Modelling dependence in joint tails

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Summary

Standard models for dependence in joint tails are based on results concerning limit distributions for componentwise maxima. We consider a wider class of joint tail dependence models that contains the standard models, and links them smoothly to weaker forms of tail dependence including independence and negative association. This approach resolves many of the theoretical problems associated with independence in extreme values, and provides practical and flexible statistical methods that offer advantages over standard techniques.

Keywords: asymptotic independence, coefficient of tail dependence, componentwise maxima, extreme value theory, maximum likelihood, slowly varying functions.

1: Introduction and Standard Methods

Modelling observations from the joint tail of an unknown multivariate distribution involves two separate issues; modelling the tail of each marginal distribution and modelling the extremal dependence structure. Marginal extremes modelling is well understood, see Davison and Smith (1990) and Smith (1989). We focus on modelling extremal dependence structures, and so remove marginal modelling issues from the discussion here by using standardised marginal variables, Z say, where Z is unit Fréchet distributed, i.e. $\text{pr}(Z \leq z) = \exp(-1/z)$ on $z > 0$. It follows that if $Z_1, \ldots, Z_n$ is an independent sequence of unit Fréchet variables, then $M_{Z,n}/n$ is unit Fréchet distributed for all $n > 0$ where $M_{Z,n} = \max(Z_i; i = 1, \ldots, n)$.

The standard approach for modelling dependence in joint tails is based on limiting results for componentwise maxima (Resnick, 1987). Here, we examine the bivariate case. Let $\{(Z_{1i}, Z_{2i}); i = 1, \ldots, n\}$ denote an independent sequence of bivariate unit Fréchet marginally distributed observations with common distribution function $F(z_1, z_2)$ but unknown dependence structure. We aim to model the extremal dependence structure of F. Standard practice is to assume that F is within the domain of attraction of a bivariate extreme value distribution, G say, with unit Fréchet margins, i.e.

$$\text{pr}\left(M_{Z_1,n}/n \leq z_1, M_{Z_2,n}/n \leq z_2\right) = F^n(nz_1, nz_2) \to G(z_1, z_2) \text{ as } n \to \infty, \tag{1}$$

where $G(z_1, z_2) = \exp\{-V(z_1, z_2)\}$, $V(z_1, z_2) = \int_0^1 \max\left\{s{z_1}^{-1}, (1-s){z_2}^{-1}\right\} dH(s)$ and H is a non-negative measure that satisfies $\int_0^1 s \; dH(s) = \int_0^1 (1-s) \; dH(s) = 1$ but is otherwise arbitrary, see Pickands (1981) and Coles and Tawn (1994). Since V is homogeneous of order -1, we have $G^n(nz_1, nz_2) = G(z_1, z_2)$, i.e. G is max-stable. Since the marginal distributions are fixed, the convergence in (1) is with respect to dependence structure only.

For statistical applications, it is usual to treat the limit (1) as an identity for large enough n for (Z_1, Z_2) within some suitably defined joint tail region, R say. That is, for all $(z_1, z_2) \in R$ and all $n > n_0$, it is assumed that $F^n(nz_1, nz_2) = G(z_1, z_2)$, which implies that $F(z_1, z_2) = G(z_1, z_2)$ over R by the max-stability property of G. This approach leads to a dependence model for componentwise maxima being used to describe the extremal dependence of the underlying process. Further insight may be obtained by studying the limit in (1). Writing $F(z_1, z_2) = 1 - \overline{F}_1(z_1) - \overline{F}_2(z_2) + \overline{F}(z_1, z_2)$ where $\overline{F}_i$ and $\overline{F}$ denote the marginal and joint survivor functions respectively, (1) becomes

$$\text{pr}(M_{Z_1,n}/n \leq z_1, M_{Z_2,n}/n \leq z_2) \sim \{1 - n^{-1}(z_1^{-1} + z_2^{-1}) + \overline{F}(nz_1, nz_2)\}^n \to G(z_1, z_2) \tag{2}$$

as $n \to \infty$. Thus $G(z_1, z_2)$ has independent components if $n\overline{F}(nz_1, nz_2) \to 0$ as $n \to \infty$, and dependent components if $n\overline{F}(nz_1, nz_2) \to \gamma(z_1, z_2) \neq 0$ as $n \to \infty$, where $\gamma(z_1, z_2) = 1/z_1 + 1/z_2 - V(z_1, z_2)$ is homogeneous of order -1. In these cases we describe Z_1 and Z_2 as asymptotically independent and asymptotically dependent respectively. Similarly, Z_1 and Z_2 are asymptotically independent and asymptotically dependent when the limiting value of $\text{pr}(Z_2 > t | Z_1 > t)$ as $t \to \infty$ is zero or greater than zero respectively (Sibuya, 1960).

2: An Alternative Approach

Ledford and Tawn (1996) showed that problems arise if models based on (1) are applied to asymptotically independent data. Ledford and Tawn (1997) developed a modelling approach that incorporated the standard methods discussed above and also handled asymptotic independence. We illustrate their approach here. Let (Z_1, Z_2) denote a bivariate random variable with unit Fréchet marginal distributions. Under suitable regularity conditions, the joint survivor function of (Z_1, Z_2) may be written

$$\mathrm{pr}(Z_1 > z_1, Z_2 > z_2) = \overline{F}(z_1, z_2) = \mathcal{L}(z_1, z_2) z_1^{-c_1} z_2^{-c_2}, \tag{3}$$

where $\mathcal{L}$ denotes a bivariate slowly varying (BSV) function (Bingham *et al.*, 1987) and $c_j \geq 0$ for $j = 1, 2$. The quantity $\eta = (c_1 + c_2)^{-1} \in (0, 1]$, i.e. the reciprocal of the index of regular variation, is called the *coefficient of tail dependence*, and effectively controls the rate of decay of $\overline{F}(tz_1, tz_2)$ as $t \to \infty$. Thus η is an important quantity for describing the extremal dependence properties of (Z_1, Z_2). In the joint tail, the marginal variables are negatively associated when $0 < \eta < 1/2$, are positively associated when $1/2 < \eta < 1$, and from (2) and (3), are asymptotically dependent when $\eta = 1$ and $\mathcal{L}(tz_1, tz_2) \not\to 0$ as $t \to \infty$, but are asymptotically independent otherwise. Ledford and Tawn (1996) exploited the relationship $\mathrm{pr}(Z_1 > r, Z_2 > r) = \mathrm{pr}\{\min(Z_1, Z_2) > r\}$ in order to estimate η as the shape parameter of the univariate variable $T = \min(Z_1, Z_2)$ using standard extreme value methods. Ledford and Tawn (1998a) give precise conditions under which model (3) holds, and go on to derive limit distributions for concomitants of extreme order statistics. Ledford and Tawn (1998b) use techniques based on (3) to examine extreme events in time-series, and Ledford (1998) examines η estimation for Gaussian dependence structures.

The structure (3) provides a smooth family of dependence models that includes asymptotically independent tail forms for both positively and negatively associated variables, and includes also the max-stable models available via the standard approach. Statistical methods based on (3) typically involve two steps: modelling the BSV function $\mathcal{L}(z_1, z_2)$ in (3), and assuming this model holds over some joint tail region, see Ledford and Tawn (1997).

3: Comparison of performance

Ledford and Tawn (1997) developed a parametric extreme values model based on (3). Here, we use maximum likelihood to fit this to four sets of simulated data, each of 25,000 points and with unit Fréchet margins, but with different dependence structures. In order of diminishing strength of extremal dependence these are:

A: the max-stable logistic dependence structure with $\alpha = 0.75$ (see Tawn, 1988),
B: the bivariate normal dependence structure with $\rho = 0.5$,
C: the Morgenstern dependence structure with $\alpha = 0.75$ (Morgenstern, 1956), and
D: the bivariate normal dependence structure with $\rho = -0.5$.

Thus A, B and C have positive extremal association, whereas the extremes of D are negatively associated. For each data set we take the fitted model and solve $\overline{F}(r_p, r_p) = p$ for a range of probabilities p. For comparison with standard methods, we construct r_p under the fitted bivariate extreme value model of Ledford and Tawn (1996). Plots showing r_p for each model and data set are given in Figure 1, together with the true r_p curves. For A, the fitted standard extreme values model agrees very well with the true r_p curve, as expected, since it is the true model for this data set. The model based on (3) performs well also, but slightly under-predicts r_p since $\hat{\eta} < 1$. For B, C and D, the standard model *greatly* over-predicts r_p, whereas the estimates of r_p for the model based on (3) are very much closer to the true r_p values.

4: Conclusions

The results here demonstrate that the parameter $\eta \in (0, 1]$, the coefficient of tail dependence, largely governs the extrapolation properties of the joint tail, and is therefore the parameter of greatest interest in statistical applications. Standard dependence models for joint tails do not offer sufficient flexibility for routine application as they have $\eta = 1$ fixed. However, models based on (3) accommodate any $\eta \in (0, 1]$ and so offer far greater flexibility for both statistical and probabilistic applications.

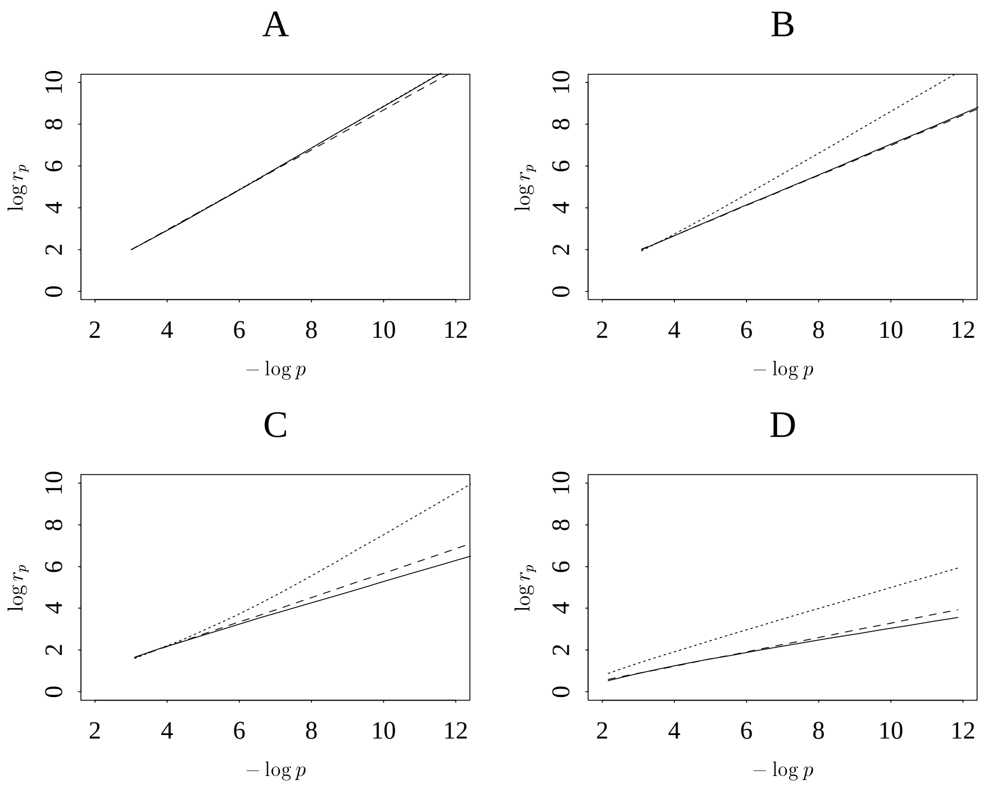

Fig 1: True values and estimates of r_p for each data set: (—) corresponds to the true distribution, (- - -) to the fitted model based on (3), and ($\cdots$) to the fitted standard extreme values model.

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