A porous particle in a shear flow. The effective viscosity of a dilute suspension

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A porous particle is a conventional model of a porous catalyst, polymeric coil, cotton ball, etc. In some practical applications, it is important to predict or treat the behavior of this sort of particles in a stream. An analysis of the effective properties of a suspension of such particles is of importance, especially in the context of models of continuum mechanics.

A number of questions naturally arise. For example, does the model of rigid particles agree sufficiently well with available experimental data or should we consider the particles deformable? Does the permeability significantly influences the behavior of the particles and the effective viscosity of a suspension of such particles as compared with the case of impermeable particles? To what extent the flow rate through the particle can vary in shear or extensional flow patterns? And so on.

To answer these questions, we study a porous spherical particle of radius a placed in a 3D shear flow, provided that the Reynolds number is small. The flow outside the particle is governed by the stationary Stokes equation and the flow inside the particle obeys Darcy's law:

$$\nabla_i p^+ = \mu \nabla_j \nabla_j v_i^+, \quad v_i^- = -\frac{k}{\mu}\nabla_i p^-.$$

Here $\nabla_i = \partial/\partial x_i$ $(i = 1, 2, 3)$; p^+, v_i^+ and p^-, v_i^- are the pressure and the fluid velocity components outside and inside the particle, respectively; x_i are the Cartesian coordinates with origin at the sphere center; k is the permeability of the particle; and μ is the fluid viscosity. Summation is assumed over repeated indices. By v_i^- the effective velocity is meant, that is, the actual fluid velocity in the porous medium times the porosity.

The incompressibility condition for the fluid has the form

$$\nabla_i v_i^+ = 0, \quad \nabla_i v_i^- = 0.$$

Far away from the particle, we have unperturbed shear flow:

$$r \to \infty, \quad v_i^+ \to e_{ij} x_j \qquad (e_{ij} = e_{ji},\ e_{ii} = 0).$$

At the particle surface, the normal velocity component is assumed to be continuous and so is the normal stress:

$$\begin{aligned} &r = a, \quad v_i^+ n_i = v_i^- n_i, \\ &r = a, \quad [\delta_{ij} p^+ - \mu(\nabla_i v_j^+ + \nabla_j v_i^+)] n_i n_j = p^-, \end{aligned}$$

where $n_i = x_i/r$ is the ith component of the outward unit normal to the particle surface ($r = \sqrt{x_i x_i}$).

For the tangential fluid velocity at the particle surface, there are three reasonable alternatives: (a) the Beavers–Joseph–Saffman boundary condition [?, ?], or, which is the same, the condition of Newtonian friction; (b) the continuity boundary condition, and (c) the no-slip condition (impermeable particle):

$$\begin{aligned}
&\text{(a)} \quad r = a, \quad \lambda\sqrt{k}\, n_k \nabla_k (v_i^+ - v_j^+ n_j n_i) = (v_i^+ - v_j^+ n_j n_i) - (v_i^- - v_j^- n_j n_i),\\
&\text{(b)} \quad r = a, \quad v_i^+ - v_j^+ n_j n_i = v_i^- - v_j^- n_j n_i,\\
&\text{(c)} \quad r = a, \quad v_i^+ - v_j^+ n_j n_i = 0,
\end{aligned}$$

where λ is a dimensionless parameter [?, ?] that depends on physical properties of the porous medium and the geometry of the surface.

Note that case (b) correspond to $\lambda = 0$, and case (c) corresponds to $k = 0$.

The 3D problem stated was solved in a closed form. For the three cases (a), (b), and (c), we calculated the surface stresses and performed a comparative analysis. In the spherical coordinates, the normal (σ_{rr}) and tangential ($\sigma_{r\theta}, \sigma_{r\varphi}$) stresses were found to have the form

$$\sigma_{rr} = \Phi(r)\Theta_1(\theta,\varphi), \quad \sigma_{r\theta} = \Psi(r)\Theta_2(\theta,\varphi), \quad \sigma_{r\varphi} = \Psi(r)\Theta_3(\theta,\varphi).$$

The table below shows some results for $\Phi(a)$ and $\Psi(a)$ in the three cases:

λ	$\sqrt{k}/a$	Φ_{a}	Φ_{b}	Φ_{c}	Ψ_{a}	Ψ_{b}	Ψ_{c}
1	0.1	1.127	0.943	1	0.607	0.943	1
10	0.1	1.444	0.943	1	0.029	0.943	1
1	0.01	1.028	0.999	1	0.951	0.999	1
10	0.01	1.213	0.999	1	0.642	0.999	1

The flow rate through a single particle is evaluated in cases (a) and (b). It was established that the flow rate divided by the second invariant of the tensor e_{ij} is practically invariable with respect to e_{ij} (the variation is less than 5%).

Based on the explicit solution for the 3D shear flow about a single particle, the effective viscosity of a dilute suspension of such particles is estimated in the spirit of the Einstein theory [?, ?]. The effective viscosity was found to be smaller compared with the classical Einstein result. In some cases, the decrease can be significant. For example, for $\lambda = 1$ and $\sqrt{k}/a = 0.1$, the decrease is about 30% in case (a), and 10% in case (b). The greater λ and $\sqrt{k}/a$, the more significant is the difference.

Acknowledgment

The author would like to express his gratitude to Dr. Kornev for fruitful discussion of this problem and useful remarks.

This work was in part supported by the Russian Foundation for Basic Research (grants No. 96-03-33574 and 97-02-17648).

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