Second Order Diversification Effects

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We give an overview of how one can employ semi-parametric extremal analysis in economics to calculate the Value at Risk (VaR) for loss probabilities which are at and below the inverse of the sample size. It is demonstrated how the highest frequency data improve the efficiency of the tail risk cum loss estimates. We show how to improve these loss estimates at the lower frequencies by correcting for second order tail effects. We also discuss first and second order benefits from portfolio diversification in hedging against extreme losses.

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Asset data sets nowadays cover millions of price quotes. These financial data sets are therefore well suited for studying the risk on extremely large losses. Regulators of the financial industry currently require that financial institutions be able to report, on a daily basis, a loss estimate for their entire portfolio given a particular risk level. The loss estimate is called the Value-at-Risk (VaR). As it happens that returns are heavy tailed distributed, the VaR can be well estimated by employing extreme value techniques, see e.g. Danielsson and De Vries (1997, 1998) and Longin (1997). The approach differs from the traditional finance based normal approach, or the non-parametric approach; see Jorion (1997). In this talk we review some of the results of the empirical literature in this area, and we investigate the gains from second order precision in tail estimation and risk analysis.

1 Heavy Tails, the Value at Risk and Portfolio Diversification

Suppose that the returns are i.i.d. and have tails which vary regularly at infinity. In that case

$$F(-x) = ax^{-\alpha}[1 + o(x^{-\alpha})] \quad \text{as } x \to \infty, \quad \text{and } a > 0. \tag{1}$$

Hence the left tail VaR $-x$ can be calculated from $ax^{-\alpha}$ for a given small risk level and given parameter estimates a and α. For the sum $\Sigma_i^n X_i$ we have by Feller's theorem (1971, VIII.8) that asymptotically

$$P\{\Sigma_i^n X_i > x\} \approx nax^{-\alpha}, \quad as\ x \to \infty, \tag{2}$$

and where the factor 'a' is as in (1). Banks often calculate for internal purposes the VaR for a one day investment horizon. Regulators, though, usually impose a longer horizon. The question is how to go from the high frequency estimate to the low frequency estimate without having to reestimate the paramers on a reduced sample size. We have the following interesting economic result.

Proposition 1 (The α-root rule) *Suppose X has finite variance, so that $\alpha > 2$. At a constant risk level p, increasing the time horizon n increases the VaR for the normal model percentagewise by more, i.e. by $\sqrt{n}$, than for the fat tailed model, where the increase is a factor $n^{1/\alpha}$.*

The above proposition can also be used to investigate the benefits from cross-sectional portfolio diversification. Suppose the following one factor model applies

$$R_i = \beta_i R + Q_i,$$

where R is the return on the market portfolio, and Q_i is the idiosyncratic risk of the return R_i on asset i. Consider a portfolio of m assets with weights w_i, $w_i > o$, $\sum_i^m w_i = 1$. Let $\overline{\beta} = \frac{1}{m}\sum_i^m \beta_i$.

Corollary 2 (Diversification benefits) *Suppose the Q_i are cross-sectionally i.i.d. distributed. For large loss levels the diversification benefits from the equally weighted portfolio are larger if the returns have finite variance but are fat tailed, than if they are normal distributed. Specifically for large $|x|$, $x < 0$ and $w_i = 1/m$, we find that*

$$P\{\sum_i^m w_i R_i < x|R\} \approx P\{\frac{1}{m^{1-1/\alpha}} Q_i + \overline{\beta} R < x|R\}. \tag{3}$$

Diversification is more effective if returns are fat tailed distributed than if the underlying distribution is normal. It has been noted in the literature that the effect of diversification is less pronounced if $\alpha < 2$ in comparison with the normal distribution. For example, Fama and Miller (1972, p. 270) discuss the case of sum stable distributions. They note that for $\alpha < 1$ diversification actually increases the dispersion. We are not aware of a discussion of the case $\alpha > 2$.

2 Estimation and High Frequency Benefits

We now turn to second order effects. Throughout we will assume that the following expansion applies:

$$F(x) = 1 - ax^{-\alpha}[1 + bx^{-\beta}] \quad \text{as } x \to \infty, \quad \text{and } a > 0.$$

We assume that $\alpha > 2$, because this is the relevant case for financial data. In this case both the mean and the variance are bounded. We first obtain a general lemma on second order convolution behavior. This result is needed because the $AMSE$ of the tail index estimator is a function of the first and second order parameters. The existing literature only gives a result on

second order convolution behavior for positive random variables, see Geluk, De Haan, Resnick and Starica (1997). But since the log-asset returns can be positive and negative, we need to analyze this case afresh. To restrict the number of different combinations that will arise, we assume that the tails are symmetric. We find that because the distribution of asset returns is two-sided, a new factor depending on $E[x^2]$ enters.

Lemma 3 (Second order convolution) *Suppose that asymptotically*

$$P\{X > x\} = P\{X \leq -x\} \approx ax^{-\alpha}(1 + bx^{-\beta}), \tag{4}$$

as $x \to \infty$. Moreover, assume that $\alpha > 2$, so that $E[x]$ and $E[x^2]$ are bounded. Suppose X_1 and X_2 are i.i.d. and satisfy (4). Then for the 2-convolution asymptotically

$$\begin{aligned} P\{X_1 + X_2 &> s\} = P\{X_1 + X_2 \leq -s\} \\ &\approx 2as^{-\alpha}\left(1 + bs^{-\beta} + \alpha E[x]s^{-1} + \frac{\alpha(\alpha+1)}{2}E[x^2]s^{-2}\right) \end{aligned} \tag{5}$$

as $s \to \infty$.

Proposition 4 *Suppose the X_i are i.i.d. with a distribution $F(x)$ that is symmetric around zero and is regularly varying with $\alpha > 2$. Then a $2w$-convolution affects the $AMSE\,(\hat{\gamma})$, where $\hat{\gamma} = \frac{1}{m}\sum_{i=1}^{m} \ln\left(\frac{X_i}{X_{m+1}}\right)$, as follows:*

(i) $\beta < 2$. There is no effect;

(ii) $\beta = 2$. The $AMSE\,(\hat{\gamma})$ changes by a factor

$$\left[\left(1 + \frac{1}{2}\alpha(\alpha+1)(2w-1)E[x^2]/b\right)^2\right]^{\alpha/(2\beta+\alpha)};$$

(iii) $\beta > 2$. The $AMSE\,(\hat{\gamma})$ changes by a factor

$$c\left[\frac{1}{2}\alpha(\alpha+1)(2w-1)E[x^2]\right]^{\frac{2\alpha}{4+\alpha}}\left(\frac{1}{b^2}\right)^{\alpha/(2\beta+\alpha)},$$

and where

$$c = \frac{4+\alpha}{2\beta+\alpha}\left(\frac{2}{\alpha+2}\right)^{\frac{2\alpha}{4+\alpha}}\left(\frac{\alpha+\beta}{\beta}\right)^{\frac{2\alpha}{2\beta+\alpha}}\left(\frac{\alpha}{4an}\right)^{\frac{4}{4+\alpha}}\left(\frac{2\beta an}{\alpha}\right)^{\frac{2\beta}{2\beta+\alpha}}.$$

The upshot of Proposition 4 is that either time aggregation has no effect, i.e. when $\beta < 2$, or that the $AMSE(\hat{\gamma})$ deteriorates, possibly only after the first few convolutions when $b < 0$ and $\beta = 2$. If $\beta > 2$ the $AMSE(\hat{\gamma})$ always deteriorates after the first convolution. While it can thus not be ruled out that higher frequencies deteriorate the $AMSE$ properties of $\hat{\gamma}$ for the first few convolutions, the majority of the cases goes into the other direction. It is therefore of interest to see what the data have to say on this matter.

Second order diversification benefits for equally weighted portfolios follow from the next result.

Proposition 5 *(i) $\beta < 2$. From (5) we see that after discarding the terms of low order*

$$P\left\{\frac{1}{2w}\sum_{i}^{2w} X_i \leq s\right\} \approx (2w)^{1-\alpha} a s^{-\alpha}[1 + b(2w)^{-\beta} s^{-\beta}].$$

(ii) $\beta = 2$. In this case the $s^{-\beta}$ and s^{-2} terms in (5) are of the same order. Iteration gives

$$P\left\{\frac{1}{2w}\sum_{i}^{2w} X_i \leq s\right\} \approx (2w)^{1-\alpha} a s^{-\alpha}[1 + \frac{b + \alpha(\alpha+1)(w - \frac{1}{2})E[X^2]}{(2ws)^2}].$$

(iii) $\beta > 2$. In this case the s^{-2} term dominates $s^{-\beta}$. Thus

$$P\left\{\frac{1}{2w}\sum_{i}^{2w} X_i \leq s\right\} \approx (2w)^{1-\alpha} a s^{-\alpha}[1 + \frac{\alpha(\alpha+1)(w - \frac{1}{2})E[X^2]}{(2ws)^2}].$$

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