

Shoaling of 6th-order Stokesian Water Waves on a Current

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1 Introduction

When water waves propagate from deep to shallower water, the wave length, wave height and wave steepness change. These changes have been predicted to 6th order for waves on a current, using a perturbed solution to the water wave problem. The work is an extension of a 4th-order solution presented by Jonsson and Arneborg (1995). The aim of this project is to improve the results by going to 6th order, and compare the solutions with numerical results. Prediction of wave properties is important in coastal and ocean engineering, when determining the forces on near-shore or offshore structures.

In this abstract a brief introduction to Fenton's 5th-order solution of the water wave problem (Fenton, 1985) is given. This theory forms the basis for calculating the wave properties. The governing equations for the shoaling calculations are presented, the 6th-order terms included. Finally some results will be presented.

2 Fenton's 5th-order Solution (1985)

The classical water wave problem can be formulated by assuming irrotational flow, neglecting the friction forces. By these assumptions potential theory can be used. We further assume plane (i.e. 2D-) flow.

In Figure 1 the used nomenclature is shown. The problem is solved in a moving frame of reference, following the wave with the absolute phase velocity c_a . The fluid velocities relative to this frame are denoted (u, v) . Fenton formulated the governing equations, using the streamfunction obeying $\frac{\partial \psi}{\partial y} = u, \Leftrightarrow \frac{\partial \psi}{\partial x} = v$.

$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$	irrotational flow
$\psi(x, 0) = 0$	impermeable bottom
$\psi(x, \eta(x)) = \Leftrightarrow Q$	kinematic boundary condition, free surface
	$Q > 0$ is the flow rate in the negative x-direction
$g\eta(x) + \frac{1}{2} \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right] = R$	dynamic boundary condition, free surface
	(the Bernoulli equation)
$\psi(x, y) = \psi(x + L, y)$	periodicity of the solution (L is the wave length)
$H = \eta(0) \Leftrightarrow \eta(\frac{L}{2})$	definition of the wave height

Fenton made an expansion of the problem with the perturbation parameter $\varepsilon = \frac{kH}{2} = \frac{\pi H}{L}$ where k is the wave number $k = \frac{2\pi}{L}$. ε is a measure of the wave steepness. The following

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expansion for ψ was assumed:

$$\frac{k\psi}{\bar{u}} = \epsilon ky + \sum_{i=1}^{\infty} \sum_{j=1}^i f_{ij} \epsilon^i \sinh(jky) \cos(jkx)$$

where $\bar{u} = \bar{u}(y) > 0$ is the mean fluid velocity (in the negative x -direction) at vertical position y (below wave trough level) in the moving frame of reference. The above expansion for ψ automatically satisfies the Laplace equation and the bottom boundary condition. The expansion was substituted into the two free surface boundary conditions, and perturbation expansions in terms of ϵ for \bar{u} , Q , R , and η were hereafter inserted in these equations. The final result is 5th-order theory expressions for ϕ (the velocity potential), \bar{u} , Q , R , and η . In a fixed frame we find the horizontal velocities by adding the phase velocity c_a .

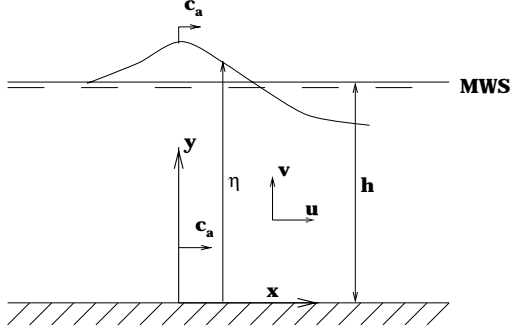


Figure 1: *Definition sketch for Fenton's solution.*

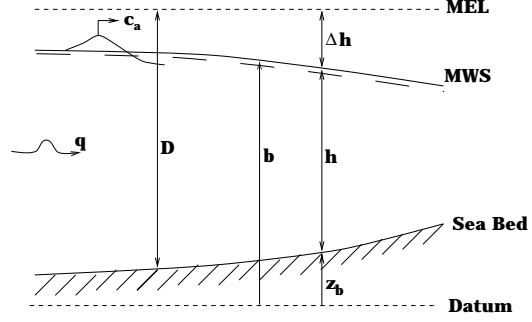


Figure 2: *Definition sketch for shoaling calculations. D is the still water depth and h is the physical depth.*

3 Shoaling Calculations

Shoaling calculations are based on the assumption that the flux of energy through a vertical cross section is conserved, when waves propagate from one depth to another. The potential energy is often calculated with the Mean Water Surface (MWS) as a datum. When waves propagate over a sloping bottom, MWS is not horizontal, and thus cannot be used as a datum for calculating the flux of potential energy, when a current is superposed on the waves.

To tackle this problem, the Mean Energy Level (MEL), which is constant, is introduced. The distance between MEL and MWS is called the set-down. Now the shoaling calculations can be based on constancy of the energy flux with MEL as datum.

Figure 2 shows the nomenclature for the shoaling calculations. Calling the volume transport velocity c_s , we have the flow $q = c_s h$. The governing equations are shown below. A_{ij} , C_i and D_i are coefficients given in Fenton (1985), except for D_6 given in Jonsson and Arneborg (1995).

Dispersion relation:

$$c_s \sqrt{\frac{k}{g}} \Leftrightarrow \frac{\omega_a}{\sqrt{gk}} + C_0 + \epsilon^2 \left(C_2 + \frac{D_2}{kh} \right) + \epsilon^4 \left(C_4 + \frac{D_4}{kh} \right) + \epsilon^6 \frac{D_6}{kh} = 0$$

Set-down:

$$\begin{aligned} k\Delta h &= \frac{k}{2g} c_s^2 + \epsilon^2 \left[C_0^2 \frac{1}{4} A_{11}^2 + \sqrt{\frac{k}{g}} c_s \frac{D_2}{kh} \right] \\ &+ \epsilon^4 \left[C_0^2 \left(\frac{1}{2} A_{11} A_{31} + A_{22}^2 \right) + \frac{1}{2} \left(\frac{D_2}{kh} \right)^2 + \sqrt{\frac{k}{g}} c_s \frac{D_4}{kh} \right] \\ &+ \epsilon^6 \left[C_0^2 \left(\frac{1}{4} A_{31}^2 + \frac{1}{2} A_{11} A_{51} + 2 A_{22} A_{42} + \frac{9}{4} A_{33}^2 \right) + \frac{D_2 D_4}{(kh)^2} + \sqrt{\frac{k}{g}} c_s \frac{D_6}{kh} \right] \end{aligned}$$

Conservation of mass:

$$\frac{d}{dx}(c_s h) = 0 \quad \Leftrightarrow \quad c_s h = \text{const}$$

Bottom topography:

$$h + \Delta h = D$$

Energy:

$$\begin{aligned} F_{MEL} = & \varepsilon^2 c_a \frac{g}{k^2} \rho \left[\frac{1}{4}(1 + G) \Leftrightarrow \sqrt{\frac{k}{g}} c_s D_2 \right] \\ & + \varepsilon^4 c_a \frac{g}{k^2} \rho \left[\Leftrightarrow \frac{3}{2} C_2 D_2 \Leftrightarrow \frac{1}{2} C_0 D_4 \Leftrightarrow \frac{D_2^2}{kh} + C_0^2 \left(\frac{1}{2} A_{11} A_{31} + A_{22}^2 \right) kh \Leftrightarrow \sqrt{\frac{k}{g}} c_s D_4 \right] \\ & + \varepsilon^6 c_a \frac{g}{k^2} \rho \left[\Leftrightarrow \frac{1}{2} C_0 D_6 \Leftrightarrow \frac{7}{6} C_2 D_4 \Leftrightarrow \frac{11}{6} C_4 D_2 \Leftrightarrow 2 \frac{D_2 D_4}{kh} \right. \\ & \left. + C_0^2 \left(\frac{1}{4} A_{31}^2 + \frac{1}{2} A_{11} A_{51} + 2 A_{22} A_{42} + \frac{9}{4} A_{33}^2 \right) kh \Leftrightarrow \sqrt{\frac{k}{g}} c_s D_6 \right] = \text{const} \end{aligned}$$

in which ρ is the density and $G \equiv 2kh / \sinh 2kh$.

The expression for the set-down to 6th order was derived by Jonsson and Arneborg (1995). The dispersion relation and the expression for F_{MEL} , however, are new to that order.

4 Results

In Figure 3 the evolution of the wave length is depicted for different currents. The chosen wave has a linear deep water steepness of $H_0/L_0 = 0.10$, and the dimensionless current has the values $q^* = \frac{c_s h}{c_0 L_0} = \{\Leftrightarrow 0.10, \Leftrightarrow 0.05, \Leftrightarrow 0.02, 0, 0.02, 0.05, 0.10\}$. H_0 is the deep-water wave height, and L_0 , c_0 the linear wave length and phase speed at deep water. Transition to a dotted curve indicates that the wave has become unphysical due to a too large steepness.

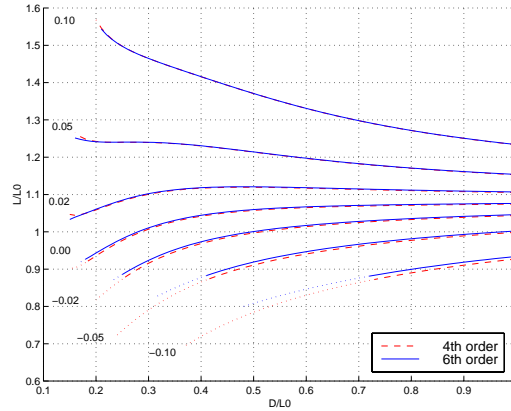


Figure 3: *Wave lengths.* $H_0/L_0 = 0.10$. Numbers in the figure are the values of the dimensionless current $q^* = \frac{c_s h}{c_0 L_0}$.

For opposing currents the wave length always decreases monotonically with decreasing depth. For strong following currents, however, the wave length increases monotonically. This can be explained by a stretching effect of the increasing current velocity on the wave profile, which in this case overrules the common shoaling effect of decreasing wave length.

The deviation between the 4th- and 6th-order results is largest for opposing currents. The prediction of the wave height and wave steepness (not shown here) shows the same tendency.

The larger the steepness, the more nonlinear the waves, and the greater the effect of the added 6th-order terms.

For the plots in Figure 3 we see that the change between monotonically increasing and decreasing wave length occurs at about $q^* = 0.02$. The waves on a small following current behave like waves on opposing currents. The reason for this is the so called return current, a wave induced ‘counter current’ below trough level, balancing the forward mass transport above wave trough level. A sufficiently high wave on a small following current can impose a return current overruling the following current under trough level.

In Figure 4 the dimensionless wave generated return current velocity $\frac{U_{ret} h}{c_0 L_0}$ for waves of 95% of the maximum 6th-order, deep-water steepness is plotted. Again dotted curves indicate waves exceeding the maximal steepness. For $q^* = 0.02$ the wave generated return current almost eliminates the following current (the return current flow is 75% of q^*). This explains quantitatively why some of the waves on a following current show the same tendencies as waves on an opposing current.

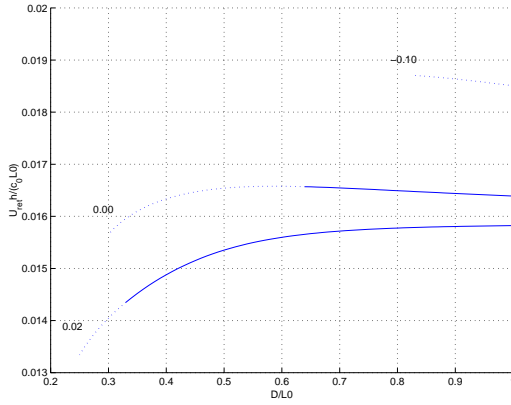


Figure 4: *Dimensionless return current velocity for a wave of 95% of the maximum 6th-order, deep-water steepness. Numbers in the figure are the values of the dimensionless current q^* .*

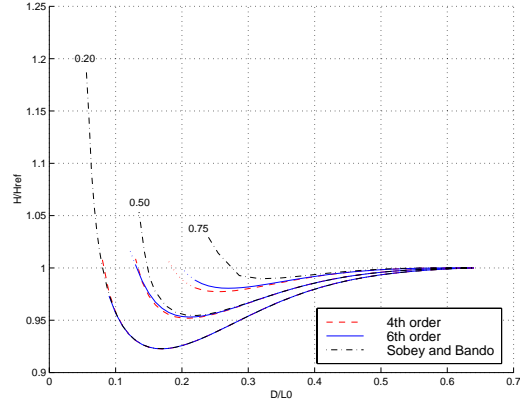


Figure 5: *Comparison of 4th- and 6th-order solution and the results from Sobey and Bando on curves are a measure of the deep-water steepness.*

In Figure 5 the 4th- and 6th-order results have been compared with some numerical results from Sobey and Bando (1991). The steeper the waves, the larger the deviation between the 4th- and 6th-order solution and the results from Sobey and Bando. The 6th-order solution is however better than the 4th-order solution. For the steepest wave shown, there is still quite a large difference between the numerical results and the 6th-order solution.

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