Electrodes in a Thin Domain

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Introduction

Let $\Omega \subset \mathbb{R}^3$ be a thin domain with a constant conductivity σ. By thin we simply mean that the size of Ω is small in one direction with respect to the two other directions, say $\Omega \sim 100 \times 100 \times 1$. Suppose we have electrodes in Ω and we want to calculate the magnetic field[1] genereted by the steady current in Ω assuming that conductivity $\sigma = 0$ out of Ω. We are interested in the effect of the shape of Ω to the magnetic field. This is the reason for the thin domain in $\mathbb{R}^3$ instead of a flat domain in $\mathbb{R}^2$.

The problem has applications in geophysics where we, for example, measure the magnetic field on a sea area and from this data want to find the shape of the bottom or, in the winter time, the shape of the ice. Solution for our direct problem is then used as a tool in solving the inverse problem.

Although the problem is a very basic one, the numerical difficulties appear when the electrodes are near the boundary $\partial\Omega$, which is allways the case in a thin domain. In the paper we derive a boundary integral equation for the problem and present a way to regularize the problem numerically.

Equations for the Electric Potential

The current J_s generated by electrodes is given by $J_s = \sigma E = -\sigma \nabla V$, where E is the electric field and V the electric potential. Corresponding magnetic field is then, by Biot-Savart law,

$$B_s(x) = \frac{\mu_0}{4\pi} \int_\Omega J_s(y) \times \frac{x-y}{|x-y|^3} dy = \frac{\mu_0 \sigma}{4\pi} \int_{\partial\Omega} V(y) \frac{n(y) \times (y-x)}{|y-x|^3} dS(y),$$

so we just need to solve the boundary values of V. Here $n(y)$ is the unit outer normal at point $y \in \partial\Omega$.

[1] The electric field or the electric potential on the boundary $\partial\Omega$ could also be of interest.

Let I_j be the current of the electrode located to point $z_j \in \Omega$, $j = 1, ..., N$. Suppose $\sum_{j=1}^N I_j = 0$. The potential V satisfies

$$\begin{cases} -\sigma \Delta V = \sum_{j=1}^N I_j \delta(\cdot - z_j) & \text{in } \Omega \\ \partial_n V = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Put

$$V_1(x) = \sum_{j=1}^N \frac{I_j}{4\pi\sigma} \frac{1}{|x - z_j|},$$

so that V_1 satisfies the Poisson part of (1), $-\sigma \Delta V_1 = \sum_{j=1}^N I_j \delta(\cdot - z_j)$. Then $V_2 = V - V_1$ satisfies

$$\begin{cases} \Delta V_2 = 0 & \text{in } \Omega \\ \partial_n V_2 = -\partial_n V_1 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

¿From Green's formulas and the jump relation $\lim_{h \to 0\pm} (Kf)(x + hn(x)) = (Kf)(x) \mp \frac{1}{2} f(x), \quad x \in \partial\Omega$, of double-layer

$$(Kf)(x) = \int_{\partial\Omega} \partial_{n(y)} \Phi(x - y) f(y) dS(y), \quad \Phi(x) = -\frac{1}{4\pi} \frac{1}{|x|},$$

we get a boundary integral equation for $V_2|_{\partial\Omega}$,

$$\frac{1}{2} V_2 - K V_2 = \frac{1}{2} V_1 + K V_1 \quad \text{on } \partial\Omega.$$

¿From this we can solve $V_2|_{\partial\Omega}$ numerically (up to a constant).

Numerical Difficulties

Although the singular part of Poisson equation (1) is removed by subtracting V_1 from V, the boundary value $\partial_n V_1$ in equation (2) for V_2 changes rapidly near the electrodes. That would require more grid points there. But the number of the grid points is very limited because of the size of the computing memory.

In some geometrically simple situations equation (1) can be solved by the method of image sources. This motivates us to put suitably chosen image sources into the exterior domain $I\!R^3 \setminus \overline{\Omega}$. Let V_1' be the potential generated by the true electrodes in Ω and the image sources. Now V_1' again satisfies the Poisson part of (1). Put $V_2' = V - V_1'$, so V_2' satisfies equation (2) with V_1' instead of V_1. If the image sources are chosen reasonably, then $\partial_n V_1'$ is much smoother than $\partial_n V_1$, which leads to a more regular numerical solution.