On existence of bifurcation points for the Vlasov-Maxwell system in the plasma theory

N.A. Sidorov
Irkutsk State University

Abstract. The existence of bifurcation points and nontrivial solutions for the stationary Vlasov-Maxwell system is studied.

1. Reduction of Vlasov-Maxwell system to the elliptic equations

Consider the Vlasov-Maxwell system(VM)

$$\partial_r f_i(r,v)\cdot v + (q_i \backslash m_i)(E + (1\backslash c)[v\times B])\cdot \partial_v f_i(r,v) = 0 \tag{1}$$

$$r\in\Omega\subseteq R^3,\ v\in R^3,\ f_i(r,v)\geq 0,\ i=1,\dots,N$$

$$\begin{aligned} &rotE = 0\\ &divB = 0\\ &divE = 4\pi\sum_{k=1}^{N} q_k\int_{R^3} f_k(r,v)\mathrm{d}v \overset{def}{=} \rho \end{aligned} \tag{2}$$

$$rotB = \frac{4\pi}{c}\sum_{k=1}^{N} q_k\int_{R^3} f_k(r,v)v\mathrm{d}v \overset{def}{=} j$$

corresponding to the distribution functions of the form[1]

$$f_i(r,v) = \lambda\hat{f}_i(-\alpha_i v^2 + \varphi_i(r),\ (v,d_i)+\psi_i(r)) \overset{\mathrm{def}}{=} \lambda\hat{f}_i(\mathbf{R},\mathbf{G}). \tag{3}$$

Here

$$\varphi_i = c_{1i} + l_i\varphi,\quad \psi_i = c_{2i} + k_i\psi;\quad \varphi : R^3 \longrightarrow R; q_1 < 0,\quad q_i > 0,\quad i = 2,\dots,N.$$

$$\psi : R^3 \longrightarrow R;\quad r\in\Omega\subseteq R^3;\ v\in R^3;\ \lambda\in R^+;\quad \alpha_i\in R^+ \overset{\mathrm{def}}{=} [0,\infty);$$

$$d_i\in R^3;\quad c_{1i},\ l_i,\ c_{2i},\ k_i\in R^1 = (-\infty,+\infty);\quad i=1,\dots,N.$$

The functions $\varphi,\ \psi$ generated the electromagnetic field $(E,\ B)$ should be defined.
Let the condition is satisfied:
I. $\hat{f}_i(\mathbf{R},\mathbf{G})$- are fixed, differentiable functions of its arguments; α_i, d_i- are free parameters, the integrals

$$\int_{R^3}\hat{f}_i\mathrm{d}v, \int_{R^3}\hat{f}_i v\mathrm{d}v$$

converge for $\forall\varphi_i,\ \psi_i$;

$$l_i = \frac{m}{2\alpha q}\frac{\alpha_i q_i}{m_i};\ \ k_i = \frac{q_i}{m_i}\frac{m}{q}\frac{(d_i, d)}{d^2};\ d \triangleq d_1;\ m \triangleq m_1;\ \alpha \triangleq \alpha_1,\ q \triangleq q_1.$$

Theorem 1. *Let the condition* **I** *is satisfied. Let the functions $\varphi,\ \psi$ satisfy the system*

$$\triangle\varphi = \mu \sum_{k=1}^{N} q_k \int_{R^3} f_k \mathrm{d}v$$

$$\triangle\psi = \nu \sum_{k=1}^{N} q_k \int_{R^3} (v, d) f_k \mathrm{d}v, \tag{4}$$

moreover

$$(\nabla\varphi, d) = 0, \quad (\nabla\psi, d) = 0, \ \ \mu = (8q\pi\alpha)\backslash m;\ \nu = -(4\pi q)\backslash(mc^2). \tag{5}$$

Let

$$E = \frac{m}{2\alpha q}\nabla\varphi,$$

$$B = \frac{d}{d^2}(\beta + \int_0^1 (d \times J(tr), r)\mathrm{d}t) - [d \times \nabla\psi]\frac{mc}{qd^2},$$

where

$$\beta -\ \ const,\ J \stackrel{def}{=} \frac{4\pi}{c}\sum_{k=1}^{N} q_k \int_{R^3} v f_k \mathrm{d}v.$$

Then $(f_i,\ E,\ B)$ is a solution of VM system.

Introduce the notations: $j_i \triangleq \int_{R^3} f_i v \mathrm{d}v, \quad \rho_i \triangleq \int_{R^3} f_i \mathrm{d}v, \quad i = 1, \ldots, N$ and condition:
II. There are the vectors $\beta_i \in R^3$ such that

$$j_i = \beta_i \rho_i, \ \ i = 1, \ldots, N.$$

If the condition **II.** is satisfied, the system (4) reduces to the form

$$\Delta\varphi = \lambda\mu \sum_{i=1}^{N} q_i A_i$$

$$\Delta\psi = \lambda\nu \sum_{i=1}^{N} q_i (\beta_i, d) A_i, \tag{7}$$

where

$$A_i(l_i\varphi,\ k_i\psi) \stackrel{\text{def}}{=} \int_{R^3} \hat{f}_i \mathrm{d}v.$$

If $\beta_i = \dfrac{d_i}{2\alpha_i}$, therefore bytheidentities $\dfrac{(d_i, d)}{\alpha_i} = \dfrac{d^2}{\alpha}\dfrac{k_i}{l_i}$

the second equation of system (6) is given by

$$\Delta\psi = \lambda \frac{\nu d^2}{2\alpha} \sum_{i=1}^{N} \frac{k_i q_i}{l_i} A_i$$

1 Main Result

Let Ω is a bounded domain in R^3 with boundary $\partial\Omega$ of $C_{2,\alpha}$ class, $\alpha \in (0,1]$ and the conditions **I., II.** are satisfied. Consider the system (6) on the subspace, defined by the conditions (5).
Let φ^0, ψ^0- are constants such that
***)**

$$\sum_{k=1}^{N} q_k A_k(l_k\varphi^0, k_k\psi^0) = 0$$
$$\sum_{k=1}^{N} q_k(\beta_k, d) A_k(l_k\varphi^0, k_k\psi^0) = 0, N \geq 3. \tag{8}$$

Therefore, the system (6) with boundary conditions

$$\varphi \mid_{\partial\Omega} = \varphi^0, \quad \psi \mid_{\partial\Omega} = \psi^0 \tag{9}$$

for $\forall\lambda \in R^+$ has a trivial solution $\varphi = \varphi^0, \ \psi = \psi^0$.

Introduce a Banach spaces $C^{2,\alpha}(\bar{\Omega})$ and $C^{0,\alpha}(\bar{\Omega}), \ \alpha \in (0,1]$.

Definition[4]. *The point λ^0 we shall call a point of bifurcation of problem (7), (9), if in $\forall$ neighborhood of point $(\varphi^0, \psi^0, \lambda^0)$ we find a point (φ, ψ, λ) satisfying the system (7), (9), and moreover*

$$\mid \varphi - \varphi^0 \mid_{2,\alpha,\Omega} + \mid \psi - \psi^0 \mid_{2,\alpha,\Omega} > 0.$$

Further, let $\varphi^0 = \psi^0 = 0$.
Introduce a Banach space E_1 of vector-functions $u = (\varphi(r), \ \psi(r)); \ \varphi,$
$\psi \in C^{2,\alpha,\Omega}$; $(\nabla\varphi, d) = 0, \ (\nabla\psi, d) = 0$ at $r \in \Omega$, $\varphi \mid_{\partial\Omega} = 0, \ \ \psi \mid_{\partial\Omega} = 0$

$$\parallel u \parallel_{E_1} = \max\left(\mid \varphi \mid_{2,\alpha,\Omega}, \ \mid \psi \mid_{2,\alpha,\Omega}\right).$$

Let E_2 is Banach space of vector-functions $U = (U_1, U_2)$, where $U_1, U_2 \in C^{0,\alpha,\Omega}$

$$\parallel U \parallel_{E_2} = \max\left(\mid U_1 \mid_{0,\alpha,\Omega}, \ \mid U_2 \mid_{0,\alpha,\Omega}\right).$$

Therefore, the problem on existence of bifurcation point λ^0 of system (6),(9) would be interpreted as a problem on the bifurcation point for the operator equation

$$(L_0 - \lambda L_1)u + \lambda R(u) = 0, \tag{10}$$

$$L_0 = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$$

$$L_1 = \begin{bmatrix} \mu T_1 & \mu T_2 \\ \nu T_3 & \nu T_4 \end{bmatrix} \triangleq \Xi \tag{11}$$

$$T_1 \triangleq \sum_{i=1}^{N} q_i l_i \frac{\partial A_i(0,0)}{\partial x}; \quad T_2 \triangleq \sum_{i=1}^{N} q_i k_i \frac{\partial A_i(0,0)}{\partial y}$$

$$T_3 \triangleq \sum_{i=1}^{N} q_i l_i (\beta_i, d) \frac{\partial A_i(0,0)}{\partial x}; \quad T_4 \triangleq \sum_{i=1}^{N} q_i k_i (\beta_i, d) \frac{\partial A_i(0,0)}{\partial y}; \quad \| R(u) \| = o(\| u \|).$$

For calculation of bifurcation points it is necessary find such values of λ^* for which $N(L_0 - \lambda_* L_1) \neq \{0\}$.
Introduce the conditions:

III. $T_1 < 0$,

IV. $T_1 T_4 - T_2 T_3 > 0$.

If $\frac{\partial f_k(\hat{0},0)}{\partial x} > 0$, then, **III** is satisfied.
Introduce a matrix

$$\| \Theta_{ij} \|_{i,j=1,\ldots,N},$$

where

$$\Theta_{ij} = q_i q_j (l_j k_i - k_j l_i)(\beta_j - \beta_i, d).$$

If the derivatives $\frac{\partial A_i}{\partial x}$, $\frac{\partial A_i}{\partial y}$ are positive and equal in the point $x = y = 0$, $\Theta_{ij} > 0$ $i \neq j$ then the conditions **III, IV** are satisfied.
By $\beta_i = \frac{d_i}{2\alpha_i}$ the elements of Θ_{ij} are nonnegative by virtue of identities

$$sign \frac{q_i}{l_i} = sign\ q, \ \frac{(d_i, d)}{\alpha_i} = \frac{d^2}{\alpha} \frac{k_i}{l_i}.$$

Lemma 2. *Let the conditions* **III, IV** *are satisfied. Therefore, the matrix* Ξ *has two single eigenvalues*

$$\chi_+ = \mu T_1 + o(1), \quad \mu = -\frac{8\pi\alpha \mid q \mid}{m}$$

$$\chi_- = \eta\epsilon \frac{T_1 T_4 - T_2 T_3}{T_1} + o(\epsilon), \quad \eta = \frac{4\pi \mid q \mid}{m} > 0 \tag{12}$$

for $\epsilon \overset{\text{def}}{=} \frac{1}{c^2} \to 0$.
Introduce the homogeneous Dirichlet problem

$$-\Delta\ e = \mu e; \quad e \mid_{\partial\Omega} = 0. \tag{13}$$

Let
IV. μ_n be an eigenvalue of Dirichlet problem (13), $\{e_{in}\}$, $i = 1, \ldots, j_n$ is an orthonormalized basis in the subspace of eigenvectors $0 < \mu_1 < \mu_2 < \ldots$.

Theorem 2. *Let the condition* ***)** *by* $\varphi^0 = \psi^0 = 0$, *the conditions* **I-IV** *are satisfied, where* μ_n *is odd-multiple eigenvalue. Let* $\chi_- < 0$ *is an eigenvalue of matrix* Ξ, *defined in Lemma 2. Then*
1) $\lambda_n = -\mu_n \backslash \chi_-$ *is a point of bifurcation of VM system.*
2) *for the point* $(\lambda_n, 0) \in R^1 \times E_1$ *there exists a component* $J_{\lambda_n} = \prec (\lambda, u) \succ$ *of solutions for the problem* (10). *Here the component* J_{λ_1} *is unbounded in the space* $R^1 \times E_1$, *and component* $J_{\lambda_n}, n \geq 2$ *either is unbounded in* $R^1 \times E_1$, *or contains except point* $(\lambda_n, 0)$ *the point* $(\lambda_m, 0)$, *where* $\lambda_m = -\mu_m \backslash \chi_-$, μ_m *is an eigenvalue of Dirichlet problem* (13), $\mu_m \neq \mu_n$.
The proof follows on the basis of Lemma 3 from the theorem 2.1 of paper [3] and the theorem 1.3 of paper [5].
The results of bifurcation theory [3-5] allow to construct the asymptotics of branches of solution in neighborhood of bifurcation points of VM system, and also to develop the iterated methods of it solution .

References

[1] Y.Markov, G.Rudykh, N.Sidorov, A.Sinitsyn, D.Tolstonogov, Steady-state solutions of the Vlasov-Maxwell system and their stability, *Acta Applicandae Mathematicae*, V.28, 253-293 (1992).

[2] N.A.Sidorov, V.A.Trenogin, The investigation of bifurcation points and continuous branches of solutions of nonlinear equations, In.:"*Differential and integral equations*", Irkutsk University, Irkutsk, N 1, 216-247 (1972).

[3] M.M.Vayenberg, V.A.Trenogin, The theory of bifurcation of solutions of nonlinear equations, Moscow, Nauka, 527pp. (1968).

[4] P.H.Rabinovich, Some global results for nonlinear eigen-value problems, *Journal of functional analysis*, N 7, 1971, 487-513 (1971).

[5] N.A.Sidorov, On explicit parametrization of solution of nonlinear equations in neighborhood of bifurcation point, *Akad. Nauk Russia Dokl.*, V.336, N 5, 592-595 (1994).