Numerical Methods for an Inverse Heat Conduction Problem

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Abstract

We consider an inverse heat conduction problem, the Sideways Heat Equation, which is a model of a problem, where one wants to determine the temperature on both sides of a thick wall, but where one side is inaccessible to measurements. We illustrate this in Figure 0.1. Mathematically it can be formulated as a Cauchy problem for the heat equation in the quarter plane, with data given along the line $x = 1$, where one wants to determine the solution to the left of that line. More precisely, we have

$$\begin{cases} \kappa T_{xx} = T_t, & x \geq 0, \quad t \geq 0, \\ T(x,0) = 0, & x \geq 0, \\ T(1,t) = g_m(t), & t \geq 0, \end{cases}$$

where g_m is a given, measured function and κ is a positive constant. Since we can obtain T for $x > 1$, by solving a well–posed quarter plane problem, also $T_x(1,\cdot)$ is determined. Thus this is a *Cauchy problem* with appropriate Cauchy data, $[T, T_x]$, given on the line $x = 1$.

The problem is ill-posed [2], in the sense that the solution (if it exists) does not depend continuously on the data. The problem can be stabilized by replacing the time derivative in the heat equation by an approximating bounded operator, e.g. a Galerkin approximation based on wavelets [3]. The resulting problem is an initial value problem for an ordinary differential equation, which can be written

$$\begin{pmatrix} T \\ \kappa T_x \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\kappa} I \\ D & 0 \end{pmatrix} \begin{pmatrix} T \\ \kappa T_x \end{pmatrix},$$

where D is a bounded approximation of the time derivative and I is the identity matrix. This is a system of ordinary differential equations, which can be solved by standard numerical methods, e.g. a Runge-Kutta method [1].

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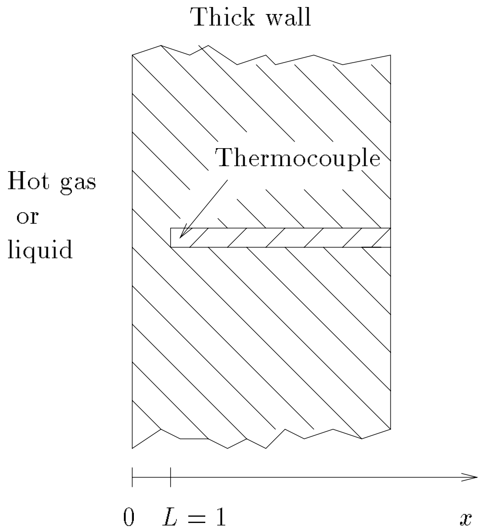

Figure 0.1: Determination of surface temperature from interior measurements.

There are several examples of industrial problems where the methods for solving the sideways heat equation can be useful. For instance consider a particle board, on which a thin lacquer coating is to be applied. It is important to estimate temperature and the temperature gradients close to the surface of the board, but direct measurements on the surface are often difficult. Instead a thermocouple placed inside the plate, as seen in Figure 0.2, gives us the temperature history, $T_{tc}(t)$. From the measured temperature we can numerically reconstruct the temperature history, $T_{ws}(t)$, on the surface of the board, by solving the sideways heat equation.

References

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[2] H.A. Levine. Continuous data dependence, regularization, and a three lines theorem for the heat equation with data in a space like direction. *Ann. Mat. Pura Appl. (IV)*, CXXXIV:267–286, 1983.

[3] T. Regińska and L. Eldén. Solving the sideways heat equation by a Wavelet-Galerkin method. *Inverse Problems*, 13:1093–1106, 1997.

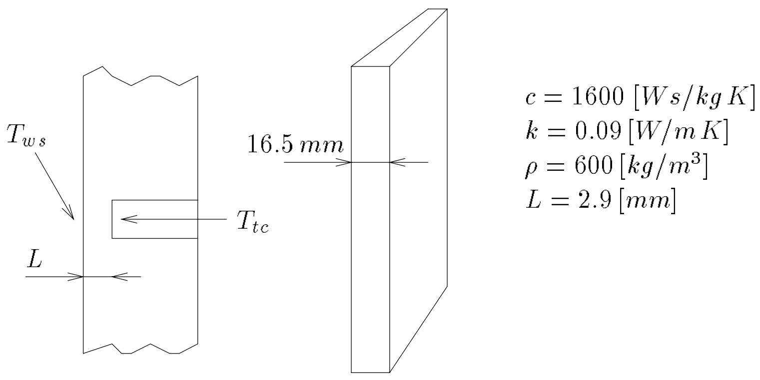

Figure 0.2: The cross–section, in principle, of the particle board. The temperature T_{tc} is measured by our thermocouple, and we seek to recover the temperature, T_{ws}, on the surface of the board.

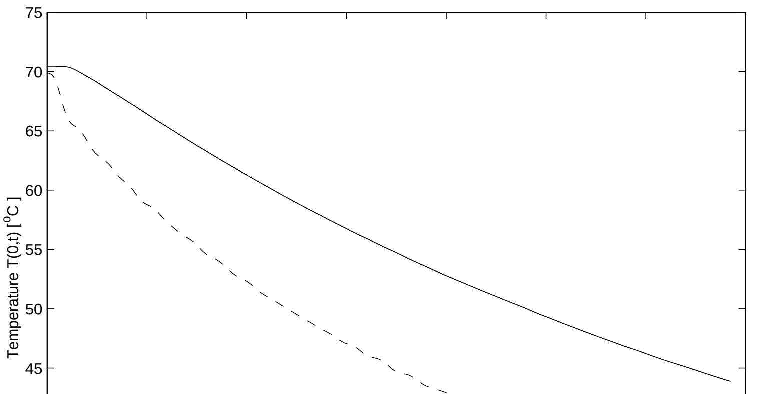

Figure 0.3: The measured temperature vector, T_{tc}, sampled at $10Hz$ (solid) and the corresponding surface temperature, T_{ws} (lower curve, dashed).