NUMERICAL METHOD FOR A PARABOLIC STOCHASTIC PARTIAL EQUATION

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Contents

Abstract

In this paper, we study the numerical approximation of a stochastic partial differential equation. We propose a finite element method, then implement backward Euler method. We show that this method is stable, derive error estimates and present numerical experiment illustrating the method and error estimates.

1. Introduction

A stochastic partial differential equation (SPDE) is a partial differential equation containing a random (noise) term. The study of SPDEs is an exciting topic which brings together techniques from probability theory, functional analysis, and the theory of partial differential equations.

Many natural phenomena and engineering applications are modeled by stochastic partial differential equations. These have been extensively analyzed. Many interesting numerical methods for approximating SPDEs have been developed and tested. The main objective of this thesis is to investigate the finite element approximation of a parabolic stochastic partial differential equation driven by white noise. This method was proposed and analyzed in [1]. We give alternative proofs by the technique developed in [2].

Let $\Omega = (0,1)$ with boundary Γ, R_+ denotes the time interval. The parabolic equation is of the form

$$\begin{aligned} u_t - au_{xx} + bu &= g, &&\text{in} \quad \Omega \times R_+, \\ u(t,x) &= 0, &&\text{on} \quad \Gamma \times R_+, \\ u(0,x) &= u_0, &&\text{in} \quad \Omega, \end{aligned} \tag{1.1}$$

where u_t denotes $\frac{\partial u}{\partial t}$ and u_{xx} the $\frac{\partial^2 u}{\partial x^2}$, and $u = u(t,x)$. In this paper we will consider the following parabolic stochastic partial differential equation,

$$\begin{aligned} u_t - au_{xx} + bu &= \partial^2 W + g, &&\text{in} \quad \Omega \times R_+, \\ u(t,x) &= 0, &&\text{on} \quad \Gamma \times R_+, \\ u(0,x) &= u_0, &&\text{in} \quad \Omega, \end{aligned} \tag{1.2}$$

where $\partial^2 W$ denotes $\frac{\partial^2 W}{\partial t \partial x}$ and $\partial^2 W$ is the mixed second-order derivative of the Brownian sheet. $W = W(t,x)$ is the *Wiener process*. We introduce a partition of $\Omega \times R_+$, let $0 = x_1 < x_2 < \cdots < x_N = 1$ and $0 = t_1 < t_2 < \cdots < t_M = T$ where $x_i = (i-1)\Delta x$ and $t_j = (j-1)\Delta t$. Let $h = \Delta x = 1/N$ and $k = \Delta t = T/M$. A reasonable approximation to $dW(t,x)$ is

$$\Delta \widehat{W} = \frac{1}{kh} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} dW ds dx, \tag{1.3}$$

Let $\hat{u} = \hat{u}(t,x)$ be the approximation of u given by

$$\begin{aligned} \hat{u}_t - a\hat{u}_{xx} + b\hat{u} &= \partial^2 \widehat{W} + g, &&\text{in} \quad \Omega \times R_+, \\ \hat{u}(t,x) &= 0, &&\text{on} \quad \Gamma \times R_+, \\ \hat{u}(0,x) &= u_0, &&\text{in} \quad \Omega, \end{aligned} \tag{1.4}$$

where $\partial^2 \widehat{W}(t,x)$ is a piecewise constant function

$$\partial^2 \widehat{W}(t,x) = \frac{1}{kh} \sum_{i=1}^{M} \sum_{j=1}^{N} \eta_{ij} \sqrt{kh} \chi_i(t) \chi_j(x), \tag{1.5}$$

where $\chi_i(t) = \begin{cases} 1, & \text{if} \quad t_i \le t \le t_{i+1}, \\ 0, & \quad \textit{otherwise}, \end{cases}$ $\chi_j(x) = \begin{cases} 1, & \text{if} \quad x_j \le x \le x_{j+1}, \\ 0, & \quad \textit{otherwise}, \end{cases}$ *and*

$$\eta_{ij} = \frac{1}{kh} \int_{t_i}^{t_{i+1}} \int_{x_j}^{x_{j+1}} dW(t,x),$$

i.e., $\eta_{ij} \in N(0,1)$ are independent identically distributed random variables.

Let HS denote the space of Hilbert-Schmidt operators from H to H, i.e.,

$$HS = \Big\{ \psi \in L(H) : \sum_{l=1}^{\infty} \|\psi e_l\|^2 < \infty \Big\}$$

with norm

$$\|\psi\|_{HS} = \Big(\sum_{l=1}^{\infty} \|\psi e_l\|^2 \Big)^{1/2}$$

where $H = L_2(\Omega)$ with the norm

$$\|v\| = \sqrt{\int_{\Omega} v^2 ds},$$

$\{e_l\}$ is an arbitrary ON-basis for H. Let E denote the expectation. Let $\psi(s) \in HS$, then

$$\int_0^t \psi(s) dW(s)$$

can be defined and have the isometry

$$E\| \int_0^t \psi(s) dW(s) \|^2 = \int_0^t \|\psi(s)\|_{HS}^2 ds.$$

According to the standard finite element method, based on the weak formulation of (1.4), find $\hat{u}_h(t) \in S_h \subset H$, such that

$$(\hat{u}_{h,t}, \chi) + a(D\hat{u}_h, D\chi) + b(\hat{u}, \chi) = (\partial^2 \widehat{W}, \chi) + (g, \chi), \quad \forall \chi \in S_h, t > 0.$$

The backward Euler method has the form

$$(\hat{U}^n - \hat{U}^{n-1}, \chi) + ka(D\hat{U}^n, D\chi) + kb(\widehat{U}^n, \chi) = k(\partial^2 \widehat{W}_n, \chi) + k(g_n, \chi),$$

we will discuss the details of this method in the following sections.

2. The Error for Approximation of the Noise

The following is Theorem 2.3 in [1].

Theorem 1. *Let $\hat{u}$ be the solution of (1.4) and u be the solution of (1.2). Then*

$$E \int_0^T \int_0^1 (u(t,x) - \hat{u}(t,x))^2 dx dt \le (c_1 k^{1/2} + c_2 \frac{h^2}{k^{1/2}}),$$

where c_1 and c_2 are constants independent of k and h, $t \in [0, \infty)$, k and h are timestep, spacestep respectively.

Proof. We write the proof for $a = 1$ and $b = 0$ only. We introduce the fundamental solution of

$$\begin{aligned} v_t(t,x) - v_{xx}(t,x) &= 0, && \text{in} \quad \Omega \times R_+, \\ v(t,0) = v(t,1) &= 0, && \text{on} \quad \Gamma \times R_+, \\ v(0,x) &= \phi(x), && \text{in} \quad \Omega, \end{aligned}$$

namely $G_t(x,y) = 2 \sum_{i=1}^{\infty} \sin n\pi x \sin n\pi y e^{-(n\pi)^2 t}$, so that

$$v(t,x) = \int_0^1 G_t(x,y) \phi(y) dy.$$

Hence, we have

$$\hat{u}(t,x) = \int_0^1 G_t(x,y) u_0(y) dy + \int_0^t \int_0^1 G_{t-s}(x,y) d\hat{W}(s,y) + \int_0^t \int_0^1 G_{t-s}(x,y) g(s,y) dy ds,$$

$$u(t,x) = \int_0^1 G_t(x,y) u_0(y) dy + \int_0^t \int_0^1 G_{t-s}(x,y) dW(s,y) + \int_0^t \int_0^1 G_{t-s}(x,y) g(s,y) dy ds,$$

and we define

$$F(t,x)=\int_0^t\int_0^1 G_{t-s}(x,y)dW(s,y)-\int_0^t\int_0^1 G_{t-s}(x,y)d\widehat{W}(s,y). \tag{2.1}$$

Thus $e(t,x)=u(t,x)-\hat{u}(t,x)=F(t,x)$. Taking expectations of both sides and letting

$$\hat{F} \quad = \quad E\int_0^T\int_0^1 F^2(t,x)dxdt$$

and $G_{t-s}(x,y)=2\sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi y)e^{-(n\pi)^2(t-s)}$, we get

$$\begin{aligned}
\hat{F}&=E\int_0^T\int_0^1 F^2(t,x)dxdt\\
&=E\int_0^T\int_0^1\Big[\int_0^t\int_0^1 G_{t-s}(x,y)dW(s,y)-\int_0^t\int_0^1 G_{t-s}(x,y)d\hat{W}(s,y)\Big]^2dxdt\\
&=E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^1\Big\{\Big[\int_0^{t_j}\int_0^1 G_{t_j-s}(x,y)dW(s,y)-\int_0^{t_j}\int_0^1 G_{t_j-s}(x,y)d\hat{W}(s,y)\Big]\\
&\quad+\Big[\int_0^t\int_0^1 G_{t-s}(x,y)dW(s,y)-\int_0^{t_j}\int_0^1 G_{t_j-s}(x,y)dW(s,y)\Big]\\
&\quad+\Big[\int_0^t\int_0^1 G_{t-s}(x,y)d\hat{W}(s,y)-\int_0^{t_j}\int_0^1 G_{t_j-s}(x,y)d\hat{W}(s,y)\Big]\Big\}^2dxdt\\
&=I+II+III.
\end{aligned}$$

We estimate I:

$$\begin{aligned}
I&=E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\sum_{l=1}^{j-1}\sum_{i=1}^{M}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}} G_{t_j-r}(x,z)dW(r,z)\\
&\quad-\sum_{l=1}^{j-1}\sum_{i=1}^{M}\int_0^{t_j}\int_0^1 G_{t_j-s}(x,y)\frac{1}{kh}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}dW(r,z)dsdy\Big]^2dxdt\\
&=\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^1\sum_{l=1}^{j-1}\sum_{i=1}^{M}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}\Big[\frac{1}{kh}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}(G_{t_j-r}(x,z)-G_{t_j-s}(x,y))dsdy\Big]^2dxdtdrdz\\
&=\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^1\sum_{l=1}^{j-1}\sum_{i=1}^{M}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}\Big[\frac{1}{kh}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}2\sum_{1}^{\infty}\sin(n\pi x)e^{-2(n\pi)^2t_j}(\sin(n\pi z)e^{(n\pi)^2r}\\
&\quad-\sin(n\pi y)e^{(n\pi)^2s})^2dsdy\Big]dxdtdrdz\\
&=\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^1\sum_{l=1}^{j-1}\sum_{i=1}^{M}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}\frac{1}{kh}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}2\sum_{1}^{\infty}\sin(n\pi x)e^{-2(n\pi)^2t_j}(\sin(n\pi z)e^{(n\pi)^2r}\\
&\quad-\sin(n\pi y)e^{(n\pi)^2s})^2dsdydtdrdzdx\\
&=\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^1\sum_{l=1}^{j-1}\sum_{i=1}^{M}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}\frac{1}{kh}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}2\sum_{1}^{\infty}\sin(n\pi x)e^{-2(n\pi)^2t_j}(\sin(n\pi z)e^{(n\pi)^2r}\\
&\quad-\sin(n\pi y)e^{(n\pi)^2s})^2dsdydtdrdzdx\\
&\leq\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j-1}\sum_{i=1}^{M}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}\frac{1}{kh}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}2\sum_{1}^{\infty}{}^{-2(n\pi)^2t_j}(\sin(n\pi z)-\sin(n\pi y))^2e^{2(n\pi)^2r}\\
&\quad+\sin(n\pi y)(e^{(n\pi)^2r}-e^{(n\pi)^2s})^2dsdydtdrdz\\
&=I_1+I_2.
\end{aligned}$$

Here we have,

$$
\begin{aligned}
I_1 &= \sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{i=1}^{M}\frac{1}{kh}\int_0^{t_j}\int_{x_i}^{x_{i+1}}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-r)}(\sin(n\pi z)-\sin(n\pi y))^2 dsdydrdzdt\\
&\leq \sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{i=1}^{M}\int_0^{t_{j-1}}\int_{x_i}^{x_{i+1}} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-r)}(n\pi h)^2 drdzdt\\
&\quad+\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{i=1}^{M}\int_{t_j-1}^{t_j}\int_{x_i}^{x_{i+1}} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-r)}4drdzdt\\
&\leq \sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{i=1}^{M}\int_{x_i}^{x_{i+1}}\frac{1}{2(n\pi)^2}(e^{-2(n\pi)^2k}-e^{2(n\pi)^2t_j})(n\pi h)^2dzdt\\
&\quad+\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{i=1}^{M}\int_{x_i}^{x_{i+1}}\frac{1}{2(n\pi)^2}(1-e^{-2(n\pi)^2k})4dzdt\\
&\leq C_1\sum_{n=1}^{\infty}e^{-2(n\pi)^2k}h^2+C_2\sum_{n=1}^{\infty}\frac{1-e^{-2(n\pi)^2k}}{2(n\pi)^2}\\
&\leq C_1\frac{h^2}{k^{1/2}}+C_2k^{1/2}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
I_2 &= \sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^{t_j}\sum_{i=1}^{M}\int_{x_i}^{x_{i+1}}\Big[\frac{1}{kh}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}\sum_{n=1}^{\infty}\sin^2(n\pi y)e^{-2(n\pi)^2(t_j-r)}(1-e^{(n\pi)^2(s-r)})dsdy\Big]^2dzdrdt\\
&\leq \sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^{t_j}\sum_{i=1}^{M}\int_{x_i}^{x_{i+1}}\sum_{n=1}^{\infty}e^{-2(n\pi)^2(t_j-r)}(n\pi)^4k^2dzdrdt\\
&\leq C_3\int_0^{t_{j-1}}\sum_{n=1}^{\infty}e^{-2(n\pi)^2(t_j-r)}(n\pi)^4k^2dr+C_4\int_{t_{j-1}}^{t_j}\sum_{n=1}^{\infty}e^{-2(n\pi)^2(t_j-r)}(n\pi)^4k^2dr\\
&= C_3\sum_{n=1}^{\infty}(n\pi)^2k^2(e^{-2(n\pi)^2k}-e^{-2(n\pi)^2t_{j-1}})+C_4\sum_{n=1}^{\infty}\frac{1-e^{-2(n\pi)^2k}}{2(n\pi)^2}\\
&\leq Ck^{1/2}.
\end{aligned}
$$

For II, we have

$$
\begin{aligned}
II &\leq E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_0^{t_j}\int_0^1(G_{t-s}(x,y)-G_{t_j-s}(x,y))dW(s,y)\Big]^2dxdt\\
&\quad+E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_{t_j}^{t}\int_0^1G_{t_s}(x,y)dW(s,y)\Big]^2dxdt\\
&=\underbrace{\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^1\int_0^{t_j}\int_0^1(G_{t-s}(x,y)-G_{t_j-s}(x,y))^2dydsdxdt}_{II_1}+\underbrace{\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^1\int_{t_j}^{t}\int_0^1G_{t-s}^2dydsdxdt}_{II_2}.
\end{aligned}
$$

Here we have,

$$
\begin{aligned}
II_1 &= \sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^1\int_0^{t_j}\int_0^1 (G_{t-s}(x,y)-G_{t_j-s}(x,y))^2\,dy\,ds\,dx\,dt\\
&= \sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^{t_j}\sum_{n=1}^{\infty}(e^{-2(n\pi)^2(t-s)}-e^{-(n\pi)^2(t_j-s)})^2\,ds\,dt\\
&=\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{n=1}^{\infty}\frac{1}{2(n\pi)^2}(e^{2(n\pi)^2t_j}-1)(e^{-(n\pi)^2t}-e^{-(n\pi)^2t_j})^2dt\\
&=\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{n=1}^{\infty}\frac{1}{2(n\pi)^2}(1-e^{2(n\pi)^2t_j})(1-e^{-(n\pi)^2(t-t_j)})^2dt\\
&\le\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{n=1}^{\infty}\frac{1}{2(n\pi)^2}(1-e^{2(n\pi)^2k})^2dt\\
&\le Nk\sum_{n=1}^{\infty}\frac{1}{2(n\pi)^2}(1-e^{2(n\pi)^2k})^2\\
&\le Ck^{1/2}.
\end{aligned}
$$

Note that $(1-e^{2(n\pi)^2t_j})\le 1$. We also have

$$
\begin{aligned}
II_2 &=\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_0^1 G_{t-s}^2(x,y)dydsdxdt\\
&=\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\int_{t_j}^{t}\frac{1}{2}\sum_{n=1}^{\infty}e^{-2(n\pi)^2(t-s)}dsdt\\
&=\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{n=1}^{\infty}\frac{1}{2(n\pi)^2}(1-e^{-2(n\pi)^2(t-t_j)})dt\\
&\le Nk\sum_{n=1}^{\infty}\frac{1}{2(n\pi)^2}(1-e^{-2(n\pi)^2k})\\
&\le Ck^{1/2}.
\end{aligned}
$$

The proof of Part III are similar. Thus, we draw the conclusion of the theorem. □

3. Numerical Method

3.1. **Numerical Method for Parabolic Differential Equations.** Firstly, we present the finite element method for (1.1). Let S_h denote the continuous piecewise linear functions which vanish on Γ and let $\{\phi_i\}_{j=1}^{M}$ be the standard basis of S_h. We pose the approximate problem to find $U_h(t)\in S_h$ for each t, such that

$$
(U_{h,t},v)+a(DU_h,Dv)+b(U_h,v)=(g,v),\quad \forall v\in S_h\subset H_0^1, t>0. \tag{3.1}
$$

Then, we introduce the backward Euler method in time. Let $0=t_0<t_1<t_2\cdots<t_N=T$ be a partition of the time interval $I=[0,T]$ (each time interval $I_n=(t_{n-1},t_n]$).

$$
\begin{aligned}
(\frac{U^n-U^{n-1}}{k},v)+a(DU^n,Dv)+b(U^n,v) &= (g_n,v)\\
(U^n-U^{n-1},v)+ka(DU^n,Dv)+kb(U^n,v) &= k(g_n,v)\\
(U^n,v)+ka(DU^n,Dv)+kb(U^n,v) &= (U^{n-1},v)+k(g_n,v)
\end{aligned}
$$

In terms of the basis $\{\phi_j\}_{j=1}^M$, we can pose

$$U(t,x)=\sum_{j=1}^{M}\xi(t)\phi_j(x),$$

from (3.1), we can get

$$\sum_{j=1}^{M}\xi_j^n(\phi_j,\phi_i)+ka\sum_{j=1}^{M}\xi_j^n(D\phi_j,D\phi_i)+kb\sum_{j=1}^{M}\xi_j^n(\phi_j,\phi_i)=\sum_{j=1}^{M}\xi_j^{n-1}(\phi_j,\phi_i)+k(g_n,\phi_i)$$

where $n=1,2,\ldots,M$ and $i=1,2,\ldots,N$. This is a linear system,

$$\xi^n=(B+kA)^{-1}(B\xi^{n-1}+\tilde{g}_n) \tag{3.2}$$

where $A_{ij}=a(D\phi_j,D\phi_i)+b(\phi_j,\phi_i)$ is the stiffness matrix, $B_{ij}=(\phi_j,\phi_i)$ is the mass matrix, $\tilde{g}_{n,i}=k(g_n,\phi_i)$ is the load vector.

3.2. **Numerical Method for Stochastic Parabolic Differential Equations.** The weak formulation of (1.4) has the form

$$(\hat{u}_t,\phi)+a(D\hat{u},D\phi)+b(\hat{u},v)=(\partial^2\widehat{W},\phi)+(g,\phi) \tag{3.3}$$

S_h denotes the continuous piecewise linear finite element space, $u\in S_h\subset H_0^1$. We than pose the approximate problem to find $\hat{u}_h(t)$, belonging to S_h for each t, such that

$$(\hat{u}_{h,t},\chi)+a(D\hat{u}_h,D\chi)+b(\hat{u},\chi)=(\partial^2\widehat{W},\chi)+(g,\chi),\quad \forall\chi\in S_h, t>0. \tag{3.4}$$

Using the backward Euler method

$$\Big(\frac{\widehat{U}^n-\widehat{U}^{n-1}}{k},\chi\Big)+a(D\widehat{U}^n,D\chi)+b(\widehat{U}^n,\chi)=\Big(\frac{\widehat{W}(t_n)-\widehat{W}(t_{n-1})}{k},\chi\Big)+(g_n,\chi). \tag{3.5}$$

The backward Euler method is considered of the linear system form

$$\sum_{l=1}^{M}\widehat{U}_j^n(\phi_j,\phi_l)+ka\sum_{l=1}^{M}\widehat{U}_j^n(D\phi_j,D\phi_l)+kb\sum_{l=1}^{M}\widehat{U}_j^n(\phi_j,\phi_l)=\sum_{l=1}^{M}\widehat{U}_j^{n-1}(\phi_j,\phi_l)+(\Delta\widehat{W}^n,\phi_l)+k(g_n,\phi_l) \tag{3.6}$$

where $\widehat{U}^n(x)=\sum_{j=1}^M\widehat{U}_j^n\phi_j(x)$ for $l=1,2,\ldots,M$.

4. Error Estimate

Denote $A_h:S_h\to S_h$ is the discrete Laplacian, defined by

$$(A_h\psi,\chi)=a(D\psi,D\chi)+b(\psi,\chi),\quad \forall\chi\in S_h \tag{4.1}$$

Denote $P_h:L_2\to S_h$ is the orthogonal projection, defined by

$$(P_hf,\chi)=(f,\chi),\forall\chi\in S_h. \tag{4.2}$$

$$\hat{u}_t+A\hat{u}=\partial^2\widehat{W}, \tag{4.3}$$

$$\hat{u}_{h,t}+A_h\hat{u}_h=P_h\partial^2\widehat{W}, \tag{4.4}$$

The mild solutions of (4.3) and (4.4) have the following form

$$\hat{u}(t)=E(t)u_0+\int_0^t E(t-s)\partial^2\widehat{W}(s)ds, \tag{4.5}$$

$$\hat{u}_h(t)=E_h(t)P_hu_0+\int_0^t E_h(t-s)P_h\partial^2\widehat{W}(s)ds, \tag{4.6}$$

where $E(t)$ is the solution operator of

$$u_t+Au=0,\quad u(0)=u_0,$$

and $E_h(t)$ is the solution operator of

$$u_{h,t} + Au_h = 0, \quad u_h(0) = P_h u_0.$$

We recall the following lemma from [2]:

Lemma 1. *Let $F_h(t) = E_h(t)P_h - E(t)$. Then*

$$\|F_h v\|_{L_\infty([0,T];H)} \le Ch^\beta |v|_\beta, \quad \textit{for} \quad v \in H^\beta, \quad 0 \le \beta \le 1, \tag{4.7}$$

and

$$\|F_h v\|_{L_2([0,T];H)} \le Ch^\beta |v|_{\beta-1}, \quad \textit{for} \quad v \in H^{\beta-1}, \quad 0 \le \beta \le 1, \tag{4.8}$$

where $|v|_\beta = \|A^{\beta/2} v\|$ for $\beta \in R$.

We now prove the following theorem.

Theorem 2. *Let $\hat{u}$ be the solution of (1.4) and $\hat{u}_h$ be the solution of (3.4). Then*

$$\sqrt{E(\|\hat{u}_h(t) - \hat{u}(t)\|^2)} \le Ch^\beta (|u_0|_\beta + \|A^{(\beta-1)/2}\|_{HS}) \le Ch^\beta,$$

when $0 \le \beta < \frac{1}{2}$, $t \in [0, \infty)$.

Proof. Let

$$\hat{e}(t) = \underbrace{E_h(t)P_h u_0 - E(t)u_0}_{\hat{e}_1(t)} + \underbrace{\int_0^t (E_h(t-s)P_h - E(t-s))\partial^2 \widehat{W}(s,\cdot)ds}_{\hat{e}_2(t)}.$$

By (4.7), we have

$$\|\hat{e}_1(t)\| \le Ch^\beta |u_0|_\beta. \tag{4.9}$$

Let $F_h(t-s) = E_h(t-s)P_h - E(t-s)$ and $K_{t-s}(x,y) = G_{h,t-s}(x,y) - G_{t-s}(x,y)$. Thus

$$\begin{aligned}
E(\|\hat{e}_2(t)\|^2) &= E\Big(\int_0^1 (\int_0^t \int_0^1 K_{t-s}(x,y)\partial^2 \widehat{W}(y,s)dyds)^2 dx\Big) \\
&= E\Big(\int_0^1 (\sum_{l=1}^N \sum_{j=1}^M \int_{I_l}\int_{I_j} K_{t-s}(x,y)\frac{1}{\sqrt{k_l h_j}}\eta_{lj} dyds)^2 dx\Big) \\
&= \int_0^1 \frac{1}{k_l h_j} \sum_{l=1}^N \sum_{j=1}^M \Big(\int_{I_l}\int_{I_j} K_{t-s}(x,y)dyds\Big)^2 dx \\
&\le \int_0^1 \int_0^t \int_0^1 K^2_{t-s}(x,y)dydsdx \\
&= \int_0^t \int_0^1 \int_0^1 K^2_{t-s}(x,y)dydxds \\
&= \int_0^t \|F_h(t-s)\|^2_{HS} ds \\
&= \int_0^t \sum_{l=1}^\infty \|F_h(t-s)e_l\|^2 ds \\
&= \sum_{l=1}^\infty \int_0^t \|F_h(s)e_l\|^2 ds \\
&\le \sum_{l=1}^\infty Ch^{2\beta}|e_l|^2_{\beta-1} \\
&= \sum_{l=1}^\infty Ch^{2\beta}\|A^{(\beta-1)/2}e_l\|^2 \\
&= Ch^{2\beta}\sum_{l=1}^\infty \|A^{(\beta-1)/2}e_l\|^2 \\
&= Ch^{2\beta}\|A^{(\beta-1)/2}\|^2_{HS},
\end{aligned}$$

where

$$\begin{aligned}
\|A^{(\beta-1)/2}\|^2_{HS} &= \sum_{j=1}^\infty \lambda_j^{\beta-1} \\
&= \sum_{j=1}^\infty (j\pi)^{2(\beta-1)},
\end{aligned}$$

$e_l(x) = \sqrt{2}\sin(l\pi x)$ and $\lambda_l = (l\pi)^2$. Thus, $\|A^{(\beta-1)/2}\|^2_{HS} < \infty$ if $\beta < 1/2$. We complete the proof of the theorem. □

Theorem 3. *Let $\hat{U}^n$ and $\hat{u}(t_n)$ be the solutions of (3.6) and (1.4), respectively. We have*

$$\sqrt{E(\|\hat{U}^n - \hat{u}(t_n)\|^2)} \le C(k^{\beta/2} + h^\beta)(1 + |u_0|_\beta), \quad for \quad 0 \le \beta < 1/2.$$

Proof. We have

$$\hat{U}^n = E^n_{kh}P_h u_0 + \sum_{j=1}^n \int_{t_{j-1}}^{t_j} E^{n-j+1}_{kh} P_h \partial^2 \widehat{W}(s,\cdot)ds,$$

and

$$\hat{u}(t_n) = E(t_n)u_0 + \int_0^{t_n} E(t_n - s)\partial^2 \widehat{W}(s,\cdot)ds.$$

Denoting $\hat{e}^n = \hat{U}^n - \hat{u}(t_n)$ and $F_n = E^n_{kh}P_h - E(t_n)$, we write

$$\hat{e}^n = \underbrace{F_n u_0}_{I} + \underbrace{\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} F_{n-j+1}\partial^2\hat{W}(s,\cdot)ds}_{II} + \underbrace{\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \big(E(t_n-t_{j-1}) - E(t_n-s)\big)\partial^2\hat{W}(s,\cdot)ds}_{III},$$

where

$$F_{h-j+1} = E^{n-j+1}_{kh} - E(t_n - t_j + 1).$$

Thus,

$$\|\hat{e}^n\| \le C(\|I\| + \|II\| + \|III\|).$$

For I, we have, (by Lemma 4.1 in [2])

$$\|I\| = \|F_n u_0\| \le C(k^{\beta/2} + h^{\beta})|u_0|_{\beta},$$

which implies that $\|I\| \le C(k^{\beta/2} + h^{\beta})|u_0|_{\beta}$. $Ae_l = \lambda_l e_l$ where $e_l(x) = \sqrt{2}\sin(l\pi x)$ and $\lambda_l = (l\pi)^2$.

For II, we have, by the isometry property

$$\begin{aligned}
&E\big(\|\sum_{j=1}^{n}\int_{t_j-1}^{t_j} F_{n-j+1}\partial^2\hat{W}(s,\cdot)ds\|\big)^2 \\
&\le \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \|F_{n-j+1}\|^2_{HS}ds \\
&= k\sum_{l=1}^{\infty}\sum_{j=1}^{n}\|F_{n-j+1}e_l\|^2_{HS} \\
&\le C\sum_{l=1}^{\infty}(k^{\beta/2} + h^{\beta})^2|e_l|^2_{\beta-1} \\
&\le C(k^{\beta} + h^{2\beta})\sum_{l=1}^{\infty}|e_l|^2_{\beta-1} \\
&= C(k^{\beta} + h^{2\beta})\sum_{l=1}^{\infty}\|A^{(\beta-1)/2}e_l\|^2 \\
&= C(k^{\beta} + h^{2\beta})\sum_{l=1}^{\infty}\|\lambda_l^{(\beta-1)/2}e_l\|^2 \\
&\le C(k^{\beta} + h^{2\beta})\sum_{l=1}^{\infty}\lambda_l^{\beta-1} \\
&= C(k^{\beta} + h^{2\beta})\sum_{l=1}^{\infty}(l\pi)^{2(\beta-1)}.
\end{aligned}$$

Thus, II is bounded when $\beta < \frac{1}{2}$.

For III, we have, by the isometry property,

$$
\begin{aligned}
&E\big(\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(E(t_n-t_{j-1})-E(t_n-s))\partial^2\hat{W}(s,\cdot)ds\|\big)^2\\
&\leq \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|E(t_n-t_{j-1})-E(t_n-s)\|_{HS}^2 ds\\
&= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|E(t_n-s)\big(E(s-t_{j-1})-I\big)\|_{HS}^2 ds\\
&= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|A^{\beta/2}E(t_n-s)A^{-\beta/2}\big(I-E(s-t_{j-1})\big)\|_{HS}^2 ds\\
&\leq \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|A^{\beta/2}E(t_n-s)\|_{HS}^2\|A^{-\beta/2}\big(I-E(s-t_{j-1})\big)\|^2 ds\\
&\leq Ck^{\beta}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|A^{\beta/2}E(t_n-s)\|_{HS}^2 ds\\
&\leq Ck^{\beta}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|A^{1/2}E(t_n-s)\|_{HS}^2 ds\\
&\leq Ck^{\beta}\|A^{(\beta-1)/2}\|_{HS}^2\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|A^{1/2}E(t_n-s)\|^2 ds.
\end{aligned}
$$

According to the Lemma 2.2 and Lemma 2.3 of [2], we obtain

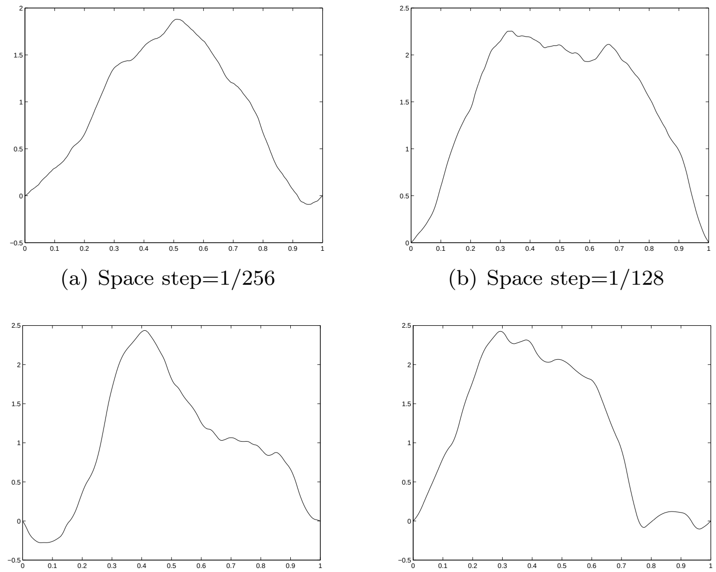
$$\|III\|^2 \leq Ck^{\beta}\|A^{(\beta-1)/2}\|^2.$$

Then, we conclude the proof. □

5. Numerical Experiments

The numerical method described in the third section is computationally tested. The experiments of the numerical method are described in this section.

Choosing $g(t,x) = 10(1+b)x^2(1-x)^2e^t - 10(2-12x+12x^2)e^t$ with $b = 0.5$ and $a = 1$. The exact solution of (1.1) would be $u(t,x) = 10e^tx^2(1-x)^2$. Now trying to get the approximation solution of (1.2). In the finite element methods, a system is solved for each time step. For each time step computation with the different scheme the results of 20 runs for different Wiener processes were averaged whereas 20 runs were averaged for each computation with the finite element methods. In the numerical experiments, we used a fine mesh to get the approximation of the exact solution. (fine mesh: $h = 1/512$ $k = 0.001$ $N = 20$ where h is the space step, k is the time step and N is the run time for different Wiener processes). Datum reported in the Table 1 are the results of error estimates when we choose different meshes. The computational results in Table 1 indicate linear rates of convergence which is in agreement with the theoretical results. Figure 1 is the result of (1.2) with different meshes.



(a) Space step=1/256 (b) Space step=1/128

(c) Space step=1/64 (d) Space step=1/32

FIGURE 1. One dimension case

TABLE 1. Computation result of one dimension case

time step k	space step h	error(with random)	error (no random)
0.001	1/256	0.1021	4.1285×10^{-5}
0.001	1/128	0.1507	1.6513×10^{-4}
0.001	1/64	0.2192	8.1171×10^{-4}
0.001	1/32	0.2933	0.0034

6. Two Dimensional Problem

6.1. **Numerical Method.** The parabolic equation in two dimension is of the form

$$\begin{aligned} u_t - a\Delta u + cu &= g, &&\text{in} \quad \Omega \times R_+, \\ u &= 0, &&\text{on} \quad \Gamma \times R_+, \\ u(0,x) &= u_0, &&\text{in} \quad \Omega. \end{aligned} \tag{6.1}$$

The parabolic stochastic partial differential equation is of the form

$$\begin{aligned} \hat{u}_t - a\Delta \hat{u} + b\hat{u} = \frac{\partial^3 W}{\partial t \partial x \partial y} + g, &\quad \text{in} \quad \Omega \times R_+, \\ \hat{u}(t,x) = 0, &\quad \text{on} \quad \Gamma \times R_+, \\ \hat{u}(0,x) = u_0, &\quad \text{in} \quad \Omega, \end{aligned} \tag{6.2}$$

where $\Omega = [0,1] \times [0,1]$ is a bounded domain in R^2 with smooth boundary Γ, $\hat{u} = \hat{u}(t,x,y)$. $\widehat{W} = \widehat{W}(t,x,y)$ is the approximation of Wiener process. $\frac{\partial^3 W}{\partial t \partial x \partial y}$ denotes the mixed third-order derivative of the Brownian sheet.

We employ the finite element method, S_h denotes the continuous piecewise linear finite element space, $\partial^3 \widehat{W}$ denotes the $\frac{\partial^3 W}{\partial t \partial x \partial y}$, find $\hat{u}_h(t)$, belonging to S_h for each t, such that

$$(\hat{u}_{h,t}, \chi) + a(\nabla \hat{u}_h, \nabla \chi) + b(\hat{u}, \chi) = (\partial^3 \widehat{W}, \chi) + (g, \chi), \quad \forall \chi \in S_h, \quad t > 0.$$

TABLE 2. Computation result of two dimension case

time step	space step	error(with random)	error (no random)
0.01	refine mesh two times	0.8383/0.8058	1.8380×10^{-9}
0.01	refine mesh one time	1.3145/1.1824	5.5096×10^{-9}
0.01	refine mesh zero time	1.4127/1.7806	4.4182×10^{-9}

Using the backward Euler method, let $\widehat{U} \in S_h \subset H_0^1$.

$$(\frac{\widehat{U}^n - \widehat{U}^{n-1}}{k}, \chi) + a(\nabla \widehat{U}^n, \nabla\chi) + b(\widehat{U}^n, \chi) = (\frac{\widehat{W}(t_n) - \widehat{W}(t_{n-1})}{k}, \chi) + (g_n, \chi) \tag{6.3}$$

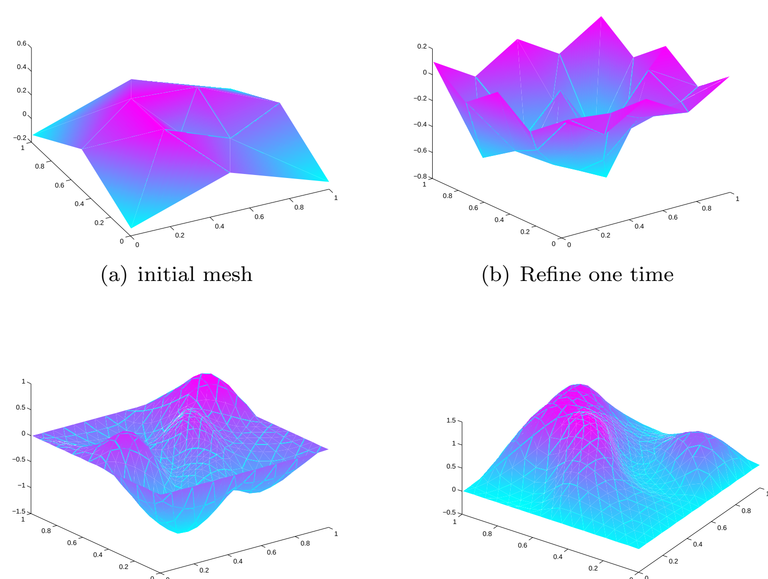
The Backward-Euler method is considered of the linear system form

(6.4)

$$\sum_{l=1}^{M} \widehat{U}_j^n(\phi_j, \phi_l) + ka\sum_{l=1}^{M} \widehat{U}_j^n(\nabla\phi_j, \nabla\phi_l) + kb\sum_{l=1}^{M} \widehat{U}_j^n(\phi_j, \phi_l) = \sum_{l=1}^{M} \widehat{U}_j^{n-1}(\phi_j, \phi_l) + (\Delta\widehat{W}^n, \phi_l) + k(g_n, \phi_l)$$

where $\widehat{U}^n(x) = \sum_{j=1}^{M} \widehat{U}_j^n \phi_j(x)$ for $l = 1, 2, \ldots, M$.

6.2. **Numerical Experiments.** In this section, I chose different meshes to compute the error. Firstly, we created a initial mesh in PDEtool, then, refined mesh three times. In the numerical experiments, we used a fine mesh to approximate the exact solution. We got the fine mesh by refining initial mesh three times. And then calculating error estimates with different meshes. The initial value of (6.2)is $u_0 = \sin(\pi x_1)\sin(\pi x_2)$. Let $g = 0$, $b = 0$ and $a = 1$, implementing solver for two dimension case. Datum reported in the Table 2 are the results of error estimates when we choose different meshes. Figure 2 is the result of (6.2) with different meshes.



(a) initial mesh

(b) Refine one time

(c) Refine two times

(d) Fine mesh

FIGURE 2. Two dimensions case

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