Reinitialization for Level Set Methods

Rainer Keck
Department of Mathematics, University of Kaiserslautern, Germany

1 Introduction

Level set methods are used to solve front propagating problems like crystal growth and multi-phase flow where a moving boundary must be determined. In contrast to front tracking methods, which are based on a Lagrangian formulation, the moving boundary is represented as the zero level set of a so-called level set function, which is given on an Eulerian co-ordinate system. Thus, the moving front is 'captured' implicitly by the level set function. This approach avoids some complex problems, such as numerical instabilities or complicated bookkeeping techniques which occur usually using front tracking methods.

2 Basic Equations

Consider the motion of two immiscible and incompressible fluids separated by its interface , , which is modelled as the zero level set of a level set function Φ, in a computational domain Ω. Using a level set approach as a description of the interface implies the existence of a mathematical relation between the evolution of the interface and the underlying flow field u. The stipulation that the level sets always move according to the velocity field means that Φ does not change along any trajectory $x(t)$ (i.e. $\Phi(t, x(t)) = \text{constant}$). Thus, the following result is easily obtained by chain rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}(\Phi(t, x(t))) &= 0 \\ \Rightarrow \Phi_t + \dot{x}(t) \cdot \nabla\Phi &= 0 \end{aligned} \tag{1}$$

This is a first-order partial differential equation of Hamilton-Jacobi type. It moves the zero level set exactly as the interface moves with the flow, which is given by its velocity field $u \equiv \dot{x}(t)$. However, it turns out that the level set function becomes 'irregular' during the computation. Therefore, it needs to be updated in order to stay 'regular'. This process is called *reinitialization* of the level set function and needs to be done after a small number of evolution time steps. Hence, reinitialization is applied many times during the entire computation.

Consider the following partial differential equation which appeared first in [2]:

$$\Phi_t = \text{sign}(\Phi_0)(1 - |\nabla\Phi|) \tag{2}$$

with initial condition $\Phi(x,0) = \Phi_0(x)$ where Φ_0 defines the interface implicitly by , $=: \{x \in \Omega : \Phi_0(x) = 0\}$. This is a non-linear hyperbolic PDE which belongs to the class of Hamilton-Jacobi equations. The idea using this equation is that a steady-state solution of (2) will be a distance function of the zero level set (i.e. $|\nabla\Phi| = 1$) with the same zero level set as the initial function Φ_0. If the level set function becomes too 'irregular' this equation will be applied to enforce the level set function to be the distance function to the zero level set. Hence, Φ will again be 'regular' enough.

Since the zero level set represents an interface of two different fluids it must not move during *reinitialization*. In fact, in numerical computations it does. Hence, it introduces an error to the areas separated by the zero level set. Moreover, the error is not qualitatively arbitrary but tends always to one direction. In particular, the area enclosed by the zero level set shrinks, i.e. area loss is introduced by reinitialization which may accumulate from application to application.

3 Analysis of Area Loss

An analysis of the the numerical scheme which is used for a discretization of equation (2) shows that a systematic error is introduced in the approximation of the absolute value of the gradient. Moreover, it turns out that this systematic error may explain the effect of area loss.

In addition to that, a modification of equation (2) is proposed, which is aimed to reduce area loss. The essential idea of this modification is to extrapolate the values of the level set function adjacent to the interface instead of computing them by the equation (2).

The results which are obtained in numerical experiments show a significant improvement as far as area loss is concerned. Furthermore, the curvature of the level sets, which is important to know in curvature-dependent problems, can be computed with less oscillations than using purely equation (2).

References

[1] Chang, Hou, Merriman, Osher: *A level set formulation of Eulerian interface capturing methods for incompressible fluid flows,* Journal of Computational Physics 124, 449-464 (1996)

[2] Sussman, Smereka, Osher: *A level set approach for computing solutions to incompressible two-phase flow,* Journal of Computational Physics 114, 146-159 (1994)