The supremum of a negative drift random walk
with dependent heavy–tailed steps

Gennady Samorodnitsky, Cornell University
jointly with Thomas Mikosch, University of Groningen

In various applied fields, such as telecommunications and queuing theory in general, insurance mathematics, finance and time series analysis among others, the model of a random walk with negative drift occurs in a natural way. In the telecommunications and queuing context, it arises as the solution to the Lindley equation. The tail distribution of the stationary solution is often viewed as an overflow probability for a buffer, and can be thought of as the distribution of the supremum of a negative drift random walk. Similarly, the probability of ruin in a homogeneous insurance portfolio can also be written in terms of the distribution of the supremum of such a random walk. The tail probability of solutions to more general stochastic recurrence equations, including the tails of ARCH and GARCH processes, can be obtained in a similar way. In the case of negative drift random walks with heavy right tail and iid steps the key result is due to Embrechts and Veraverbeke (1982). Let X_n, $n \in Z$, be iid subexponential random variables with a common law F and common negative mean $-\mu < 0$. If

$$S_0 = 0\,, \quad S_n = X_1 + \cdots + X_n\,, \quad n \geq 1\,,$$

then

$$P\Big(\sup_{n\geq 0} S_n > \lambda\Big) \sim \frac{1}{\mu}\int_\lambda^\infty (1 - F(x))\,dx \quad \text{as } \lambda \to \infty\,.$$

In most applications (except, perhaps, insurance) the assumption of independent step sizes is, clearly, unrealistic. For example, in the queuing context a typical model has steps distributed as the difference between service times and interarrival times of successive customers, and the independence assumption is universally believed not to hold. Rather, one hopes that the dependence existing in the data does not matter as far as quantities of interest, such as the ruin probability or the overflow probability, are concerned. Certain results available to date confirm this hope. For example, Asmussen, Schmidli and Schmidt (1997) show that the Embrechts and Veraverbeke result remains valid (in the queuing context) under fairly general dependence structure of the interarrival times if the service times are still independent. However, in the present work we show that if the step sizes exhibit clustering of exceedances of high thresholds, then the Embrechts and Veraverbeke result may no longer be valid.

We choose to model the steps X_n, $n \in Z$, of the random walk as a two–sided linear process:

$$X_n = -\mu + \sum_{j=-\infty}^{\infty} \varphi_{n-j}\,\varepsilon_j\,, \quad n \in Z\,,$$

where $(\varepsilon_n)_{n\in Z}$ is a sequence of zero mean iid random variables and $\mu > 0$ is a constant. We assume that $\varepsilon = \varepsilon_0$ satisfies the following regular variation and tail balance conditions:

$$P(|\varepsilon| > \lambda) = L(\lambda)\,\lambda^{-\alpha}\,,$$

$$\lim_{\lambda\to\infty} \frac{P(\varepsilon > \lambda)}{P(|\varepsilon| > \lambda)} = p, \quad \lim_{\lambda\to\infty} \frac{P(\varepsilon < -\lambda)}{P(|\varepsilon| > \lambda)} = q\,,$$

as $\lambda \to \infty$, for some $\alpha > 1$ and $0 < p \le 1$. Here L is a slowly varying (at infinity) function. The coefficients φ_j, not all of which are equal to zero, are assumed to satisfy the following condition:

$$\sum_{j=-\infty}^{\infty} |j\; \varphi_j| < \infty\,.$$

Under the above assumptions we show that

$$P(\sup_{n\ge 0} S_n > \lambda) \sim \frac{[p\,(m_\varphi^+)^\alpha + q\,(m_\varphi^-)^\alpha]}{\alpha - 1}\frac{1}{\mu}\;\lambda\; P(|\varepsilon| > \lambda)\,,$$

where

$$m_\varphi^+ = \sup_{-\infty<n<\infty} \sum_{k=-\infty}^{n} \varphi_k$$

and

$$m_\varphi^- = \sup_{-\infty<n<\infty} \sum_{k=-\infty}^{n} (-\varphi_k)\,.$$