Analysis of the large time behaviour of a diffusive - convective transport in a n-layer stratified medium

Gabriela Marinoschi*, **Dorel Homentcovschi***, **Harry Vereecken****

The paper deals with the study of the scale effect on the transport of a solute concentration in a stratified porous medium, with the final goal of obtaining a characterization of the motion at large time and a description of the solute asymptotic behaviour. The paper is also intended to offer an explanation of the reasons that lead to the occurrence of a Fickian character of the large time diffusive convective transport in such a medium. The results may be used to interpret diffusion phenomena met within other physical or industrial problems which are described by similar models. From the mathematical point of view the problem comes to the study of the Laplace-Fourier transform of the concentration and to the particularity of its singular points.

The physical model is represented by a stratified medium consisting of n parallel horizontal infinite layers, each characterized by the height h_j and the porosity Φ_j. A horizontal transport of a stable pollutant of mass equal to μ, instantaneously released in the point $(0, y^*)$, is considered and its properties are defined by the velocity $(V_j, 0)$, the local horizontal dispersion coefficient K_{L_j} and the local vertical dispersion coefficient K_{T_j}, $j = \overline{1, n}$. The mathematical model is represented by n equations of diffusion-advection written for each concentration function $C_j(t, x, y)$ corresponding to the layer $[y_{j-1}, y_j]$

$$\frac{\partial C_j}{\partial t} + V_j \frac{\partial C_j}{\partial x} = K_{L_j} \frac{\partial^2 C_j}{\partial x^2} + K_{T_j} \frac{\partial^2 C_j}{\partial y^2} \tag{1}$$

accompanied by the boundary conditions

$$\frac{\partial C_1}{\partial y} \mid_{y=y_0} = 0 \ ; \ \frac{\partial C_n}{\partial y} \mid_{y=y_n} = 0 \tag{2}$$

$$C_1(t, x, y) \mid_{y=y^*-0} = C_1(t, x, y) \mid_{y=y^*+0} \tag{3}$$

$$C_j(t, x, y) \mid_{y=y_j-0} = C_{j+1}(t, x, y) \mid_{y=y_j+0}, \quad j = \overline{1, n-1} \tag{4}$$

$$\Phi_j K_{T_j} \frac{\partial C_j}{\partial y} \mid_{y=y_j-0} = \Phi_{j+1} K_{T_{j+1}} \frac{\partial C_{j+1}}{\partial y} \mid_{y=y_j+0}, \quad j=\overline{1,n-1} \tag{5}$$

and C_j and their derivatives w.r to must vanish when $|x|$ or $t \to \infty$.

The purpose is to prove that at large time this model can be replaced by a unique equation with well defined coefficients written for an equivalent homogeneous medium characterized by parameters resulted from the properties of the initial medium. The first step is to apply a Fourier-Laplace transform denoted $L(s,k,y)$ to the system (1)-(5). Hence its study will be replaced by the study of the corresponding system written for the Fourier-Laplace transform. The final expression of $L\,(s,k,y)$ is determined using an interesting relationship between two components and reads as

$$L(s,k,y) = \begin{cases} B\ ch[r_1(y-y_0)], & y \in [y_0, y^*] \\ B\ ch[r_1(y-y_0)] - \frac{\alpha_1}{r_1}\ sh[r_1(y-y^*)], & y \in [y^*, y_1] \\ \varepsilon_2(y)\ B + \mu_2(y), & y \in [y_1, y_2] \\ \quad .... & \\ \varepsilon_{n-1}(y)\ B + \mu_{n-1}(y), & y \in [y_{n-2}, y_{n-1}] \\ \varepsilon_n(y)\ B + \mu_n(y), & y \in [y_{n-1}, y_n] \end{cases}$$

where $\alpha_1 = \mu / K_{T_1}$, $\kappa_n = \Phi_n K_{T_n}$, $r_j = \sqrt{[s - (-k^2 K_{L_j} + \imath k V_j)]/K_{T_j}}$ and

$$\varepsilon_n(s,k,y) = \varepsilon_{n-1}(y_{n-1}) ch[r_n(y-y_{n-1})] + \frac{\kappa_{n-1}}{\kappa_n} \frac{1}{r_n} sh[r_n(y-y_{n-1})] \varepsilon'_{n-1}(y_{n-1})$$

$$\mu_n(s,k,y) = \mu_{n-1}(y_{n-1}) ch[r_n(y-y_{n-1})] + \frac{\kappa_{n-1}}{\kappa_n} \frac{1}{r_n} sh[r_n(y-y_{n-1})] \mu'_{n-1}(y_{n-1})$$

$$\varepsilon_1 = ch(r_1 h_1)\ ; \qquad \mu_1 = -\alpha_1 / r_1 \cdot sh[r_1(y_1 - y^*]$$

The value of B, depending on s and k, will be determined from the boundary condition at $y = y_n$, as $B(s,k) = -\delta_n^1 / \delta_n$ and the forms of δ_n and δ_n^1 are obtained by recurrence from the relationships

$$\delta_n = \mathcal{A}_1 \delta_{n-1} + \sum_{j=2}^{n-1} \mathcal{A}_j \delta_{n-j} + \mathcal{A}_n \varepsilon_1\ ;\ \delta_n^1 = \mathcal{A}_1 \delta_{n-1}^1 + \sum_{j=2}^{n-1} \mathcal{A}_j \delta_{n-j}^1 + \mathcal{A}_n \mu_1$$

where

$$\mathcal{A}_1 = \frac{\kappa_{n-1}}{\kappa_n} ch(r_n h_n) \ ; \ \mathcal{A}_n = r_n sh(r_n h_n) ch(r_{n-1} h_{n-1}) ... ch(r_2 h_2) ch(r_1 h_1)$$

$$\mathcal{A}_j = \frac{\kappa_{n-j}}{\kappa_{n-j+1}} \frac{r_n}{r_{n-j+1}} sh(r_n h_{n)} ch(r_{n-1} h_{n-1}) ... ch(r_{n-j+2} h_{n-j+2}) sh(r_{n-j+1} h_{n-j+1})$$

for $j = \overline{2, n-1}$.

The next step is to determine from the Laurent expansion of $L(s,k,y)$ around $s = 0$, the behaviour of its original $F(t,k,y)$ for $t \to \infty$, which will be denoted $F_\infty(t,k)$. Then using the moments of the concentration function, taking into account their significance and applying the inverse Fourier transform one proves that the equivalent equation that characterizes the mean transport in the considered stratified medium at large time is represented by

$$\frac{\partial \overline{C}_\infty}{\partial t} + V_\infty \frac{\partial \overline{C}_\infty}{\partial x} - D_\infty \frac{\partial^2 \overline{C}_\infty}{\partial x^2} = \overline{f}(t,x)$$

where $\overline{C}_\infty(t,x)$ represents the mean concentration at large time, V_∞ is the mean velocity and D_∞ is the effective longitudinal coefficient, these two being found to depend on the statistical properties of the original medium. The expressions of $\overline{f}(t,x)$ and $\overline{C}_\infty(t,x)$ are also determined.

Specific results are obtained for general boundary conditions in the case of a bounded medium and for an infinite medium too. The determinant role of the boundary conditions upon the occurrence or not of a Fickian character at large time is discussed.

Authors'addresses:

* Romanian Academy, Institute of Applied Mathematics "Caius Iacob", P.O.Box 1-24, 70700, Bucharest, Romania

** Forschungszentrum Jülich GmbH, ICG-4, Jülich, Germany