PREDICTION OF THE MAXIMUM SIZE IN WICKSELL'S CORPUSCLE PROBLEM

Rinya TAKAHASHI and Masaaki SIBUYA

Kobe University of Mercantile Marine and Takachiho University

1. Introduction

Metal fatigue of high-strength steel starts from defects due to nonmetallic inclusions. It is known by experience that the fatigue strength is strongly related to the maximum size of inclusions in a high stress region of the steel. Hence, it is necessary to predict the maximum size based on the size data measured in microscopic view-fields of planar sections of a specimen.

Murakami (1993, 1994) developed a procedure of the prediction to control the quality of high-strength steel. In his problem, nonmetallic inclusions are round and scattered in the steel. To estimate the size distribution and the spatial density of random spheres from those of circles on the sectional plane is Wicksell's corpuscle problem (Wicksell, 1925), a classical one in stereology.

A feature of Murkami's procedure is to use only the maximum sectional sizes of nonmetallic inclusions in microscopic view-fields. In this paper, based on a statistical parametric model we propose some extended Murakami's procedure according to ways of measurement on sectioning plane, and examine there performance mainly by simulations.

2. A parametric model of Wicksell's corpuscle problem

2.1 *Wicksell's corpuscle problem.* Spherical particles of random size are randomly scattered in a space. The centers of the spheres constitute a Poisson process of intensity λ_V, and the size S_V of the sphere is independent of its position. The spheres are cut by a sectional plane, and the centers of the sectional circles constitute a Poisson process of intensity λ_A, and the size S_A of the circle is independent of its position. Assume $E(S_V)$ is finite and small enough compared with λ_V, and the spheres are actually disjoint.

Usually, the size S_ω, $\omega = A$ and V, is diameter. In this paper, S_A is the area of the sectional circle and S_V is the area of the great circle of sphere. Their p.d.f.'s (probability density functions), distribution functions and survival functions are denoted by $f_\omega(s)$, $F_\omega(s)$ and $\overline{F}_\omega(s) = 1 - F_\omega(s)$ ($\omega = A$ and V), respectively.

It is known that

$$f_A(s) = \frac{1}{2\mu_0}\int_s^\infty \frac{1}{\sqrt{v-s}} f_V(v)dv, \quad \overline{F}_A(s) = \frac{1}{2\mu_0}\int_s^\infty \frac{1}{\sqrt{v-s}} \overline{F}_V(v)dv. \tag{2.1}$$

where $\mu_0 = E(\sqrt{S_V})$. The probability distribution of S_A is called Wicksell transform of that of S_V.

2.2 *Generalized gamma model and prediction problem.* Let the generalized gamma distribution with the p.d.f.

$$\frac{1}{\Gamma(\alpha)} \cdot \frac{\gamma}{\xi^{\alpha\gamma}} x^{\alpha\gamma-1} e^{-(x/\xi)^\gamma} \mathbf{1}[0 < x < \infty], \quad \alpha,\ \gamma,\ \xi > 0, \tag{2.2}$$

be denoted by $\mathsf{Ga}(\alpha,\ \gamma,\ \xi)$. Suppose the area S_V of the great circle of the sphere to follow $\mathsf{Ga}(\alpha,\ \gamma,\ \xi)$. Generally, the parameters $\alpha,\ \gamma,\ \xi$ and the intensity of the sphere λ_V are unknown. We observe the area S_A of the circles in k parts of identical area A of the sectional plane. The distribution of S_A denoted by $\mathsf{WGa}(\alpha,\ \gamma,\ \xi)$. The number N_A of the circles in a part of area A is the Poisson variable with mean $\lambda_A A$.

We consider the following problem: Predict the square root of the maximum area, $\sqrt{W_V}$, of the great circles of the spheres in a part of volume V, where the stress of tension or compression of a test piece is maximum. The expected number of spheres in the part is $\lambda_V V$.

1.3 *Basic of prediction method.* Our prediction methods are based on the following facts. (For the maximum of Wicksell transforms, a general result was obtained by Drees and Reiss (1992).)

Proposition. *If S_V follows $\mathsf{Ga}(\alpha,\ \gamma,\ \xi)$ and $\lambda_V V \to \infty$, then the distribution of the square root of the maximum area $\sqrt{W_V}$ is approximated by the Gumbel distribution $\Lambda((t - b_V)/a_V)$, where the scale and the location parameters are determined as follows*:

$$\begin{aligned} a_V \Big/ \sqrt{\xi} &= \tau_V^{\frac{1}{2\gamma}-1} \Big/ 2\gamma, \qquad\qquad \tau_V = \log(\lambda_V V), \\ b_V \Big/ \sqrt{\xi} &= \tau_V^{\frac{1}{2\gamma}} + \frac{a_V}{\sqrt{\xi}} \Big((\alpha - 1) \log \tau_V - \log \Gamma(\alpha) \Big). \end{aligned}$$

Remark that

$$\tau_V = \tau_A + \log \frac{V}{A} - \delta, \quad \delta = \log\left(\frac{2\,\mu_0}{\sqrt{\pi}} \right) = \frac{1}{2} \log \xi + \log \left\{ \frac{2}{\sqrt{\pi}}\, \Gamma\left(\alpha + \frac{1}{2\gamma} \right) \Big/ \Gamma(\alpha) \right\},$$

where $\tau_A = \log(\lambda_A A)$.

Based on these facts, we predict $\sqrt{W_V}$ as follows. First, we estimate unknown parameters in $\mathsf{WGa}(\alpha,\ \gamma,\ \xi)$ and τ_A by data. Further, from these estimates we estimate δ, τ_V, a_V and b_V. Finally, we estimate the mean and quantiles of $\sqrt{W_V}$ by linear expressions

$$\widehat{b_V} + c\, \widehat{a_V}. \tag{2.3}$$

For the mean $c = 0.5772...$, Euler's constant, and for the pth quantile $c = -\log(-\log p)$.

3. Measurement and Prediction Methods

We consider the following five ways of measurement on sectioning planes and corresponding prediction methods:

(1) To measure the maximum area W_A within each of k parts of area A on the sectioning plane.

(2) To measure the maximum areas W_A and the number of the circles within each of k parts of area A on the sectioning plane.
(3) To measure the r largest maximum areas and the number of the circles within a part of area kA on the sectioning plane.
(4) To measure the whole areas which exceed the some threshold and the number of the circles within a part of area kA on the sectioning plane.
(5) To measure the whole areas within a part of area kA on the sectioning plane.

In the cases (1)-(4), Takahashi and Sibuya (1996, 1998) assumed the two shape parameters α and γ to be known, and used the Gumbel approximations of W_V and W_A to predict W_V. The accuracy of the prediction methods was examined by simulations. They recommended the prediction method based on the way of measurement (4) because of its satisfactory performance under the practically probable assumption (S_A follows $\mathsf{WGa}(1,\ 1/2,\ 1)$, that is the diameter of the sphere follows the eponential distribution).

In the case (5), the maximum likelihood method for estimating α, γ and ξ does not work well, because the computations of the maximum likelihood equations are troublesome. So, we consider some estimation methods based on the moments of S_A or $\log S_A$. These estimation methods are evaluated by their asymptotic variance-covariance matrices and simulations under the practically probable assumption. The case where α, γ, ξ are unknown, the estimation method based on the first three cumulants of $\log S_A$ is preferable to the one based on the moments of S_A. Unfortunately, its error is rather large. The p.d.f.'s of $\mathsf{WGa}(\alpha,\ 1/2,\ 1)$ are similar for close α values. So, we consider the case where α is known. In this case, the estimation method based on the moments $S_A^{1/2}$ and S_A is good. The performanse of the prediction method based on this good moment estimation is satisfactory.

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