Time-domain analysis of a beam with discrete inhomogeneous supports under moving load

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Abstract

In the last few years several papers have been published on the investigation of discretely supported continuous beam problems. Most of them analyse the motion forms of the beam under moving load in the frequency domain, see e.g. Belotserkovskiy [1] in the periodic case and Kalker [2] in the inhomgeneous case. In contrast to the references above the present authors have published a finite closed–form time–domain solution for the periodically supported beam problem [3]. In the present paper we intend to generalize our previous time–domain results in order to give the actual motion forms and loading conditions for the inhomogeneously supported beam, which can be treated in a finite number of steps without involving Fourier transforms or integration.

Keywords : time–domain analysis of vehicle systems, discretely supported beam problem, transient load

The partial differential equation of the Bernoulli–Euler beam with discrete inhomogeneous supports under the action of the complex phasor excitation $F_0 \mathrm{e}^{wt}\delta(x-vt)$, $w \in \mathbb{C}$ moving at constant velocity v has the form

$$EI\frac{\partial^4 z}{\partial x^4} + \rho A\frac{\partial^2 z}{\partial t^2} + \sum_{j=-\infty}^{\infty} (m_{sj}\ \ddot{Z}_j + k_{bj}\ \dot{Z}_j + s_{bj} Z_j)\delta(x-l_j) = F_0 \mathrm{e}^{wt}\delta(x-vt). \quad (1)$$

Equation (1) is coupled with the system of ordinary differential equations

$$m_{sj}\ \ddot{Z}_j + k_{bj}\ \dot{Z}_j + s_{bj} Z_j = k_{pj}\left(\frac{\partial}{\partial t} z(l_j,t) - \dot{Z}_j\right) + s_{pj}\left(z(l_j,t) - Z_j\right),\ j \in \mathbb{Z} \quad (2)$$

governing the pad/sleeper/ballast elements of the track. Here $z(x,t)$ is the vertical displacement of the beam, while $Z_j(t)$ stands for the vertical position of the jth sleeper located at $x = l_j$. System (1–2) must satisfy boundary condition

$$\lim_{|x|\to\infty} z(x,t) = 0. \tag{3}$$

In practice the number of sleepers is finite. The main difficulty with problem (1–3) is that we cannot guarantee the existence of any continuous solutions under the action of a moving load in case of finitely many sleepers. Even in the fixed load case $v = 0$ a finite number of sleepers is not able to retain the infinite beam if the exciting phasor is harmonic, altough for $\mathrm{Re}(w) \neq 0$ the time–domain solution of problem (1–3) has been constructed by the present authors and Zoltán Zábori in [4]. Kalker [2] has solved problem (1–3) with finitely many sleepers in the frequency domain by alterating the differential operator in (1) by a small additional term $\varepsilon\frac{\partial z}{\partial t}$, where finally ε tends to zero. In this paper we choose another way for obtaining the time–domain solution to the problem.

We suppose that we have finitely many different sleepers of inhomogeneous parameters $m_{sj}, k_{bj}, s_{bj}, k_{pj}, s_{pj}$, $j = 0, 1, ..., N$ and different distances between them, and infinitely many uniform sleepers distributed under the rest of the infinite beam homogeneously, i.e.

$$\begin{gathered} L := \frac{l_N - l_0}{N},\ l_j := l_0 + jL, \\ m_{sj} := m_s,\ k_{bj} := k_b,\ s_{bj} := s_b,\ k_{pj} := k_p,\ s_{pj} := s_p \\ \text{for } j < 0 \text{ or } j > N, \end{gathered} \tag{4}$$

where m_s,k_b,s_b,k_p,s_p are the average values of the corresponding parameters for $j = 0, 1, ..., N$ computed e.g. by the trapezoid rule as

$$m_s := \frac{1}{2(l_N - l_0)}\left\{m_{s0}(l_1 - l_0) + \sum_{j=1}^{N-1} m_{sj}(l_{j+1} - l_{j-1}) + m_{sN}(l_N - l_{N-1})\right\}.$$

Let λ_i denote the root of characteristic polynomial $P(\lambda) := EI\lambda^4 + \rho A(w - \lambda v)^2$ for $i = 1, ..., 4$. Then the transition matrix of the $\mathrm{e}^{\epsilon_j\lambda_i x + (w-\lambda_i v)t}$ component of the solution from the $(n-1)$st part $[l_{n-1}, l_n]$ to the nth part can be written as

$$(\mathbf{A}_{in})_{jk} := \mathrm{e}^{\lambda_i\epsilon_k(l_n - l_{n-1})}\left(\delta_{jk} - \frac{\gamma_{in}\epsilon_j}{4EI\lambda_i^3}\right), \quad i, j, k = 1, ..., 4,$$

where

$$\frac{1}{\gamma_{in}} = \frac{1}{m_{sn}(w - \lambda_i v)^2 + k_{bn}(w - \lambda_i v) + s_{bn}} + \frac{1}{k_{pn}(w - \lambda_i v) + s_{pn}}, \quad i = 1, ..., 4,$$

and $\epsilon_j = \mathrm{i}^{j-1}$, see [3]. The transition matrix from the 0th part to the nth part is given then by

$$\mathbf{P}_{in} := \left\{\begin{array}{c} \mathbf{A}_{i,-1}^n,\ n \le 0, \\ \mathbf{A}_{in}\mathbf{A}_{i,n-1}...\mathbf{A}_{i1},\ 1 \le n \le N, \\ \mathbf{A}_{i,-1}^{n-N}\mathbf{A}_{iN}\mathbf{A}_{i,N-1}...\mathbf{A}_{i1},\ n > N. \end{array}\right.$$

Solutions of the homogeneous equation corresponding to PDE (1) in the region $\mathbb{R} \times (\frac{l_m}{v}, \frac{l_{m+1}}{v})$ have the form

$$z_{\text{hom}}(l_n + x, \frac{l_m}{v} + t) = \sum_{i=1}^{4} \mathrm{e}^{(w-\lambda_i v)(\frac{l_m}{v}+t)} \mathbf{e}_i(x)^T \mathbf{P}_{in} \mathbf{c}_{im},$$

where the coordinate functions of vector function $\mathbf{e}_i$ are defined by

$$(\mathbf{e}_i(x))_j := \mathrm{e}^{\lambda_i \epsilon_j x},$$

while $\mathbf{c}_{im}$ is an arbitrary vector depending on m. A particular solution of PDE (1) can be given by the methods of [3] as

$$z_{\text{part}}(l_n + x, \frac{l_m}{v} + t) = \sum_{i=1}^{4} \mathrm{e}^{(w-\lambda_i v)(\frac{l_m}{v}+t)} \mathbf{e}_i(x)^T \mathbf{P}_{in} \mathrm{e}^{\lambda_i l_m} \mathrm{H}(l_n - l_m + x - vt) \mathbf{P}_{im}^{-1} \mathbf{f}_i,$$

where H stands for Heaviside's unit jump function, while $\mathbf{f}_i := [\frac{F_0}{P'(\lambda_i)}, 0, 0, 0]^T$.

Boundary condition (3) implies $\lim_{|n|\to\infty} \mathbf{P}_{in}(\mathbf{c}_{im} + \mathrm{e}^{\lambda_i l_m} \mathrm{H}(l_n - l_m + x - vt) \mathbf{P}_{im}^{-1} \mathbf{f}_i) = 0$. From this, by the linear independence of the eigenvectors of matrix $\mathbf{A}_{i,-1}$ one can deduce $\mathbf{c}_{im} = \mathrm{e}^{\lambda_i l_m} \mathbf{B}_i \, \mathbf{P}_{im}^{-1} \mathbf{f}_i$, where $\mathbf{B}_i := (\mathbf{D}_{12} \mathbf{U}_i^{-1} - \mathbf{D}_{34} \mathbf{U}_i^{-1} \mathbf{P}_N)^{-1} \mathbf{D}_{34} \mathbf{U}_i^{-1} \mathbf{P}_N$ is a constant matrix of size 4×4, $\mathbf{U}_i$ is the matrix composed of the eigenvectors of matrix $\mathbf{A}_{i,-1}$ in the way $\sum_{s=1}^{4} (\mathbf{A}_{i,-1})_{js} (\mathbf{U}_i)_{sk} = \xi_{ik} (\mathbf{U}_i)_{jk}$, $|\xi_{ik}| < 1$ for $k = 1, 2$, while diagonal matrices $\mathbf{D}_{12}$ and $\mathbf{D}_{34}$ are defined by $\mathbf{D}_{12}$:=diag$(1, 1, 0, 0)$ and $\mathbf{D}_{34}$:=diag$(0, 0, 1, 1)$.

Utilizing the results above the solution of problem (1–4) takes the form

$$z(l_n + x, \frac{l_m}{v} + t) = \mathrm{e}^{w(\frac{l_m}{v}+t)} \sum_{i=1}^{4} \mathrm{e}^{-\lambda_i v t} \mathbf{e}_i(x)^T \mathbf{P}_{in} (\mathbf{B}_i + \mathrm{H}(l_n - l_m + x - vt) \mathbf{I}) \mathbf{P}_{im}^{-1} \mathbf{f}_i.$$

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