$

The conditions (1) and (2) ensure that $\mathbb{F}_{W,n}(.)$ is a distribution function, whereas (3), (4) and (5) are similar to those of Mason and Newton (1992), and Arenal-Gutiérrez and Matrán (1996).

Now, we study the influence of the maxima in the construction of limit theorems. We choose a resample size $m(n)$ and we derive conditions on $m(n)$ for the "in probability" and "with probability 1" consistency of $U_{m(n),m(n)}^W$.

* Université Paris VI, L.S.T.A., Tour 45-55, 3ème étage, 4 Place Jussieu, 75252 Paris Cedex 05, FRANCE.

E-mail: guillou@ccr.jussieu.fr

When $W_{i,n} := 1$, $1 \leq i \leq n$, the problem was studied by Deheuvels, Mason and Shorack (1993). In this paper, we assume that we do **not** have $W_{i,n} := 1, 1 \leq i \leq n$. Our first result is stated in the following theorem.

Theorem 1.

I. Assume that $m(n) \to \infty$ and

$$\frac{m(n)}{n} \to 0,$$

as $n \to \infty$. Then, under the assumptions (1), (2), (3), (4) and (5), we have in probability

$$\mathbb{P}\left(m(n) \frac{U_{n,n} - U^W_{m(n),m(n)}}{U_{n,n}} > x \middle| U_1, ..., U_n \right) \longrightarrow \exp(-x), \;\; x \geq 0,$$

as $n \to \infty$.

II. Assume that $m(n) \to \infty$ and

$$\frac{m(n)}{\sqrt{n}} \to 0,$$

as $n \to \infty$. Then, under the assumptions (1), (2), (4) and (5), we have with probability 1

$$\mathbb{P}\left(m(n) \frac{U_{n,n} - U^W_{m(n),m(n)}}{U_{n,n}} > x \middle| U_1, ..., U_n \right) \longrightarrow \exp(-x), \;\; x \geq 0,$$

as $n \to \infty$.

At present, we consider the general problem where $X_1, ..., X_n$ are i.i.d. random variables from a distribution function F. We study the influence of weights on extremes. We refer to De Haan (1970), and Galambos (1987) for further details on this field.

The following theorem shows that whenever F is in the domain of attraction of a Gumbel distribution, then the distribution of $X^W_{m(n),m(n)}$ is consistent.

Theorem 2.

Assume that F is in the domain of attraction of a Gumbel distribution. Whenever $m(n) \to \infty$ and $\frac{m(n)}{\sqrt{n}} \to 0$ as $n \to \infty$, then, under the assumptions (1), (2), (4) and (5), we have, with probability 1, for all x,

$$\mathbb{P}\left(\frac{X^W_{m(n),m(n)}-B_n}{A_n}\leq x\,\middle|\,X_1,...,X_n\right)\longrightarrow \exp\Big(-\exp(-x\log 2)\Big),\ \text{ as } n\to\infty,$$

where

$$A_n = X_{n-[\frac{n}{m(n)}],n} - X_{n-2[\frac{n}{m(n)}],n} \text{ and } B_n = X_{n-[\frac{n}{m(n)}],n}.$$

We establish similar theorems when F is in the domain of attraction of a Fréchet and Weibull distribution. The detailed proofs of the above results are given in El-Nouty and Guillou (1998).

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