Temperature effects on viscous flows in channels with porous walls

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The Navier-Stokes equations for the flow of a viscous and incompressible fluid in a porous channel, with constant and uniform injection or suction of fluid through the walls, admit a similarity solution [1]. The problem is thus reduced to a fourth order differential equation with boundary conditions at the channel walls. The equation depends on one parameter, the Reynolds number R, defined in terms of the channel half width, and the velocity of the fluid at the walls.

The influence of a temperature dependent viscosity when the channel walls are at different temperatures is analised, neglecting thermal dissipation effects. This problem admits a similarity solution also, leading to a system of two coupled differential equations for a variable related to the velocity and for the temperature. There are three boundary conditions at each wall. The problem now depends on two parameters such as R and the Péclet number, P, and on the functional dependence of viscosity on temperature.

As mentioned above, the similarity solution for the time-independent problem without temperature effects was studied by Berman [1] for small values of R in 1953, and has been considered by many authors. Succesive papers (see, e.g., Refs. 2–7) proved the existence of symmetric solutions for all values of R. In fact, for the suction case there are three symmetric solutions above a certain critical value of R. The existence of asymmetric similarity solutions for the suction case with constant viscosity has also been established in Ref. 8, where an analysis of the stability of the various solutions is given.

Basic equations. The x cooordinate is taken along the channel and the y coordinate perpendicular to the channel walls. The velocity, temperature and pressure are written as

$$\mathbf{u}(x,y,t) = x\frac{\partial f(y,t)}{\partial y}\hat{\mathbf{x}} - f(y,t)\hat{\mathbf{y}}\,, \tag{1}$$

$$T(x,y,t) = \theta(y,t)\,, \tag{2}$$

$$p(x,y,t) = \pi(y,t) + A(t)x^2/2\,. \tag{3}$$

For the dimensionless variables the units of length and velocity are the channel half width, and injection or suction velocity at the walls, respectively. Temperature is measured with respect to the value at one wall, in units of the temperature difference between walls. All other scalings of variables are obvious. Under these assumptions, the following system of equations for the unknowns $f(y,t)$ and $\theta(y,t)$ holds,

$$R(ff''' - f'f'') + (\mu(\theta)f'')'' = R\frac{\partial f''}{\partial t}\,, \tag{4}$$

$$Pf\theta' + \theta'' = P\frac{\partial \theta}{\partial t}\,, \tag{5}$$

where ' $\equiv \partial/\partial y$. The boundary conditions are: $f(-1,t)=1$, $f(1,t)=-1$, $f'(-1,t)=0$, $f'(1,t)=0$, $\theta(-1,t)=0$, $\theta(1,t)=1$. After solving this system, the pressure field is obtained using the relations

$$\begin{aligned} \pi'(y,t) &= -(f^2/2)' - (\mu f')'/R + \frac{\partial f}{\partial t}\,, && (6) \\ A(t) &= f f'' - f'^2 + (\mu f'')'/R - \frac{\partial f'}{\partial t}\,. && (7) \end{aligned}$$

Numerical methods. The system **??**–**??** was solved with an algoritm based on spectral methods, where the unknowns are expanded in a basis of Tchebyschev polynomials and the spatial derivatives are evaluated using the recurrence properties of the coeficients in transform space. The time evolution was implemented with a fourth order Runge-Kutta algorithm.

The stationary case was solved entirely in the transform space. The boundary conditions were enforced using a spectral tau method.[9]

Accuracy of the algoritms was tested by comparing with known results of the temperature independent case.

Stationary solutions. An exponential dependence with temperature was selected for the viscosity: $\mu(T)=\exp(-\gamma T)$. The limit $\gamma \to 0$ represents the temperature independent case. Taking the Péclet number $P=1$ and small values for γ the effect of the temperature gradient on the flow is analised. For each value of R, the solutions are described in terms the value of $f''(-1)$, which is proportional to the viscous stress on one wall. The bifurcation diagram for the $\gamma=0$ case is characterised by the presence of a 'pitchfork' bifurcation. At this value of R the symmetric solution becomes unstable and two asymmetric stable solutions appear.[8] As γ increases, the unfolding of the pitchfork is observed.

Approximate analytic expressions for the behaviour of $f''(-1)$ as a function of R in the neighbourhood of the bifurcation point have been derived, for different values of γ. These results are in good agreement with those obtained from the numerical solutions.

Stability analysis. An analysis of the temporal stability of the stationary solutions f_0, θ_0 was performed taking small perturbations espressed by $f=f_0+f_s\exp(st)$ and $\theta=\theta_0+\theta_s\exp(st)$. The following eigenvalue problem is obtained

$$\begin{aligned} R\,(f_0 f_s''' + f_0''' f_s - f_0' f_s'' - f_0'' f_s') + (\mu_0 f_s''' + \dot{\mu}_0 f_0'' \theta_s)'' &= R s f_s''\,, && (8) \\ P(f_0\theta_s' + \theta_0' f_s) + \theta_s'' &= P s \theta_s\,, && (9) \end{aligned}$$

where the functions $\mu_0(y)$ and $\dot{\mu}_0(y)$ denote $\mu(\theta_0(y))$ and $d\mu/d\theta(\theta=\theta_0(y))$, respectively.

Of the three solutions that exist for values of R above the bifurcation, one solution is unstable and the other two are stable in a certain range of r, losing stability at a Hopf bifurcation. The critical value of R where stability is lost takes place at different values of R for each branch and depends on γ. It decreases with increasing γ. For example, for $\gamma=0.3$ the critical values for each branch are $R=9.989$ and $R=12.428$. When $\gamma=0$ both critical values coincide and are equal to 12.963. The stability analysis shows that the $R-\gamma$ plane can be divided into four regions, each one characterised by the type (stationary or periodic) and number of stable solutions that exist there.

Periodic solutions. For values of R above a critical value, there exist periodic solutions. These solutions have been obtained using the time dependent numerical integration scheme mentioned above. For $\gamma = 0.3$ and $R = 10$ the period obtained is 7.37. The corresponding value obtained from the eigenvalue equations **??**–**??** is given by $2\pi/\mathrm{Im}(s) = 7.33$. Similar calculations for $\gamma = 1$ and $R = 5.45$ give 6.30 for the numerical integration, also in good agreement with the result $2\pi/\mathrm{Im}(s) = 6.23$ from equations **??**–**??**.

A detailed analysis of the results and graphics will be presented.

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